§6 Some further properties of $\mathbb{N}, \mathbb{N}', \mathbb{N}^*$.

Set: $\mathbb{P}_0 = \mathbb{N}$, $\mathbb{P}_1 = \mathbb{N}'$, $\mathbb{P}_2 = \mathbb{N}^*$ and $\mathbb{P}_3 = \mathbb{C} * \mathring{\mathbb{N}}$ (where $\mathring{\mathbb{N}}$ is a name for Namba forcing in $V^{\mathbb{C}}$).

We say that $\gamma = \langle \gamma_i \mid i < \omega \rangle$ is $\mathbb{P}_h$ - conforming ($h \leq 3$) iff

- γ is monotone and cofinal in ω_2^V
- $\gamma \in V[G]$ for a G which is $\mathbb{P}_h$-generic.

We prove: (assuming CH)

Theorem (As a sufficient collapse to ω_1)
If γ is $\mathbb{P}_h$ conforming, it is not $\mathbb{P}_j$ - conforming for any $j \neq h$.

But this implies:

Corollary Let $\mathbb{B}_i = BA(\mathbb{P}_i)$ be the canonical complete Boolean algebra over $\mathbb{P}_i$. Then $\mathbb{B}_i, \mathbb{B}_j$ are not isomorphic for $i \neq j$.

Def Working in any extension of V, define:
$\delta = \langle \delta_i \mid i<\omega \rangle$ is *magical* iff
δ is monotone, cofinal in $\omega_2 =: \omega_2^V$
and whenever $A \in V$ is club in ω_2,
then $\bigvee n \bigwedge i \geq n\ \delta_i \in A$.

Lemma 1 Let $\gamma = \langle \gamma_i \mid i<\omega \rangle$ be $\mathbb{N}'$-generic. Then γ is not magical. In fact, there is $A \in V$ club in ω_2 s.t. $\gamma_i \notin A$ for $i<\omega$.

proof.
We show that for every $T \in \mathbb{N}'$
there is $T' \leq T$, $A \in V$ s.t.
A is club and $\bigwedge t \in T'\ \mathrm{rng}(t) \cap A = \emptyset$.
We first thin T as follows.
Let $s = \mathrm{stem}(T)$. We construct
T_n s.t. $T = T_0 \geq T_1 \geq \cdots \geq T_n \geq$
and $T_m \mid n = T_n \mid n$ for $m \geq n$,
(where $T \mid n = \{ t \in T \mid |t| \leq n \}$)

For $i \leq |u|$ set $T_i = T$. Now let $m \geq 1$. For each $t \in T_i$, $|t| = i$, select A_t club in ω_2 s.t. $\{\alpha \mid t^\frown\langle\alpha\rangle \in T_i\} \setminus A_t$ is unbounded in ω_2.

Set $T_{i+1} =$ (the set of $t' \in T_i$ s.t. $t'(i) \notin A_{t \restriction i}$).

$$= \bigcup\{T_{t^\frown\langle\alpha\rangle} \mid t^\frown\langle\alpha\rangle \in T_i \wedge |t| = i \wedge \alpha \notin A_t\},$$

Then $T' = \bigcup_i T_i \mid i' = \bigcap_i T_i$ has the property that:

$$t \in T',\ |t| = i \to \{\alpha \mid t^\frown\alpha \in T\} \cap A_t = \emptyset.$$

For each $m \geq |u|$ sets:

$$A_m = \{\lambda \mid \bigwedge t \in \lambda^{<\omega} (t \in T' \to \lambda \in A_t)\},$$

Then A_m is club in ω_2 and:

$$(t \in T',\ |t| = i,\ t^\frown\langle\alpha\rangle \in T') \to \alpha \notin A_m.$$

Set $A = \bigcap_m A_m$. Then:

$$t \in T' \to \bigwedge_i t(i) \notin A.$$

But we have shown that the set of such T' is dense in $\mathbb{N}'$. Hence if G is $\mathbb{N}'$-generic,

there is such T', A with $T' \in G$.
Hence $\Lambda i\, \gamma_i \notin A$. QED (Lemma 1).

