§ 3 The successor step

We are given $\langle \mathbb{B}_i \mid i \leq \mu \rangle$ satisfying (a)–(h) and wish to construct $\mathbb{B}_{\mu+1}$ s.t. (a)–(h) continue to hold. Let $\gamma = \gamma_{\mu+1}$. Then $\gamma = \omega_2^{V^{\mathbb{B}_\mu}}$, $2^{\gamma} = \gamma$, and $\gamma = \beta_\mu^+$ if β_μ exists. Otherwise $\gamma = \mu = \gamma_\mu$ is inaccessible in V. In either case we know that $\mathbb{B}_\mu \subset H_\gamma$.

§ 3.1 The first successor case

Suppose that $\gamma \in A_0$. Then GCH holds below κ and we wish to make γ ω_1-cofinal without collapsing $\beta_{\mu+1} = \gamma^+$. Thus we force with $\mathrm{coll}(\omega_1, \omega_2)$ over $V[G]$, where G is $\mathbb{B}_\mu$-generic. In other words we let $\mathbb{B}_{\mu+1} \simeq \mathbb{B}_\mu * \mathring{\mathbb{B}}$, where $\Vdash_{\mathbb{B}_\mu} \mathring{\mathbb{B}} = \mathrm{coll}(\omega_1, \omega_2)$.

We verify (a)–(h). (a) is straightforward. (b) holds, since $\Vdash_{\mathbb{B}_\mu} \mathring{\mathbb{B}}$ is subcomplete. (c) is immediate, since $\mu+1$ is not a limit point and (a)–(h) hold of $\langle \mathbb{B}_i \mid i \leq \mu \rangle$. (d) follows by § 2 Lemma 4.1. Similarly for (d). (e) follows by § 2 Lemma 4.2. We now sketch the proof of (f). Let B be $\mathbb{B}_h$-generic, where $h \leq \mu$.

Set $\tilde{\mathbb{B}}_i = \mathbb{B}_i / B$ for $h \le i \le \mu+1$. (Thus $\langle \tilde{\mathbb{B}}_{h+j} \mid j \le \mu+1-h \rangle$ is the new iteration in $V[B]$.) We first note that if B' is $\mathbb{B}_\mu$-generic over V, then

$$\mathbb{B}_{\mu+1} / B' \simeq \mathring{\mathbb{B}}^{B'} = BA(\mathrm{coll}(\omega_1, \omega_2))$$

in $V[B']$. (Recall that $\mathbb{B}_{\mu+1} \simeq \mathbb{B}_\mu * \mathring{\mathbb{B}}$ where $\Vdash_{\mathbb{B}_\mu} \mathring{\mathbb{B}} = BA(\mathrm{coll}(\omega_1, \omega_2))$.)

Now let $\tilde{B}$ be $\tilde{\mathbb{B}}_\mu$-generic over $V[B]$. Then $\tilde{\mathbb{B}}_{\mu+1} / \tilde{B} = (\mathbb{B}_{\mu+1}/B)/\tilde{B} \simeq \mathbb{B}_{\mu+1}/B'$, where $B' = B * \tilde{B} = \{ b \in \mathbb{B}_\mu \mid b/B \in \tilde{B} \}$ is $\mathbb{B}_\mu$-generic over V. Hence there is $\dot{\sigma} \in V[B]^{\tilde{\mathbb{B}}_\mu}$ s.t.

$$\Vdash_{\tilde{\mathbb{B}}_\mu} \dot{\sigma} : \tilde{\mathbb{B}}_{\mu+1} / \dot{B} \overset{\sim}{\longleftrightarrow} BA(\mathrm{coll}(\omega_1, \omega_2))$$

in $V[B]$, ($\dot{B}$ being the canonical generic name). Let

$$\Vdash_{\tilde{\mathbb{B}}_\mu} \mathring{\mathring{\mathbb{B}}} = BA(\mathrm{coll}(\omega_1, \omega_2))$$

Define $\sigma : \tilde{\mathbb{B}}_{\mu+1} \overset{\sim}{\longleftrightarrow} \tilde{\mathbb{B}}_\mu * \mathring{\mathring{\mathbb{B}}}$ in $V[B]$ by $\sigma(a) =$ that $\mathring{a}'$ s.t. $\Vdash_{\tilde{\mathbb{B}}_\mu} \mathring{a}' = \dot{\sigma}(\check{a}/\dot{B})$

Then for $b \in \tilde{\mathbb{B}}_\mu$ we have:

$$\sigma(b) = b' \text{ where } \Vdash_{\tilde{\mathbb{B}}_\mu} b' = \begin{cases} 1 \text{ if } b^\vee \in \mathring{B} \\ 0 \text{ if } b^\vee \in \mathring{B} \end{cases}$$

- i.e. $\sigma \upharpoonright \tilde{\mathbb{B}}_\mu$ is the natural injection.

Thus $\tilde{\mathbb{B}}_{\mu+1}$ satisfies precisely those conditions which we had placed upon $\mathbb{B}_{\mu+1}$ in V. Thus we can carry out all of our proofs in $V[B]$ with $\langle \tilde{\mathbb{B}}_{h+j} \mid j \leq \mu+1-h \rangle$ in place of $\langle \mathbb{B}_i \mid i \leq \mu+1 \rangle$.

Finally, we note that, since GCH holds below κ, elementary consideration give us: $\mathbb{B}_\mu * \mathring{\mathbb{B}}$ has cardinality $\leq \gamma^+$. Hence we can choose $\mathbb{B}_{\mu+1}$ s.t. $\mathbb{B}_{\mu+1} \subset H_{\gamma^+}$.

This completes the first successor case. The second will be much harder, since we shall need $\mathbb{B}_{\mu+1} \simeq \mathbb{B}_\mu * \mathring{\mathbb{B}}$ for a $\mathring{\mathbb{B}}$ which has yet to be defined.

The second successor case

Now suppose $\gamma \notin A_0$, where $\gamma = \gamma_{\mu+1} = \omega_2^{V^{\mathbb{B}_\mu}}$. Then γ must acquire cofinality ω at the next stage. But then all regular cardinals in $[\gamma, \beta_{\mu+1})$ must become ω-cofinal. Recall that $\beta_{\mu+1}$ is the least β' s.t. either $\mathrm{cf}(\beta') = \omega_1$ and $2^{\underset{\smile}{\beta'}} = \beta'$, or $\beta' \in A_0$ is regular. In the latter case $\beta' = \beta^+$ is a successor cardinal, $\beta \geq \gamma$, and $2^{\underset{\smile}{\beta}} = \beta$, since GCH holds below κ if $A \neq \emptyset$. From now on let β be defined by:

$$\beta = \beta_{\mu+1} \quad \text{if } \mathrm{cf}(\beta_{\mu+1}) = \omega_1$$

$$\beta^+ = \beta_{\mu+1} \quad \text{if not,}$$

where β is a cardinal.

Now let B be $\mathbb{B}_\mu$-generic. We work in $V[B]$ to define a set of conditions $\mathbb{P} = \mathbb{P}_B$ which collapses all regular $\tau \in [\gamma, \beta]$ to ω. If $\mathrm{cf}(\beta) = \omega_1$, then β^+ becomes ω_2 in $V[B]^{\mathbb{P}}$. Otherwise β^{++} becomes ω_2. We then take:

$\mathbb{B}_{\mu+1} \simeq \mathbb{B}_\mu * \mathring{\mathbb{B}}$ where $\Vdash_{\mathbb{B}_\mu} \mathring{\mathbb{B}} = BA(\mathbb{P}_{\mathring{B}})$, $\mathring{B}$ being the canonical generic name.

Let $A \in V$ s.t. $A \subset H_\xi$ and $H_\eta^V = L_\xi^A$ *)
whenever $\eta \leq \beta$ s.t. $2^\xi = \eta$. Set:

<u>Def.</u> $M = L_\beta^{A, \mathbb{B}_\mu} =_{Df} \langle L_\beta[A, \mathbb{B}_\mu], A, \mathbb{B}_\mu \rangle$.

$N = \langle H_{\beta^+}^V, M, <, m \rangle$

where $<$ is a well ordering of N.

$Q = H_\gamma^V$.

<u>Def</u> Working in $V[B]$ where B is $\mathbb{B}_\mu$ - generic set:

$M^B = L_\beta^{A, \mathbb{B}_\mu, B}$, $N^B = \langle H_{\beta^+}^{V[B]}, M^B, < \rangle$

(<u>Note</u> M^B has the same sets as $H_\beta^{V[B]}$)

$Q^B = Q[B] =_{Df} \bigcup_{x \in Q} L_\gamma[x, B]$.

(<u>Note</u> $Q^B = H_\gamma^{V[B]}$ since $\gamma = \omega_2^{V[B]}$ is regular in $V[B]$. Hence if B, B' are $\mathbb{B}_\mu$ - generic and $V[B] = V[B']$, then $Q^B = Q^{B'}$.)

Working in $V[B]$ we now define:

$\Gamma_* =$ the collection of $\langle S, C \rangle$ s.t.

- S is a transitive set
- $S \models (ZFC^- + \omega_1$ is the largest cardinal)
- $C \subset S$ cofinally
- C is countable

(Recall that "$C \subset S$ cofinally" means $\bigcup C = S$.)