<u>Lemma 2</u> Let W be an extension of V with $H_{\omega_1}^W = H_{\omega_1}^V$. Let $\gamma = \langle \gamma_i \mid i < \omega \rangle$, $\gamma' = \langle \gamma'_i \mid i < \omega \rangle \in W$ be monotone ~~&~~ cofinal in ω_2. Then

$$\gamma' \in V[\gamma].$$

proof.

Let $f_i \in V$ s.t. $f_i : \omega_1 \xrightarrow{\text{onto}} \gamma_i$

Set $X_\alpha = \bigcup_i f_i''\alpha$ for $\alpha \leq \omega_1$

Then $\langle X_\alpha \mid \alpha \leq \omega_1 \rangle \in V[\gamma]$ and

For $i < \omega$ let α_i be least s.t. $\gamma'_i \in X_{\alpha_i}$. Then $\alpha = \sup_i \alpha_i < \omega_1$. ~~Hence~~ But there is $g \in V[\gamma]$ s.t. $g : \omega_1 \xrightarrow{\text{onto}} X_\alpha$. Hence, $\{\gamma'_i \mid i < \omega\} = g''a \in V[\gamma]$ for an $a \subset \omega_1$. QED (Lemma 2)

<u>Lemma 3</u> There is no magical sequence in $V[\gamma]$ where γ is $\mathbb{N}'$-generic.

proof.

Let $\delta = \langle \delta_i \mid i < \omega \rangle$ be a counterexample. Let $A \in V$ s.t. $\gamma_i \notin A$ for $i < \omega$. Let $n < \omega$ s.t. $\delta_i \in A$ for $i \geq n$. Then $\langle \delta_i \mid i \geq n \rangle$ is a counter example. which is disjoint from γ. Hence we may w.l.o.g. assume $\{\delta_i \mid i < \omega\} \cap \{\gamma_i \mid i < \omega\} = \emptyset$. By thinning the sequence δ we may also assume that there is a monotone sequence $\langle n_i \mid i < \omega \rangle$ in ω s.t.

$\gamma_{n_i} < \delta_i < \gamma_{n_i + 1}$ for $i < \omega$. Let $B \in V$

and $B \subset \omega_2$ s.t. $H_{\omega_1} = L_{\omega_1}[B]$ and $\omega_1 =$ the largest cardinal in $L_{\omega_2}[B]$.

Then there is $\alpha < \omega_1$ s.t. $\{\delta_i \mid i < \omega\} \subset X$, where $X =$ the smallest $X \prec \langle L_{\omega_2}[B], B \rangle$ s.t. $\alpha \cup \{\gamma_n \mid n < \omega\} \subset X$. We can assume w.l.o.g. that $\alpha = \omega_1 \cap X$.

Let $\pi : \langle L_\beta[\bar{B}], \bar{B} \rangle \xrightarrow{\sim} X$.

Let $\pi(\bar{\gamma}_n, \bar{\delta}_i) = \gamma_n, \delta_i$.

Then $\bar{\gamma}_{n_i} < \bar{\delta}_i < \bar{\gamma}_{n_i+1}$ for $i < \omega$.

Let $T \in G$ (where $\gamma = \bigcup \cap G$ and G is $\mathbb{N}'$-generic) s.t.

$T \Vdash \big(\dot{\pi} : \langle L_\beta[\breve{\bar{B}}], \bar{B}\rangle \prec \langle L_{\omega_2}[B], B\rangle$

and $\dot{\pi}(\breve{\bar{\gamma}}_n) = \dot{\gamma}_n$ for $n < \omega$

and $\dot{\pi}(\breve{\bar{\delta}}_n) = \dot{\delta}_n$ for $n < \omega\big)$

where $\dot{\gamma}$ is the canonical name for γ and $\dot{\delta}^G = \delta_p$ and $\dot{\pi}$ is defined from $\dot{\alpha}$, $\dot{\gamma}$ as above.

Then

(1) If $t \in T$, $|t| = n_i + 2$, then there is $\eta < t(n_i+1)$ s.t.

$T_t \Vdash \check{\eta} = \dot{\delta}_i$.

proof.

Let $\bar{f}_n$ = the $\langle L_\beta[\bar{B}], \bar{B}\rangle$-least f s.t. $f : \bar{\alpha} \xrightarrow{\text{onto}} \bar{\gamma}_n$ $(n < \omega)$

Let f_t = the $\langle L_{\omega_2}[B], B\rangle$-least f s.t. $f : \omega_1 \xrightarrow{\text{onto}} t(n)$

for $t \in T$, ...

Whenever $G \ni T$ is $\mathbb{N}'$-generic, then $\dot{\pi}^G(\bar{f}_n(\nu)) = f_{\langle\gamma_0, \ldots, \gamma_n\rangle}(\nu)$

for $\nu < \bar{\alpha}_n$

Thus, setting $\pi_t = f_t \circ \bar{f}_n^{-1}$ for $|t| = n$, $t \in T$, we have:

$$T_t \Vdash \check{\pi} \upharpoonright \check{\bar{\delta}}_n = \check{\pi}_t \quad \text{for } n < \omega.$$

In particular: If $|t| = n_i + 2$, $t \in T$ and $\eta = \pi_t(\bar{\delta}_i)$, then $T_t \Vdash \check{\eta} = \mathring{\delta}_i$. QED(1)

We now thin T as follows. For $i < \omega$ we define T_i s.t.

$T_{i+1} \leq T_i$, $T_i \mid n_i+1 = T_j \mid n_i+1$ and $\operatorname{stem}(T_i) = \operatorname{stem}(T_j)$ for $i \leq j$.

We then set: $T' = \bigcap_i T_i = \bigcup_i T_i \mid n_i+1$.

Thus $T' \in \mathbb{N}'$. Define T_i by:

$T_i = T$ if $n_i + 1 < |s|$ where $s = \operatorname{stem}(T)$

If $n_i + 1 \geq |s|$, proceed as follows:

We pick for each $t \in T_i$ s.t. $|t| = n_i + 1$ a set

$$A_i = A_i^t \subset \{\alpha \mid t^\frown\langle\alpha\rangle \in T_i\}$$

as follows:

Set: $\eta_\alpha = \eta_\alpha^t = \pi_{t^\frown\langle\alpha\rangle}(\bar{\delta}_i)$.

(Hence $t(n_i) < \eta_\alpha < \alpha$).

Case 1 For all $\mu < \omega_1$ there is α s.t. $\eta_\alpha > \mu$.

Successively pick α_i s.t. $\eta_{\alpha_i} > \sup_{j < i} \alpha_j$.

Set: $A = \{\alpha_i \mid i < \omega_2\}$.

Case 2 Case 1 fails.

Then there is η s.t.

$$\mathrm{card}(\{\alpha \mid \eta_\alpha = \eta\}) = \omega_2.$$

Set $A = \{\alpha \mid \eta_\alpha = \eta\}$

We then set:

$$T_{i+1} = \{t \mid t \in T_i \text{ and } t(n_i + 1) \in A_i^t \text{ if } |t| \geq n_i + 2\}$$

$$= \bigcup \{T_{t^\frown\langle\alpha\rangle} \mid |t| = n_i + 1 \wedge t \in T_{i+1} \wedge \alpha \in A_i^t\}$$

If $t \in T'$ and $|t| = n_i + 1$, we then have for $t^\frown\langle\alpha\rangle \in T'$:

Either $\eta_\alpha^t < \min\{\alpha \mid t^\frown\langle\alpha\rangle \in T'\}$

or $\eta_\alpha^t > \sup\{\beta \mid t^\frown\langle\beta\rangle \in T' \wedge \beta < \alpha\}$

If we let B_t = the closure of $\{\alpha \mid t^\frown \alpha \in T'\}$ in ω_2, then $\eta^t_\alpha \notin B_t$. Set:

$B_i = \{\lambda \mid \bigwedge t \in \lambda^{n_i+1} (t \in T' \to \lambda \in B_t)\}$

Then $\eta^t_\alpha \notin B_i$, since otherwise $\eta^t_\alpha \in B_t$, since $t(h) < \eta^t_\alpha$ for $h \leq n_i$.

If we set: $B = \bigcap_{i<\omega} B_i$, it follows that $\eta^t_\alpha \notin B$ whenever $t^\frown\langle\alpha\rangle \in T'$ and $|t| = n_i + 1$.