Def For $u = \langle s_u, c_u \rangle$, $v = \langle s_v, c_v \rangle \in \Gamma_*$ set:

$$\pi : u \triangleleft_* v \underset{Df}{\longleftrightarrow} (\pi : s_u \prec s_v \wedge \pi '' c_u = c_v)$$

Def $u \triangleleft_* v \underset{Df}{\longleftrightarrow} \bigvee \pi \; \pi : u \triangleleft_* v$

Def For $u = \langle s_u, c_u \rangle \in \Gamma_*$ set: $\alpha_u = \alpha_{s_u} = \omega_1^{s_u}$.

The following facts are readily verified and will be stated here without proof:

Fact 1 Let $\langle s, c \rangle \in \Gamma_*$, $\alpha = \alpha_s$. Then
$s = \{ f(\nu) \mid f \in c \wedge \nu < \alpha \}$

Fact 2 If $\pi : u \triangleleft_* v$, then $\alpha_u \leq \alpha_v$ and
$\mathrm{rng}(\pi) = \{ f(\nu) \mid f \in c_v \wedge \nu < \alpha_u \}$

Hence:

Fact 3 For any $\alpha \leq \alpha_v$ there is at most one pair $\langle u, \pi \rangle$ s.t.
$\pi : u \triangleleft_* v$ and $\alpha_u = \alpha$.

Hence:

Fact 4 Let $u \triangleleft_* v$. There is exactly one π s.t. $\pi : u \triangleleft_* v$.

Def $\pi_{uv} \simeq_{Df}$ that π s.t. $\pi : u \triangleleft_* v$.

Fact 5 $\langle \pi_{uv} \mid u \triangleleft_* v \rangle$ is a continuous commutative system.

Note "continuous" means that if $u_i \triangleleft_* u_j \triangleleft v$ for $i \leq j < \lambda$, then the transitivized direct limit $\langle u, \langle \pi_{u_i u} \mid i < \lambda \rangle \rangle$ of $\langle \langle u_i \mid i < \lambda \rangle, \langle \pi_{u_i u_j} \mid i \leq j < \lambda \rangle \rangle$ exists and There is $\pi : u \triangleleft_* v$ defined by

$$\pi \pi_{u_i u} = \pi_{u_i v} \quad (i < \lambda).$$

Hence:

Fact 6 $\{d_u \mid u \triangleleft_* v \wedge u \neq v\}$ is closed in d_v.

We now define:

Def R is a smooth model iff

- $R = L_\beta^{\vec{A}}$ for some $A_1, \ldots, A_n, \beta$
- R models ZFC^- or Zermelo set theory
- $L_\eta[\vec{A}] = H_\eta^R$ whenever $2^\xi = \eta$ in R.

Def Γ = the set of $\langle R, C \rangle$ s.t.

- R is a smooth model
- $\langle Q, C \rangle \in \Gamma_*$ where $Q = H_{\omega_2}^R$.

We also write $Q_R = Q_{\langle R, C \rangle} = H_{\omega_2}^R$.

Def Let $u = \langle R_u, C_u \rangle$, $v = \langle R_v, C_v \rangle \in \Gamma$.
$\pi : u \lhd v$ iff

- $\pi : R_u \prec R_v$
- $\pi \upharpoonright Q_u : \langle Q_u, C_u \rangle \lhd_* \langle Q_v, C_v \rangle$
- There is R_{uv} s.t. $\langle R_{uv}, \pi \rangle$ is the liftup of $\langle R_u, \pi \upharpoonright Q_u \rangle$

(Hence the map π is wholly determined by $\pi \upharpoonright Q_u$.)

Def $u \lhd v$ iff $\bigvee \pi \; \pi : u \lhd v$.

It follows easily that:

Fact 7 Let $u \lhd v$. There is exactly one π s.t. $\pi : u \lhd v$.

Def $\pi_{uv} \simeq$ that π s.t. $\pi : u \lhd v$

Fact 8 $\langle \pi_{uv} \mid u \lhd v \rangle$ is a continuous commutative system.

However, the analogue of Fact 3 does not hold for Γ, since $\{u \mid u \lhd v\}$ need not be linearly ordered by $\lhd$.

Nonetheless we do have:

<u>Fact 9</u> Let $u, w \lhd v$, $rng(u,v) \subset rng(w,v)$. Then $u \lhd w$ and $\pi_{wv}\pi_{uw} = \pi_{uv}$.

The following fact will often be used tacitly:

<u>Fact 10</u> Let $\pi : \langle R, C\rangle \lhd \langle R', C'\rangle$. Let $\gamma \in R$ s.t. $2^{\tau} = \gamma$ in R and $\gamma \geq \omega_2^R$. Let $\pi(\gamma) = \gamma'$, $R = L_\beta^{\vec{A}}$, $R' = L_{\beta'}^{\vec{A}'}$.

Set: $\bar{R} = L_\gamma^{\vec{A}}$, $\bar{R}' = L_{\gamma'}^{\vec{A}'}$, $\bar{\pi} = \pi \upharpoonright \bar{R}$,

Then $\bar{\pi} : \langle \bar{R}, C\rangle \lhd \langle \bar{R}', C\rangle$.

proof.

Clearly $\bar{\pi} : \bar{R} \prec \bar{R}'$ and $\bar{R}$ models ZFC$^-$ or Zermelo. Moreover $\bar{\pi}''C = C'$ and $Q_{\bar{R}} = Q_R$, $Q_{\bar{R}'} = Q_{R'}$. Hence $\bar{\pi} \upharpoonright Q_{\bar{R}} : \langle Q_{\bar{R}}, C\rangle \lhd_* \langle Q_{\bar{R}'}, C'\rangle$.

<u>Claim</u> Let $\bar{\pi} : \bar{R} \to \tilde{R}$ cofinally. Then $\langle \tilde{R}, \bar{\pi}\rangle$ is the liftup of $\langle \bar{R}, \pi \upharpoonright Q_R\rangle$.

prf.

We must show that $\bar{\pi} : \bar{R} \to \tilde{R}$ is $\omega_2^{\bar{R}}$-cofinal. Let $x \in \tilde{R}$. Then $x \in \pi(w)$ where $w \in \bar{R}$, $\bar{\bar{w}} < \omega_2$ in $\bar{R}$. Let $x \in L_{\pi(\nu)}^{\vec{A}'}$, where $\nu \in \bar{R}$. Set $z = w \cap L_\nu^{\vec{A}}$. Then $z \in \bar{R}$, $\bar{\bar{z}} < \omega_2$ in $\bar{R}$ and $x \in \bar{\pi}(z) = \pi(w) \cap L_{\pi(\nu)}^{\vec{A}'}$.

QED (Fact 10)

We return now to Q^B, M^B, N^B as defined above. We shall use an infinitary language $\mathcal{L}_B$ on N^B to define an $\mathcal{L}$-forcing $\mathbb{P}_B = \mathbb{P}_{\mathcal{L}_B}$ in $V[B]$. $\mathbb{P}_B$ is intended to add a C s.t. $\langle Q^B, C\rangle \in \Gamma_*$ without adding new reals. However, C should make not only γ ω-cofinal, but every regular $\tau \leq \beta$.

$\mathcal{L} = \mathcal{L}_B$ is the infinitary language on N^B with:

<u>Predicate</u> $\in$; <u>Constants</u> $\underline{x}$ $(x \in N^B)$, $\mathring{c}$

<u>Axioms</u>: ZFC^-, $\bigwedge v\,(v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$

for $x \in N^B$, $H_{\omega_1} = \underline{H_{\omega_1}}$, $\mathring{c} \prec \underline{M^B}$, and

$(*)$ $\bigwedge x \in \underline{M^B}\ \bigvee u \in \underline{H_{\omega_1}}\ \bigvee \pi\,(\pi : u \triangleleft \langle \underline{M^B}, \mathring{c}\rangle \wedge$
$\wedge\ x \in \mathrm{rng}(\pi) \wedge \Psi(\pi))$ *)

(This says, in particular, that every $x \in \underline{M^B}$ can be found in the liftup of a countable $\bar{M}$ by a π' s.t.
$\pi' : \langle Q_{\bar{M}}, \bar{C}\rangle \triangleleft_* \langle M^B, \mathring{c}\rangle$.)

*) where $\Psi(\pi) = \begin{cases} \bigcup \mathrm{rng}(\pi) = \underline{M^B} & \text{if } \beta \text{ is regular} \\ \pi = \pi & \text{if not} \end{cases}$

Lemma 1 $\mathcal{L}$ is consistent.

proof.