This means that if $G \ni T'$ is $\mathbb{N}$-generic, given the generic sequence $\langle \gamma_n \mid n < \omega \rangle$, then $\delta_i \notin B$ for $i < \omega$, (since $\delta_i = \eta^{\langle \gamma_0, \ldots, \gamma_{n_i} \rangle}_{\gamma_{n_i+1}}$).

Thus $\langle \delta_i \mid i < \omega \rangle$ is not magical, Contr.! QED (Lemma 3)

Lemma 4. Let γ be $\mathbb{N}^*$-generic. Then γ is a maximal magical sequence in the following sense: Let $\delta = \langle \delta_i \mid i<\omega \rangle$, where $\delta \in V[\gamma]$ is magical. Then

$$\bigvee n \bigwedge m \geq n \quad \delta_m \in \{\gamma_j \mid j<\omega\}.$$

proof.

Suppose not. Let δ be a counterexample. Then there is a monotone function $\langle n_i \mid i<\omega \rangle$ s.t. $\bigvee_j \; \delta_j \in (\gamma_{n_i}, \gamma_{n_i+1})$ for $i<\omega$. Pick $\delta_{m_i} \in (\gamma_{n_i}, \gamma_{n_i+1})$. Then $\langle \delta_{m_i} \mid i<\omega \rangle$ is magical. Hence we may assume w.l.o.g. that $\delta_i \in (\gamma_{n_i}, \gamma_{n_i+1})$ for $i<\omega$, where $\langle n_i \mid i<\omega \rangle$ is monotone. We again let $B \subset \omega_2$ s.t. $L_{\omega_2}[B] = H_{\omega_2}$ and ω_1 is the largest cardinal in $L_{\omega_2}[B]$ (where $B \in V$). Define α, X, $\pi : \langle L_\rho[\bar{B}], \bar{B} \rangle \xrightarrow{\sim} X$ exactly as in Lemma 3. Let $\pi(\bar{\gamma}_n, \bar{\delta}_i) = \gamma_n, \delta_i$. Then $\bar{\gamma}_{n_i} < \bar{\delta}_i < \bar{\gamma}_{n_i+1}$ as before. Let G be the $\mathbb{N}^*$-generic set giving the sequence γ, ... $\bar{G}$

There is $T \in G$ s.t.

$$T \Vdash (\overset{\smile}{\pi'} : \langle L_\beta[\bar{B}], \bar{B} \rangle \prec \langle L_{\omega_2}[B], B \rangle \wedge$$

$$\wedge\ \pi'(\check{\gamma}_n) = \gamma'_n \text{ for } n < \omega$$

$$\wedge\ \pi'(\check{\delta}_n) = \delta'_n \text{ for } n < \omega)$$

where $\dot{\gamma}^G = \gamma$ and $\dot{\delta}^G = \delta$ and π' is defined from $\check{\alpha}, \dot{\gamma}$ as above.

Then just as before we get:

(1) For $t \in T$, $|t| = n_i + 2$, there is $\eta < t(n_i + 1)$ s.t. $T_t \Vdash \check{\eta} = \dot{\delta}_i$.

For $t \in T$, $|t| = n_i + 1$, $t^\frown\langle\alpha\rangle \in T$, let $\eta_\alpha = \eta^t_\alpha =:$ that η s.t. $T_{t^\frown\langle\alpha\rangle} \Vdash \check{\eta} = \dot{\delta}_i$.

Imitating the proof in Lemma 3 we then define successive thinnings T_i of T & set: $T' = \bigcap_i T_i$.

For $n_i + 1 < |s|$, $s = \operatorname{stem}(T)$, we set $T_0 = T$. Now let $n_i + 1 \geq |s|$ & let T_i be given. For $t \in T_i$, $|t| = n_i + 1$, we define:

$$A = A_t = \{\alpha \mid t^\frown\langle\alpha\rangle \in T_i\}$$

as follows: Since $\eta_\alpha < \alpha$ for $t^\frown\langle\alpha\rangle \in T_i$, there must be

η s.t. $\{\alpha \mid t^\frown\alpha \in T_i \wedge \eta_\alpha = \eta\}$ is stationary.

Set $A = \{\alpha \mid t^\frown\alpha \in T_i \wedge \eta_\alpha = \eta\}$.