Let $\sigma : \bar{N} \prec N^B$ where $\bar{N}$ is countable and transitive. Set $\bar{Q} = H_{\omega_2}^{\bar{N}}$. Let $\sigma : \bar{Q} \prec \tilde{Q}$ cofinally. Let $\langle \tilde{N}, \tilde{\sigma} \rangle$ be the liftup of $\langle \bar{N}, \sigma \restriction \bar{Q} \rangle$. Let $k : \tilde{N} \prec N^B$ s.t. $k\tilde{\sigma} = \sigma$ and $k \restriction \tilde{Q} = \mathrm{id}$. Let $\tilde{\mathcal{L}}$ be defined on $\tilde{N}$ like $\mathcal{L}$ on N. It suffices to show:

Claim $\tilde{\mathcal{L}}$ is consistent,

since this is a $\Pi_1(\tilde{N})$ statement. We show that $\langle H_{\omega_2}, \tilde{C} \rangle$ models $\tilde{\mathcal{L}}$, where $\tilde{C} = \mathrm{rng}(\sigma \restriction \bar{Q})$. (Note that $\tilde{N} \in H_{\omega_2}$.) All axioms other than (*) are trivial. We verify (*). Set:

$D =$ the set of $\alpha < \omega_1$ s.t. there is a $u_\alpha = \langle Q_\alpha, C_\alpha \rangle$ with $\langle Q_\alpha, C_\alpha \rangle \triangleleft_* \langle \tilde{Q}, \tilde{C} \rangle$

Then D is club in ω_1 and $\alpha_0 = \omega_1^{\bar{Q}}$ is minimal in D with $u_{\alpha_0} = \langle \bar{Q}, \bar{Q} \rangle$. Since $\langle \tilde{N}, \tilde{\sigma} \rangle$ is the liftup of $\langle \bar{N}, \sigma \restriction \bar{Q} \rangle$ and $\sigma \restriction \bar{Q} = \pi_{u_{\alpha_0}, \langle \tilde{Q}, \tilde{C} \rangle}$, we see that $\langle \bar{N}, \pi_{u_{\alpha_0} u_\alpha} \rangle$ has a transitive liftup $\langle N_\alpha, \tilde{\sigma}_{\alpha_0 \alpha} \rangle$. Moreover, there is a map $\tilde{\sigma}_{\alpha, \omega_1} : N_\alpha \prec \tilde{N}$ defined by $\tilde{\sigma}_{\alpha, \omega_1}(\tilde{\sigma}_{\alpha_0 \alpha}(f)(\nu)) = \tilde{\sigma}(f)(\nu)$, where $\nu < \alpha$ and $f \in \bar{N}$ s.t. $f : \omega_1 \to \bar{N}$.

Set: $\tilde{\sigma}_{\alpha\beta} = (\tilde{\sigma}_{\beta, \omega_1})^{-1} \tilde{\sigma}_{\alpha \omega_1}$ for $\alpha \leq \beta$, $\alpha, \beta \in D \cup \{\omega_1\}$. It is clear by these definitions that:

$\langle N_\beta, \tilde{\sigma}_{\alpha\beta} \rangle$ = the liftup of $\langle N_\alpha, \pi_{u_\alpha u_\beta} \rangle$ for $\alpha \leq \beta$, $\alpha, \beta \in D \cup \{\omega_1\}$ (with $u_{\omega_1} = \langle \tilde{Q}, \tilde{C} \rangle$).

Now set: $\bar{M} = \sigma^{-1}(M)$, $M_\alpha = \tilde{\sigma}_{\alpha_0 \alpha}(\bar{M})$ for $\alpha \in D \cup \{\omega_1\}$. Then $M_{\omega_1} = \tilde{M} = \tilde{\sigma}(\bar{M})$.

Set $\sigma'_{\alpha\beta} = \tilde{\sigma}_{\alpha\beta} \upharpoonright M_\alpha$ $(\alpha \leq \beta, \alpha, \beta \in D \cup \{\omega_1\})$.

Clearly $\tilde{M} = \bigcup_{\alpha \in D} \text{rng}(\sigma'_{\alpha, \omega_1})$, so it suffices to show:

<u>Claim</u> $\sigma'_{\alpha, \omega_1} : \langle M_\alpha, C_\alpha \rangle \lhd \langle \tilde{M}, \tilde{C} \rangle$.

Clearly $\sigma'_{\alpha, \omega_1} : M_\alpha \prec \tilde{M}$ and $\sigma'_{\alpha \omega_1}{}'' C_\alpha = \tilde{C}$. Now let:

$\sigma'_{\alpha, \omega_1} : M_\alpha \to \tilde{M}_\alpha$ cofinally, (Then $\tilde{M}_\alpha = \tilde{M}$ if β is regular.)

But $\sigma'_{\alpha,\omega_1}$ is $\omega_2^{M_\alpha}$-cofinal, since $\tilde{\sigma}_{\alpha,\omega_1} : N_\alpha \to \tilde{N}$ is $\omega_2^{N_\alpha}$-cofinal. (To see this let $x \in \tilde{M}_\alpha$. Then $x \in \tilde{\sigma}_{\alpha,\omega_1}(a)$ where $a \in N_\alpha$, $\bar{\bar{a}} < \omega_2$ in N_α. Since $\sigma'_{\alpha,\omega_1} : M_\alpha \to \tilde{M}_\alpha$ cofinally, there is $b \in M_\alpha$ s.t. $x \in \sigma'_{\alpha,\omega_1}(b)$. But then $a \cap b \in M_\alpha$ + $x \in \sigma'_{\alpha,\omega_1}(a \cap b)$, where $\overline{\overline{a \cap b}} < \omega_2$ in M_α. QED (Lemma 1)

We shall make heavy use of the following lemma:

<u>Lemma 2</u> Let $\mathfrak{A}$ be a solid model of $\mathcal{L}$. Let $\langle A_n \mid n < \omega \rangle \in \mathfrak{A}$ s.t. $A_n \subset M^B$ for $n < \omega$. Then there is $u = \langle s, c \rangle \in \Gamma \cap H_{\omega_1}$ and $\pi \in \mathfrak{A}$ s.t.

- $\pi : \langle s, c \rangle \lhd \langle M^B, \mathring{c}^{\mathfrak{A}} \rangle$
- $\pi : \langle s, \bar{A}_n \rangle \prec \langle M^B, A_n \rangle$ for $n < \omega$

where $\bar{A}_n =_{\text{df}} \pi^{-1} {}'' A_n$.

proof of Lemma 2

Set $M^* = \langle M^B, A_1, A_2, \dots \rangle$. Then $M^* \in \mathfrak{A}$.

Working in $\mathfrak{A}$ we successively pick

$X_i \prec M^*$, $\pi_i : u_i \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$ s.t. $u_i \in H_{\omega_1}$

as follows: Let $<$ wellorder H_{ω_1}.

X_0 = the smallest $X \prec M^*$

Let $\langle x^j_n \mid n < \omega \rangle$ enumerate X_j.

u_i = the $<$-least $u \in H_{\omega_1}$ s.t.

$u \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$ and $x^j_n \in \mathrm{rng}(\pi_i)$

for $j, n < i$, where $\pi_i = \pi_{u, \langle M^B, \dot{c}^{\mathfrak{A}} \rangle}$.

X_{i+1} = the smallest $X \prec M^*$ s.t.

$X_i \cup \mathrm{rng}(\pi_i) \subset X$.

Let $X = \bigcup_i X_i$. Then $X = \bigcup_i (\pi_i)$

Let $\pi^* : \bar{M}^* \overset{\sim}{\longleftrightarrow} X$ where $\bar{M}^*$ is transitive.

Hence $\bar{M}^* \in H_{\omega_1}$. Let $\bar{M}^* = \langle \bar{M}, \bar{A}_1, \bar{A}_2, \dots \rangle$.

It suffices to show:

<u>Claim</u> $\pi : \langle \bar{M}, \bar{C} \rangle \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$,

where $\pi = \pi^* \upharpoonright \bar{M}$, $\bar{C} = \pi^{-1} {}'' \dot{c}^{\mathfrak{A}}$.

proof:

Clearly $\dot{c}^{\mathfrak{A}} \subset \mathrm{rng}(\pi_0) \subset X$. Hence

$\pi \upharpoonright \bar{Q} : \langle \bar{Q}, \bar{C} \rangle \triangleleft_* \langle Q^B, \dot{c}^{\mathfrak{A}} \rangle$ where $\pi^*(\bar{Q}) = Q$.

Now let $\pi : \bar{M} \to \tilde{M}$ cofinally.

It suffices to show:

<u>Claim</u> $\langle \tilde{M}, \tilde{\pi} \rangle$ is the liftup of $\langle \bar{M}, \pi \upharpoonright \bar{Q} \rangle$.

prf.
We must show that $\pi : \bar{M} \to \tilde{M}$ is $\omega_2^{\bar{M}}$-cofinal. Let $x \in \tilde{M}$. Then $x \in \pi_i(a_i)$ for an $i < \omega$, where $\bar{\bar{a}}_i < \omega_1$ in M_{a_i}, since $\tilde{M} = \bigcup X$ and $X = \bigcup_i rng(\pi_i)$.
Hence $a = \pi_i(a_i) \in X$ and $\bar{\bar{a}} \leq \omega_1$ in M.
Hence $x \in a = \pi(\bar{a})$ for an $\bar{a} \in \bar{M}$ s.t. $card(\bar{a}) \leq \omega_1$ in $\bar{M}$. QED (Lemma 2)

" " " " "

We are now ready to define the set of conditions $\mathbb{P}_B = \mathbb{P}_{\mathcal{L}_B}$.
We first set:

<u>Def</u> $\tilde{\mathbb{P}}$ = the set of $p = \langle p_0, p_1 \rangle$ s.t.
$p_0 = \langle M_p, C^p \rangle \in \Gamma \cap H_{\omega_1}$
$p_1 = F^p$ is an at most countable set of pairs $\langle a, \bar{a} \rangle$ s.t. $\bar{a} \subset M_p$, $a \subset M^B$.

<u>Def</u> For $p \in \tilde{\mathbb{P}}$ let φ_p be the conjunction of:
- $p_0 \triangleleft \langle \underline{M}^B, \dot{c} \rangle$ (Let $\pi_p =_{Df} \pi_{p_0, \langle \underline{M}^B, \dot{c} \rangle}$)
- $\pi_p : \langle \underline{m}_p, \underline{\bar{a}} \rangle \prec \langle \underline{M}^B, \underline{a} \rangle$ for all $\langle a, \bar{a} \rangle \in F^p$
- $\pi_p : \underline{M}_p \to \underline{M}^B$ cofinally if β is regular

<u>Def</u> $\mathcal{L}(p) = \mathcal{L} + \varphi_p$.

Def $R^p = \text{rng}(F^p)$, $D^p = \text{dom}(F^p)$

Def $\mathbb{P} = \mathbb{P}_B = \mathbb{P}_{\mathcal{L}_B} =_{Df} \{p \in \tilde{\mathbb{P}} \mid \mathcal{L}(p)$ is consistent$\}$

For $p, q \in \mathbb{P}$ set:
$p \leq q$ iff the following hold:

- $R^q \subset R^p$
- $q_0 \lhd p_0$
- $\pi_{q_0 p_0} : \langle M_q, \bar{a} \rangle \prec \langle M_p, a' \rangle$ whenever $\langle a, \bar{a} \rangle \in F^q$, $\langle a, a' \rangle \in F^p$.

<u>Lemma 3.1</u> Let $p, q \in \mathbb{P}$. Then $p \leq q$ iff

- $R^q \subset R^p$
- $\mathcal{L}(p) \vdash (\mathcal{L}(q) \wedge \text{rng}(\pi_q) \subset \text{rng}(\pi_p))$.

prf.
$(\rightarrow)$ Let $p \leq q$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. It follows easily that $\text{rng}(\pi_q^{\mathfrak{A}}) \subset \text{rng}(\pi_p^{\mathfrak{A}})$ and $\mathfrak{A} \models \mathcal{L}(q)$.

$(\leftarrow)$ Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Then $\text{rng}(\pi_q^{\mathfrak{A}}) \subset \text{rng}(\pi_p^{\mathfrak{A}})$. Hence by Fact 9, $q_0 \lhd p_0$ and $\pi_q^{\mathfrak{A}} = \pi_p^{\mathfrak{A}} \circ \pi_{q_0 p_0}$.

Since $\mathfrak{A} \models \mathcal{L}(q)$, we then have:
$\pi_{q_0 p_0} = (\pi_p^{\mathfrak{A}})^{-1} \pi_q^{\mathfrak{A}} : \langle M_q, \bar{a} \rangle \prec \langle M_p, a' \rangle$
for $\langle a, \bar{a} \rangle \in F^q$, $\langle a, a' \rangle \in F^p$.

QED (3.1)

We set: $\pi_{qp} =_{\mathrm{df}} \pi_{q_0 p_0}$ if $p \leq q$.

Exactly as in [LF] §0.1 – §0.3 we prove:

Lemma 3.2 Let $p \in \mathbb{P}$. Then

- $(F^p)^{-1}$ is a function
- If R^p is closed under set difference, then $F^p : D^p \leftrightarrow R^p$
- $\pi^p =_{\mathrm{df}} F^p \restriction M_p$ is injective into M^B.

The following lemma expresses a strong form of "revisability" in the sense of [LF].

Lemma 3.3 Let $p \in \mathbb{P}$. Let $C \prec M_p$ cofinally. Then $p' \in \mathbb{P}$ where:

$p'_0 = \langle M_p, C \rangle$, $p'_1 = p_1$.

proof

Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Form $\mathfrak{A}'$ by replacing $\dot{c}^{\mathfrak{A}}$ with $c' = \pi_p^{\mathfrak{A}}{}'' C$.

Claim $\mathfrak{A}' \models \mathcal{L}(p')$

We first show: $\mathfrak{A} \models \mathcal{L}$.

Note that if

$u = \langle Q_u, C_u \rangle \triangleleft_* \langle Q^B, \dot{c}^{\mathfrak{A}} \rangle$ and $\alpha_u \geq \alpha_p$,

then $\langle Q_p, C_p \rangle \triangleleft_* u \triangleleft_* \langle Q^B, \dot{c}^{\mathfrak{A}} \rangle$

and $\mathrm{rng}(\pi_{u, \langle Q^B, \dot{c}^{\mathfrak{A}} \rangle}) \supset \mathrm{rng}(\pi_p^{\mathfrak{A}} \restriction Q_p)$

(where, of course, $Q_p = H_{\omega_2}^{M_p}$.)

Set: $u' = \langle Q_u, C_{u'} \rangle$ where $C_{u'} = \pi^{-1}_{u, \langle Q^B, \dot{c}^{\mathfrak{A}} \rangle} {}'' c'$ $= \pi_{\langle Q_p, C_p \rangle, u} {}'' c$. Then $u' \triangleleft_\gamma \langle Q, c' \rangle$ and $\pi_{u', \langle Q^B, c' \rangle} = \pi_{u, \langle Q^B, \dot{c}^{\mathfrak{A}} \rangle}$, as is easily seen. But this means that if $v = \langle S_v, C_v \rangle \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$ with $\alpha_v \geq \alpha_p$, then $v' = \langle S_v, C_{v'} \rangle \triangleleft \langle M^B, c' \rangle$ where $C_{v'} = \pi_{\langle Q_p, C_p \rangle, \langle Q_v, C_v \rangle} {}'' c$, with $\pi_{v', \langle M^B, c' \rangle} = \pi_{v, \langle M^B, \dot{c}^{\mathfrak{A}} \rangle}$, since $\pi = \pi_{v', \langle M^B, c' \rangle}$ is uniquely determined by $\pi \upharpoonright Q_v$. Thus $(*)$ continues to hold in $\mathfrak{A}'$. The other axioms are trivial.

Our argument shows, in particular, that $\pi^{\mathfrak{A}}_p = \pi_{\langle M_p, c \rangle, \langle M^B, c' \rangle}$.

Hence $\pi_{\langle M_p, c \rangle, \langle M^B, c' \rangle} : \langle M_p, \bar{a} \rangle \prec \langle M^B, a \rangle$ whenever $\langle a, \bar{a} \rangle \in F^p = F^{p'}$.

Thus $\mathfrak{A} \models \mathcal{L}(p')$. QED (3.3)

We now prove the main lemma on extendability of conditions.

<u>Lemma 3.4</u> $\mathbb{P} \neq \emptyset$. Moreover, if $p, q \in \mathbb{P}$ and $\mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent, then there is an r s.t. $r \leq p, q$. If $R \subset \mathcal{P}(M^B)$ is any countable set we may, in fact, choose r s.t. $R \subset R^r$.

proof

To see $\mathbb{P} \neq \emptyset$, let $\mathfrak{A}$ be a solid model of $\mathcal{L}$. Let $u \lhd \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$, $u \in H_{\omega_1}$. Then $p \in \mathbb{P}$ where $p_0 = u$, $p_1 = \emptyset$.

Now let $\mathfrak{A} \models \mathcal{L}(p) \cup \mathcal{L}(q)$. Set:

$X = \mathrm{rng}(\pi_p^{\mathfrak{A}}) \cup \mathrm{rng}(\pi_q^{\mathfrak{A}}) \cup F^p \cup F^q \cup R$,

Then $X \in \mathfrak{A}$ is countable in $\mathfrak{A}$ with $X \subset \mathcal{P}(M)$. By Lemma 2 there is $\langle \bar{M}, \bar{c} \rangle \lhd \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$ s.t. $\langle \bar{M}, \bar{c} \rangle \in H_{\omega_1}$ and $\pi : \langle \bar{M}, \bar{A} \rangle \prec \langle M, A \rangle$ for all $A \in X$, where $\pi = \pi_{\langle \bar{M}, \bar{c} \rangle, \langle M^B, \dot{c}^{\mathfrak{A}} \rangle}$ and $\bar{A} = \pi^{-1}{}''A$,

Define r by: $r_0 = \langle \bar{M}, \bar{c} \rangle$,

$r_1 =$ the set of $\langle A, \bar{A} \rangle$ s.t. $A \in R^p \cup R^q \cup R$ and $\bar{A} = \pi^{-1}{}''A$,

Then $\mathfrak{A} \models \mathcal{L}(r)$. Hence $r \in \mathbb{P}$. But $\pi_p^{\mathfrak{A}} : p_0 \lhd \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$; $\pi_r^{\mathfrak{A}} : r_0 \lhd \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$ and $\mathrm{rng}(\pi_p^{\mathfrak{A}}) \subset \mathrm{rng}(\pi_p^{\mathfrak{A}})$, Hence $\pi : p_0 \triangleleft r_0$ where $\pi = \pi_r^{\mathfrak{A}} \circ (\pi_p^{\mathfrak{A}})^{-1}$, by Fact 9. But then, if $\langle a, \bar{a} \rangle \in F^p$, $\langle a, a' \rangle \in F^r$, we have:

$$\pi : \langle M_p, \bar{a} \rangle \prec \langle M_r, a' \rangle.$$

Hence $r \leq p$ with $\pi = \pi_{pr}$.