Set $T_{i+1} = \{t \in T_i \mid t(n_i+1) \in A_{t \upharpoonright n_i+1}$ if $|t| \geq n_i+1\}$.

It follows as before that if $B = B_t =$ the closure of $\{\alpha \mid t^\frown\langle\alpha\rangle \in T'\}$ for $|t| = n_i+1$, then $\eta_{t\langle\alpha\rangle} \notin B_t$ for $t^\frown\langle\alpha\rangle \in T'$. Setting

$$B_i = \{\lambda \mid \bigwedge t \in \lambda^{n_i+1}\,(t \in T' \to \lambda \in B_t)\}$$

we again have: $\eta_{t^\frown\langle\alpha\rangle} \notin B_i$. But then $\eta_{t^\frown\langle\alpha\rangle} \notin B$ where $B = \bigcap_i B_i$.

Hence if $T' \in G$ & G is $\mathbb{N}^*$-generic, we conclude that $\delta_i \notin B$ for $i < \omega$. Contradiction! QED (Lemma 4)

Now define $\delta = \langle \delta_i \mid i < \omega \rangle$ to be quasi-magical iff whenever $F \in V$ and $F : \omega_2 \to \omega_2$, then $\forall n \,\Lambda m \geq n \;\delta_{m+1} \geq F(\delta_m)$.

We recall the following known theorem about IN:

Lemma 5 Let G be IN-generic. Then:

(a) If $\delta = \langle \delta_i \mid i < \omega \rangle \in V[G]$ is monotone + cofinal in ω_2, then δ is an IN-generic sequence.

(b) No IN-generic sequence is quasi magical.

Finally we note:

Lemma 6 Let G be $\mathbb{C} \times \mathrm{IN}$-generic. Let $\delta = \langle \delta_i \mid i < \omega \rangle \in V[G]$ be magical. Then there is a magical $\delta' = \langle \delta'_i \mid i < \omega \}$ s.t.

$$\{\delta_i \mid i < \omega\} \cap \{\delta'_i \mid i < \omega\} = \emptyset.$$

proof.

Let $V[G] = V[c][\gamma]$ where c is $\mathbb{C}$-generic and γ is IN-gen. over $V[c]$. Then there is a monotone sequence

$\langle n_i \mid i<\omega\rangle$ s.t. $(\delta_{n_i}, \delta_{n_{i+1}}) \cap C \neq \emptyset$.

This follows from the $\mathbb{C}$-genericity of C. Pick $\delta_i' \in (\delta_{n_i}, \delta_{n_{i+1}})$. Then $\delta' = \langle \delta_i' \mid i<\omega\rangle$ has the desired property.

QED (Lemma 6).

Now set: $\mathbb{P}_0 = \mathbb{N}$, $\mathbb{P}_1 = \mathbb{N}'$, $\mathbb{P}_2 = \mathbb{N}^*$, $\mathbb{P}_3 = \mathbb{C} * \hat{\mathbb{N}}$.

Def $\delta = \langle \delta_i \mid i<\omega\rangle$ is $\mathbb{P}_n$ - <u>conforming</u> iff δ is monotone and cofinal in ω_2 and $\delta \in V[G]$ for $\mathbb{P}_n$-generic G.

<u>Theorem</u> If δ is $\mathbb{P}_n$ - conforming ($n \leq 3$), then it is not $\mathbb{P}_m$ conforming for an $m \neq n$.

proof.

(1) Let δ be $\mathbb{P}_0$ - conforming. Then δ is not $\mathbb{P}_n$ - conforming for $n > 0$.

proof.

If $\delta \in V[G]$ + G is $\mathbb{P}_n$ - generic, then there is a quasi magical $\gamma \in V[G]$. ∴ Hence $\gamma \in V[\delta]$ by Lemma 2, contradicting Lemma 5.

(It follows, of course, that no $\mathbb{P}_n$-conforming sequence is $\mathbb{P}_0$-conforming if $n > 0$.)

Using Lemma 1, we then get

(2) Let σ be $\mathbb{P}_1$-conforming. Then σ is not $\mathbb{P}_n$ conforming for $n > 1$.

Using Lemma 4 and Lemma 6 we then get

(3) Let σ be $\mathbb{P}_2$-conforming. Then σ is not $\mathbb{P}_3$-conforming.

QED