Similarly $r \leq q$. QED (3.4)

<u>Cor 3.5</u> p, q are compatible in $\mathbb{P}$ iff $\mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent.

prf:

($\leftarrow$) by Lemma 3.4

($\rightarrow$) If $r \leq p, q$, then $\mathcal{L}(r) \vdash \mathcal{L}(p) \cup \mathcal{L}(q)$.

QED (3.5)

<u>Cor 3.6</u> Let $p \in \mathbb{P}$, $R \subset \mathbb{P}(M^B)$ where R is countable. There is $q \leq p$ s.t. $R \subset R^q$.

<u>Cor 3.7</u> Let $p \in \mathbb{P}$, $u \subset M^B$, where u is countable. There is $q \leq p$ s.t. $u \subset \text{rng}(\pi^q)$

<u>Lemma 3.8</u> Let $p \in \mathbb{P}$, $u \subset M_p$, u finite. There is $q \leq p$ s.t. $q_0 = p_0$ and $u \subset \text{dom}(\pi^q)$.

prf: Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$.

Set: $q_0 = p_0$, $F^q = F^p \cup (\pi_p^{\mathfrak{A}} \upharpoonright u)$.

QED (3.8)

Using the extension lemma we get:

<u>Lemma 3.9</u> Let G be $\mathbb{P}$-generic. Then

(a) $\langle P_p \mid p \in G \rangle, \langle \pi_{pq} \mid q \le p \text{ in } G \rangle$ is a directed system with the limit: $\langle M^B, c^G \rangle, \langle \pi_p^G \mid p \in G \rangle$

(Moreover $\pi_p^G = \bigcup \{ \pi^q \mid q_0 = p_0 \wedge q \le p \wedge q \in G \}$.)

(b) $\pi_p^G : P_0 \triangleleft \langle M^B, c^G \rangle$ for $p \in G$

(c) $\pi_p^G : \langle M_p, \bar{a} \rangle \prec \langle M^B, a \rangle$ whenever $\langle a, \bar{a} \rangle \in F^p$.

The proof is left to the reader.

" " " " " "

Using the "revisability" lemma 3.3 we get:

<u>Lemma 3.9</u> $\mathbb{P}$ adds no reals.

prf.

Let $\Vdash \mathring{f} : \check{\omega} \to 2$. It is enough to show:

<u>Claim</u> The set Δ of p s.t. $\bigvee f\ p \Vdash \mathring{f} = \check{f}$ is dense in $\mathbb{P}$.

Let $r \in \mathbb{P}$. We construct $q \le r$ s.t. $q \in \Delta$. Let:

$N^* = \langle H_\theta, N, <, \mathring{f}, \mathbb{P}, r, \ldots \rangle$ where $\theta > 2^B$ and $<$ wellorders N^*.

Let $p \in \mathbb{P}$ conform to N^* (as defined in [LF] § 3). Set:

$\bar{N}^* = \bar{N}^*(p, N^*) = \langle H', N'^{B'}, <, \mathring{f}', \mathbb{P}', r', \ldots \rangle$

Pick $G' \ni r$ which is $\mathbb{P}'$-generic over $\bar{N}^*$.

Define q by: $q_0 = \langle M_p, c^{G'} \rangle$, $q_1 = p_1$

Then $q \in \mathbb{P}$ by the revisability lemma.

Let $f = \mathring{f}'^{G'}$. It suffices to show:

<u>Claim</u> $q \Vdash \mathring{f} = \check{f}$ and q is compatible with r.

We first show: $q \Vdash \mathring{f} = \check{f}$. Suppose not.

Then there is $q' \le q$ s.t. $q' \Vdash \mathring{f}(\check{n}) \neq \check{f}(n)$

for some n. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(q')$. Let $\pi^* \supset \pi_q^{\mathfrak{A}} \cup F^{\mathfrak{t}}$ s.t.

$\pi^*: \bar{N}^* \prec N^*$. Let $s' \in G'$ s.t. $s' \Vdash_{\mathbb{P}'} \mathring{f}'(\check{n}) = \check{f}(n)$. Set $s = \pi^*(s')$.

Then $s \Vdash_{\mathbb{P}} \mathring{f}(\check{n}) = \check{f}(n)$. Hence s, q are incompatible. We obtain a contradiction by proving:

<u>Claim</u> $\mathfrak{A} \models \mathcal{L}(q') \cup \mathcal{L}(s)$.

Prf.

$\mathfrak{A} \models \mathcal{L}(q')$ is trivial. We prove $\mathfrak{A} \models \mathcal{L}(s)$.

Note that $s_0 = s'_0$ and F^s = the set of $\langle a, \bar{a} \rangle$ s.t. $a = \pi^*(a')$ and $\langle a', \bar{a} \rangle \in F^{s'}$ for some a'. Clearly

$\pi^{G'}_{s'} : s_0 \lhd \langle M'^{B'}, c^{G'} \rangle$, since $s' \in G'$.

But $q_0 = \langle M'^{B'}, c^{G'} \rangle$ and $\pi_q^{\mathfrak{U}} : q_0 \triangleleft \langle M^B, \dot{c}^{\mathfrak{U}} \rangle$.
Set $\pi = \pi_q^{\mathfrak{U}} \circ \pi_{r'}^{G'} = \pi^* \circ \pi_{r'}^{G'}$. Then $\pi \in \mathfrak{U}$
and $\pi : r_0 \triangleleft \langle M^B, \dot{c}^{\mathfrak{U}} \rangle$. It remains only
to show:

<u>Claim</u> $\pi : \langle M_r, a \rangle \prec \langle M^B, a \rangle$ for $\langle a, \bar{a} \rangle \in F_r^r$,
since then $\mathfrak{U} \models \mathcal{L}(r)$ with $\pi = \pi_r^{\mathfrak{U}}$.
Let $a = \pi^*(a')$. Then $\langle a', \bar{a} \rangle \in F^{r'}$. Then
$\pi_{r'}^{G'} : \langle M_r, \bar{a} \rangle \prec \langle M_q, a' \rangle$ and
$\pi_q^{\mathfrak{U}} : \langle M_q, a' \rangle \prec \langle M^B, a \rangle$ since $\pi^*(\langle M_q, a' \rangle) = \langle M^B, a \rangle$

This proves: $q \Vdash \dot{f} = \check{f}$. But the last part of the proof shows that for <u>every</u> $r' \in G'$, $r = \pi^*(r')$ is compatible with q' for any $q' \leq q$ s.t. $\mathfrak{U} \models \mathcal{L}(q')$.
But $r' \in G'$ and $r = \pi^*(r')$ since $\pi^* : \bar{N}^* \prec N^*$. Hence r is compatible with q. QED (3.9)

An immediate corollary is:

<u>Cor 3.10</u> Let $\theta \geq 2^\beta$ be regular. If $G \ni p$ is $\mathbb{P}$-generic and $C = C^G$, then
$\langle H_\theta^{V[B][G]}, c \rangle$ models $\mathcal{L}(p)$.

proof:
The only problematical axiom was $H\omega_1 = \underline{H\omega_1}$, which is now seen to hold.

QED (3.10)

Def Let $C \prec M^B$ be countable and cofinal, $G^C =_{Df}$ the set of $p \in \mathbb{P}$ s.t. $p_0 \lhd \langle M^B, C\rangle$ and, letting $\pi = \pi_{p_0, \langle M^B, C\rangle}$, we have:

$\pi : \langle M_p, \bar{a}\rangle \prec \langle M^B, a\rangle$ whenever $\langle a, \bar{a}\rangle \in F^p$.

and $\pi : M_p \to M^B$ is cofinal if β is regular.

Lemma 3.11 Let G be $\mathbb{P}$-generic. Then $G = G^C$ where $C = C^G$.

proof

$G \subset G^C$ is trivial. We prove $(\supset)$

Let $p \in G^C$. If $p \notin G$ there is $q \in G$ which is incompatible with p. But then $\langle H_\theta^{V[G]}, C\rangle \models \mathcal{L}(p) \cup \mathcal{L}(q)$ for regular $\theta \geq 2^\beta$. QED (3.11)

Lemma 3.12 Let G be $\mathbb{P}$-generic. Then $\bar{\bar{\beta}} \leq \omega_1$ in $V[B][G]$

proof

For each $\xi < \beta$ there is $\langle \bar{M}, \bar{C}, \bar{\xi}\rangle \in H_{\omega_1} \cap \Gamma$ s.t. $\langle \bar{M}, \bar{C}\rangle \lhd \langle M^B, C^G\rangle$ and $\pi(\bar{\xi}) = \xi$ where $\pi = \pi_{\langle \bar{M}, \bar{C}\rangle, \langle M^B, C^G\rangle}$. This maps a subset of H_{ω_1} onto β. QED (3.12)

<u>Lemma 3.13</u> Let G be $\mathbb{P}$-generic. If $\omega_1 < \tau \le \beta$ and τ is regular in $V[B]$, then $cf(\tau) = \omega$ in $V[B][G]$

proof.

If $\tau = \beta$, then for any $p \in G$ we have $\sup \pi_p^G{}''\beta_p = \beta$ where β_p is countable.

Now let $\tau < \beta$. Let $p \in G$ s.t. $\pi_p^G(\bar\tau) = \tau$. Then each $\xi < \tau$ lies in $\pi_p^G(u)$ for a $u \in M_p$ s.t. $\bar{\bar{u}} \le \omega_1$ in M_p. But the set U of such u is countable. Set:
$\mu_u = \sup u \cap \bar\tau$ for $u \in U$. Then $\mu_u < \bar\tau$ and $\{\pi_p^G(\mu_u) \mid u \in U\}$ is cofinal in τ.

QED (3.13)

<u>Cor 3.14</u> If $\omega_1 < \delta \le \beta$ and $cf(\delta) \ne \omega_1$,*) then $cf(\delta) = \omega$ in $V[B][G]$.

We now recall [LF] §4 Lemma 4.1 which says:

<u>Fact 11</u> Let β be a cardinal in an inner model W s.t. $2^{\beta} = \beta$ in W. Let $\gamma = 2^{\beta}$ in W. Assume that in V we have: $2^{\omega} = \omega_1$, $\bar{\bar{\beta}} = \omega_1$, $cf(\beta) = \omega$. Then $\bar{\bar{\gamma}} \le \omega_1$ in V.

*) We are working over $V[B]$, so statements like $cf(\delta) \ne \omega_1$ are understood to be in the sense of $V[B]$.

Hence:

Cor 3.14.1 If $cf(\beta) \neq \omega_1$, then $\bar{\bar{\mu}} = \omega_1$ in $V[B][G]$, where $\mu = 2^\beta$.

(μ^+ remains a cardinal, however, since $\bar{\bar{\mathbb{P}}} \leq \mu$. Hence if $2^\mu = \mu$, we can conclude $cf(\mu) = \omega_1$, since otherwise μ^+ would be collapsed by Fact 11. In particular, $\mu = \beta^+$ + $cf(\mu) = \omega_1$ in $V[G]$ if GCH holds in V.)

The case $cf(\beta) = \omega_1$ is quite different as shown by:

<u>Lemma 3.15</u> Let $cf(\beta) = \omega_1$ in $V[B]$. Then β^+ remains a cardinal in $V[B][G]$ (Hence $\beta^+ = \omega_2$ in $V[B][G]$)

proof.

We imitate the proof of [LF] §4 Lemma 3.1 to show:

<u>Sublemma 3.15.1</u> $BA(\mathbb{P})$ has a dense subset of size β.

prf. Work in $V[B]$.

Set $H = H_{(\beta)^+}$. Then $\langle H[G], c^G \rangle$ models $\mathcal{L}$ whenever G is $\mathbb{P}$-generic (interpreting $\underline{x}$ by x). Let $\Vdash_{\mathbb{P}} \dot{c} = c^{\dot{G}}$, where $\dot{G}$ is the canonical generic name. We can give every $\mathcal{L}$ sentence φ an interpretation $[\![\varphi]\!] \in \mathbb{B} = BA(\mathbb{P})$ in $H^{\mathbb{P}}$, interpreting $\dot{c}$ by $\dot{c}$ and $\underline{x}$ by $\check{x}$.

We then have:

$\langle H[G], c^G \rangle \Vdash \varphi(x_1, \ldots, x_n) \longleftrightarrow$

$\longleftrightarrow [\![\varphi(\underline{x}_1, \ldots, \underline{x}_n)]\!] \cap G \neq \emptyset$

for $x_1, \ldots, x_n \in N$ and G a $\mathbb{P}$-generic set.

Thus it suffices to prove:

<u>Claim</u> For each $p \in \mathbb{P}$ there is an $\mathcal{L}$-statement $\varphi \in M^B$ s.t. $[\![\varphi]\!] \neq 0$ and $[\![\varphi]\!] \subset [p]$. ($[p]$ being the smallest $b \in \mathbb{B}$ s.t. $p \in b$).

As G is $\mathbb{P}$-generic, we have:
$[p] \cap G \neq \emptyset \longleftrightarrow p \in G \longleftrightarrow \langle H[G], c^G \rangle \models \varphi_p$.
$\longleftrightarrow [\![\varphi_p]\!] \cap G \neq \emptyset$. Hence
$[p] = [\![\varphi_p]\!]$ and it suffices to show:

<u>Claim</u> $\Vdash_{\mathbb{P}} \psi \to \varphi_p$ for a $\psi \in M^\beta$ s.t. $[\![\psi]\!] \neq 0$

Set $N^* = \langle H, N^\beta, < \rangle$ where $<$ well orders H.
We may assume w.l.o.g. that
p conforms to N^*, since the set
of such p is dense in $\mathbb{P}$. Let G be
$\mathbb{P}$-generic with $p \in G$. Let $\tilde{\beta} = \sup \pi_p^{G} {}'' \beta_p$.
Then $\tilde{\beta} < \beta$ since $\tilde{\beta}$ is ω-cofinal.
Set $\tilde{M} = J_{\tilde{\beta}}^{A, B_{\omega}, B}$. For $a \in \mathbb{R}^p$ set $\tilde{a} = a \cap \tilde{M}$.
Then $\pi_p^G : \langle \bar{M}, \bar{a} \rangle \to \langle \tilde{M}, \tilde{a} \rangle$ is cofinal
and Σ_0-preserving whenever $\langle a, \bar{a} \rangle \in F^p$.
But then
(1) $\tilde{a} = \bigcup_{z \in M_p} \pi_p^G(z \cap \bar{a})$.
Let $\langle a_i \mid i < \omega \rangle$ enumerate $\mathbb{R}^p$ in V,
Then $\langle \bar{a}_i \mid i < \omega \rangle \in H_{\omega_1}$, where $\langle a_i, \bar{a}_i \rangle \in F^p$.
Then $\langle \tilde{a}_i \mid i < \omega \rangle \in M$, since $\tilde{a}_i \in M$
Moreover $\langle \tilde{a}_i \mid i < \omega \rangle \in M$, Let ψ be the sentence:
and $\mathrm{cf}(\beta) > \omega$.

There are π, σ s.t. $\sigma : \langle \underline{Q}_p, \underline{c}^p \rangle \lhd_* \langle \underline{Q}^B, \dot{c} \rangle \wedge$
$\wedge \langle \tilde{\underline{M}}, \pi \rangle$ is the liftup of $\langle \underline{M}_p, \sigma \rangle \wedge$
$\wedge \bigwedge_{i<\omega} (\tilde{\underline{a}}_i = \bigcup_{z \in \underline{M}_p} \pi(z \cap \bar{\underline{a}}_i))$,

Clearly $\psi \in M^B$. Moreover,

(2) $[\![\psi]\!] \neq 0$, since $\langle H[G], c^G \rangle \models \psi$
(since then ψ holds with $\sigma = \pi_p^G \upharpoonright Q_p$, $\pi = \pi_p^G$).

We show:

(3) $\langle H[G], c^G \rangle \models \psi \to \varphi_p$
whenever G is $\mathbb{P}$-generic.

Let $\langle H[G], c^G \rangle \models \psi$. Let $\sigma : \langle Q_p, c^p \rangle \lhd_* \langle Q, c^G \rangle$
and $\langle \tilde{M}, \pi \rangle$ = the liftup of $\langle M_p, \sigma \rangle$.
It remains only to show:
$\pi : \langle M_p, \bar{a} \rangle \prec \langle M^B, a \rangle$ whenever
$\langle a, \bar{a} \rangle \in F^p$, since then we have
$\pi : \langle M_p, C^p \rangle \lhd \langle M^B, c^G \rangle$ and
hence: $\quad p \in G^{c^G} = G$ with $\pi = \pi_p^G$,
Let $b = \{\vec{z} \in M \mid \langle M, a \rangle \models \chi(\vec{z})\}$. Then $b \in R^p$
by the N^*-conformity of p. Let
$\langle b, \bar{b} \rangle \in F^p$. Then by N^*-conformity:
$\bar{b} = \{\vec{z} \in M_p \mid \langle M_p, \bar{a} \rangle \models \chi(\vec{z})\}$.
Hence: $\langle M_p, \bar{a} \rangle \models \chi(\vec{z}) \leftrightarrow \vec{z} \in b \leftrightarrow$
$\leftrightarrow \pi(\vec{z}) \in \tilde{b} = b \cap \tilde{M} \leftrightarrow \langle M, a \rangle \models \chi(\pi(\vec{z}))$,
since $\tilde{b} = \bigcup_{u \in M_p} \pi(u \cap \bar{b})$. QED (3.15)

<u>Note</u> $cf(\beta) = \omega_1$ is the only case to consider if $A_0 = 0$.

We also note that we could have defined $\mathcal{L}$ (and hence $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$) somewhat differently: Let $\mathcal{L}'$ be like $\mathcal{L}$ except that in (*) we omit: $\mathrm{Urng}(\pi) = M$ if β is regular, and instead add the axiom:

(*) If β is regular, then whenever $u \in H_{\omega_1}$ and $\pi : u \triangleleft \langle \underline{M}, \dot{c} \rangle$, we have: $\sup \mathrm{On} \cap \mathrm{rng}(\pi) < \beta$.

It turns out that $\mathcal{L}'$ is also consistent. If $\mathbb{P}' = \mathbb{P}_{\mathcal{L}'}$ and β is regular, we can modify the proof of Lemma 3.1 to get: $\mathbb{B}' = BA(\mathbb{P}')$ contains a dense subset of size β. Hence $\beta^+ = \omega_2$ and $cf(\beta) = \omega_1$ in $V[G']$, where G' is $\mathbb{P}'$-generic. We omit the proof, since this is not relevant to the present paper.

We are now ready to prove that $\mathbb{P}$ is subcomplete. Since we are working in $V[B]$ we shall again write V for $V[B]$ and - for the sake of simplicity - we also write Q, M, N for Q^B, M^B, N^B.

Lemma 4 $\mathbb{P}$ is subcomplete.

prf. (We work in $V[B]$)
Let $W = L_\tau^{A'}$ where $2^\beta < \theta < \tau$, τ is regular, and $H_\theta \subset W$. Let $\sigma : \bar{W} \prec W$ s.t. $\bar{W}$ is countable and full with:

$\sigma(\bar{\theta}, \bar{\mathbb{P}}, \bar{M}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{P}, M, s, \lambda_i \quad (i = 1, \dots, n)$

where $\mathbb{P} \in H_{\lambda_i}$ (hence $N \in H_{\lambda_i}$), $\lambda_i < \theta$, and λ_i is regular for $i = 1, \dots, n$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{W}$.

Claim There is $q \in \mathbb{P}$ s.t. whenever $G \ni q$ is $\mathbb{P}$-generic, then there is $\sigma_0 \in V[G]$ with:

(a) $\sigma_0 : \bar{W} \prec W$

(b) $\sigma_0(\bar{\theta}, \bar{\mathbb{P}}, \bar{M}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{P}, M, s, \lambda_i \quad (i = 1, \dots, n)$

(c) $\sup \sigma_0 {}'' \bar{\lambda}_i = \sup \sigma {}'' \bar{\lambda}_i \quad (i = 0, \dots, n)$,
where $\bar{\lambda}_0 = On \cap \bar{W}$

(d) $\sigma_0 {}'' \bar{G} \subset G$.

We first show by standard methods:

Sublemma 4.1 Let δ be least s.t. $L_\delta(W)$ is admissible. The following language $\mathcal{L}^*$ on $L_\delta(W)$ is consistent:

Predicate $\in$, Constants $\underline{x}$ $(x \in L_\delta(W))$, $\dot{\sigma}$

Axioms: ZFC$^-$, $\bigwedge v (v \in \underline{x} \leftrightarrow \bigvee_{z \in x} v = \underline{z})$ for $x \in L_\delta(W)$,

$\dot{\sigma} : \underline{\bar{W}} \prec \underline{W}$, $\dot{\sigma}(\underline{\bar{\theta}}, \underline{\bar{\mathbb{P}}}, \underline{\bar{M}}, \underline{\bar{s}}, \underline{\bar{\lambda}_i}) = \underline{\theta}, \underline{\mathbb{P}}, \underline{M}, \underline{s}, \underline{\lambda_i} \quad (i = 1, \dots, n)$,

$\sup \dot{\sigma} {}'' \underline{\bar{\lambda}_i} = \underline{\sup \sigma {}'' \bar{\lambda}_i} \quad (i = 0, \dots, n)$, and:

$\dot{\sigma} \upharpoonright \underline{\bar{Q}} : \underline{\bar{Q}} \prec \underline{Q}$ cofinally (where $\sigma(\bar{Q}) = Q$),

$\langle \underline{N}, \dot{\sigma} \upharpoonright \underline{\bar{N}} \rangle$ is the liftup of $\langle \bar{N}, \dot{\sigma} \upharpoonright \underline{\bar{Q}} \rangle$

<u>Note</u> $\mathcal{L}^*$ does <u>not</u> posit that $H_{\omega_n} = \underline{H_{\omega_1}}$,

prf. (sketch) of 4.1

Let $\mathcal{L}_0$ be like $\mathcal{L}^*$ except that the axiom

$$\sup \dot{\sigma}''\bar{\lambda}_i = \underline{\sup \sigma''\bar{\lambda}_i} \quad (i=0,\dots,n)$$

is replaced by:

$$\sup \dot{\sigma}''\bar{\lambda}_i = \lambda_i \quad (i=0,\dots,n) \quad (\text{where } \lambda_0 =_{Df} \tau)$$

Let $\sigma \upharpoonright \bar{Q} : \bar{Q} \prec \tilde{Q}$ cofinally ($\bar{Q} = H^{\bar{W}}_{\omega_2}$) and let $\tilde{\sigma} : \bar{W} \prec W$ be the liftup of $\bar{W}$ by $\sigma \upharpoonright \bar{Q}$.

Let $k : \tilde{W} \prec W$ s.t. $k \upharpoonright \tilde{Q} = id$ and $k\tilde{\sigma} = \sigma$.

Let $\tilde{\mathcal{L}}_0$ be defined on $L_{\tilde{\delta}}(\tilde{W})$ like $\mathcal{L}_0$ on $L_\delta(W)$ in the obvious sense, where $\tilde{\delta}$ is least s.t. $L_{\tilde{\delta}}(\tilde{W})$ is admissible. (More precisely, $\mathcal{L}_0$ is defined in the parameter W and parameters $Q, M, N, \theta, s, \lambda_i \dots \in W$. $\tilde{\mathcal{L}}_0$ has the same definition over $L_{\tilde{\delta}}(\tilde{W})$ in the parameter $\tilde{W}$ and the parameters $k^{-1}(Q), k^{-1}(M), \dots, k^{-1}(s), k^{-1}(\lambda_i)$ $(i=1,\dots,n)$.

Then $\langle H_{\omega_2}, \hat{\sigma} \rangle$ models $\tilde{\mathcal{L}}_0$. Assume w.l.o.g. $\lambda_0 > \dots > \lambda_n$ and let

$$\sigma \upharpoonright H^{\bar{W}}_{\bar{\lambda}_n} : H^{\bar{W}}_{\bar{\lambda}_n} \prec H' \text{ cofinally.}$$

(Here $H^{\bar{W}}_{\lambda_n} = \bar{W}$ if $n=0$.) Let $\sigma' : \bar{W} \prec W'$ be the liftup of $\bar{W}$ by $\sigma \upharpoonright H^{\bar{W}}_{\bar{\lambda}_n}$. Let $k' : W' \prec W$ s.t. $k' \upharpoonright H' = id$ and $k'\sigma' = \sigma$.

There is then $\tilde{k} : \tilde{W} \prec W'$ s.t.

$\tilde{k} \upharpoonright \tilde{Q} = id$ and $\hat{k}\tilde{\sigma} = \sigma'$. We then have $k'\tilde{k} = k$. Let δ' be least s.t. $L_{\delta'}(W')$ is admissible and let $\mathcal{L}_0'$ be defined over $L_{\delta'}(W')$ as $\mathcal{L}_0$ was defined over $L_\delta(W)$ (in the obvious sense). The statement that $\tilde{\mathcal{L}}_0$ is consistent is $\Pi_1(L_{\tilde{\delta}}(\tilde{W}))$ in the parameter $\tilde{W}$ and parameters $\vec{p} \in \tilde{W}$. The statement that $\mathcal{L}_0'$ is consistent is then $\Pi_1(L_{\delta'}(W'))$ in W' and $k'(\vec{p})$. Hence $\mathcal{L}_0'$ is consistent. Note that $k'(N) = N$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}'$ which lies in some generic extension $V[G]$ of V. Let $\mu > \bar{\tau}$ be regular in $V[G]$. Then $\langle H_\mu^{V[G]}, k'\circ\sigma' \rangle$ models $\mathcal{L}^*$, where $\sigma' = \dot{\sigma}^{\mathfrak{A}}$. QED (4.1)

Now let $N^* = \langle H_\delta, W, N, \sigma, \lambda_1, \ldots, \lambda_m, s, \mathbb{P}, \ldots \rangle$ where $\delta > \Omega_W$. Let p conform to N^*. Set: $\bar{N}^* = \bar{N}^*(p, N^*) = \langle H', W', N', \sigma', \lambda_1', \ldots, \lambda_m', s', \mathbb{P}', \ldots \rangle$. Let $\bar{\mathcal{L}}^*$ be defined in $\bar{N}^*$ like $\mathcal{L}^*$ in N^*. Let $\mathfrak{A} \in H_{\omega_1}$ be a solid model of $\bar{\mathcal{L}}^*$. Set: $\sigma^* = \dot{\sigma}^{\mathfrak{A}}$. Set $\bar{C} = C^{\bar{G}}$, $C' = \sigma^{*}{}''\bar{C}$. Since $\sigma^* \upharpoonright \bar{Q} : \bar{Q} \prec Q'$ cofinally (where $Q' = Q_p$ is defined in $\bar{N}^*$ like Q in N^*), we have: $q \in \mathbb{P}$ where q is defined by:

$q_0 = \langle M_p, c' \rangle$, $q_1 = P_1$.

We show that this q satisfies the claim.

Let $G \ni q$ be $\mathbb{P}$-generic. Note that, since $\langle N', \sigma^* \upharpoonright \bar{N} \rangle$ is the liftup of $\langle \bar{N}, \sigma^* \upharpoonright \bar{Q} \rangle$ and $\sigma^* \upharpoonright \bar{Q} : \bar{Q} \prec Q'$ with $\sigma^{*}{}''\bar{c} = c' = c_q$, we have: $\langle \bar{M}, \bar{c} \rangle \lhd q_0 = \langle M_q, c' \rangle$ with $\pi_{\langle \bar{M}, \bar{c} \rangle, q_0} = \sigma^* \upharpoonright \bar{M}$. But $q_0 \lhd \langle M, c \rangle$ with $\pi_{q_0, \langle M, c \rangle} = \pi_q^G$. (Here $c = c^G$.) Thus $\langle \bar{M}, \bar{c} \rangle \lhd \langle M, c \rangle$ with $\pi_{\langle \bar{M}, \bar{c} \rangle, \langle M, c \rangle} = \pi_q^G \circ \sigma^* \upharpoonright \bar{M}$.

Now let $\pi^* \supset \pi_q^G \cup F^q$ s.t. $\pi^* : \bar{N}^* \prec N^*$. Set $\sigma_0 = \pi^* \sigma^*$. Then (a)–(c) are readily established. We show:

(d) $\sigma_0{}''\bar{G} \subset G$.

Let $\bar{r} \in \bar{G}$, $r = \sigma_0(\bar{r})$. Then $r_0 = \bar{r}_0$. But then $r_0 \lhd \langle \bar{M}, \bar{c} \rangle$ and $\pi_{r_0, \langle \bar{M}, \bar{c} \rangle} = \pi_{\bar{r}_0}^{\bar{G}}$, since $\bar{r} \in \bar{G}$. Hence $r_0 \lhd \langle M, c \rangle$ and $\pi_{r_0, \langle M, c \rangle} = \sigma_0 \circ \pi_{\bar{r}}^{\bar{G}}$ since $\pi_{\langle \bar{M}, \bar{c} \rangle, \langle M, c \rangle} = \sigma_0 \upharpoonright \bar{M}$ by the above.

It remains only to show:

<u>Claim</u> $\sigma_0 \pi_{\bar{r}_0}^{\bar{G}} : \langle M_r, \bar{a} \rangle \prec \langle M, a \rangle$ whenever $\langle a, \bar{a} \rangle \in F^r$.

prf

Let $\langle a, \bar{a} \rangle = \sigma_0(\langle a', \bar{a} \rangle)$ where $\langle a', \bar{a} \rangle \in F^{\bar{r}}$. Then $\pi_{\bar{r}}^{\bar{G}} : \langle M_r, \bar{a} \rangle \prec \langle \bar{M}, a' \rangle$. But $\sigma_0 \upharpoonright \bar{M} : \langle \bar{M}, a' \rangle \prec \langle M, a \rangle$ since $\sigma_0(\langle \bar{M}, a' \rangle) = \langle M, a \rangle$.

QED (Lemma 4)

Note that $\overline{\overline{BA(\mathbb{P}_{\mathbb{B}})}} \leq 2^{\beta_{\mu+1}}$, since either $\beta = \beta_{\mu+1}$, $cf(\beta) = \omega_1$, and $BA(\mathbb{P}_{\mathbb{B}})$ has a dense subset of size β, by Sublemma 3.15.1, or else $\beta_{\mu+1} = \beta^+$, $\overline{\overline{\mathbb{P}_{\mathbb{B}}}} = 2^{\beta} = \beta^+$ (since then GCH holds below κ). Now let $\Vdash_{\mathbb{B}_\mu} \mathring{\mathbb{B}} = BA(\mathbb{P}_{\mathring{\mathbb{B}}})$, $\mathring{\mathbb{B}}$ being the canonical generic name. We then form $\mathbb{B}_\mu * \mathring{\mathbb{B}}$, which also has cardinality $\leq 2^{\beta_{\mu+1}}$, since $\mathbb{B}_\mu$ has cardinality $\leq 2^{\beta_\mu} \leq \beta_{\mu+1}$, since $2^{\beta_\mu+1} = \beta_{\mu+1}$. Let $k : \mathbb{B}_\mu \to \mathbb{B}_\mu * \mathring{\mathbb{B}}$ be the natural injection. Choose $\mathbb{B}_{\mu+1} \supseteq \mathbb{B}_\mu$ s.t. there is an isomorphism ℓ with:

$$\begin{array}{ccc} \mathbb{B}_{\mu+1} & \xleftarrow{\ \ell\ } & \mathbb{B}_\mu * \mathring{\mathbb{B}} \\ & \nwarrow \quad \nearrow_{k} & \\ & \mathbb{B}_\mu & \end{array}$$

We ensure that $\mathbb{B}_{\mu+1} \subset H_{\beta^+_{\mu+1}}$. By Lemma 4 we know that $\mathbb{B}_{\mu+1}$ is subcomplete. However, we have found it necessary to devise another representation of $\mathbb{B}_{\mu+1}$ in order to elicit its deeper properties. Working in V define as before:

$Q = H_\gamma$, $M = L_\beta^{A, \mathbb{B}_\mu}$, $N = \langle H_{\beta^+}, M, <, \ldots \rangle$.

We then define a class Γ' of triples as follows:

Def Γ' = the set of $u = \langle R, \mathbb{B}, C\rangle$ s.t.

- $R = L_\beta^{A,\mathbb{B}}$ models ZFC$^-$ or Zermelo
- $\mathbb{B} \in R$ is a complete BA in R and B is $\mathbb{B}$-generic over R.

Set: $R^B = L_\beta^{A,\mathbb{B},B}$; $\gamma = \gamma_u = \omega_2^{R^B}$, $Q = Q_u = H_\gamma^R$,
$Q^B = Q_u^B = H_\gamma^{R^B}$; $u^* = \langle R^B, C\rangle$

- $\langle R^B, C\rangle \in \Gamma$ with $\langle Q^B, C\rangle \in \Gamma^*$
- $\mathbb{B} \subset Q$ satisfies γ-cc in R.

Def Let $u = \langle R_u, \mathbb{B}^u, C^u\rangle$, $v = \langle R_v, \mathbb{B}^v, C^v\rangle \in \Gamma'$

$\pi : u \triangleleft' v$ iff

- $\pi : R_u \prec R_v$ s.t. π"$\mathbb{B}^u \subset \mathbb{B}^v$.
- $\pi \upharpoonright Q_u : Q_u \prec Q_v$ cofinally
- Let $\pi : R_u \longrightarrow R_{uv}$ cofinally. Then $\langle R_{u,v}, \tilde{\pi}\rangle$ is the liftup of $\langle M_u, \pi \upharpoonright Q_u\rangle$
- Let $\pi^* \supset \pi$ be the unique extension of π s.t. $\pi^* : R_u^{B^u} \prec R_v^{B^v}$. Then π^*"$C_u = C_v$.

We then get:

<u>Lemma 5.1</u> 1. $\pi : u \triangleleft' v$, then $\pi^* : u^* \triangleleft v^*$.

The proof is straightforward.

<u>Lemma 5.2</u> Let $\pi : u \triangleleft v^*$ where $v \in \Gamma'$

There is a unique pair $\langle \tilde{\pi}, \tilde{u}\rangle$ s.t. $\tilde{u} \in \Gamma'$ and $\tilde{\pi} : \tilde{u} \triangleleft' v$ and $u = \tilde{u}^*$, $\pi = \tilde{\pi}^*$.

proof of 5.2

Let $v = \langle M_v, B^v, C^v \rangle$, $u = \langle M_u, C^u \rangle$.

Let $M_u = L_{\beta_u}^{A_u, \mathbb{B}_u, B_u}$. Set:

$M_{\tilde{u}} = L_{\beta_u}^{A_u, \mathbb{B}_u}$, $B^{\tilde{u}} = B^u$, $C^{\tilde{u}} = C^u$.

It follows easily that $\tilde{u} \in \Gamma'$.

Moreover $Q_{\tilde{u}} = H_\gamma^{M_{\tilde{u}}}$, $Q_u = Q_{\tilde{u}}^{B^{\tilde{u}}}$, where $\gamma = \gamma_{\tilde{u}} = \omega_2^{M_u}$. Clearly $M_u = M_{\tilde{u}}^{B^{\tilde{u}}}$.

Set $\tilde{\pi} = \pi \upharpoonright M_{\tilde{u}}$. We verify

<u>Claim</u> $\tilde{\pi} : \tilde{u} \triangleleft' v$.

$\tilde{\pi} : M_{\tilde{u}} \prec M_v$ and $\tilde{\pi}'' B_{\tilde{u}} \subset B_v$ is trivial, as is: $\tilde{\pi} \upharpoonright Q_{\tilde{u}} : Q_{\tilde{u}} \prec Q_v$ cofinally.

Now let $\tilde{\pi} : M_{\tilde{u}} \to \tilde{M}$ cofinally.

<u>Claim</u> $\langle \tilde{M}, \tilde{\pi} \rangle$ is the liftup of $\langle Q_{\tilde{u}}, \tilde{\pi} \upharpoonright Q_{\tilde{u}} \rangle$.

proof.

Let $x \in \tilde{M}$. Then $x \in \pi(a)$ for an $a \in M_u$ s.t. $\bar{\bar{a}} < \gamma$ in M_u. Let $f \in M_u$ s.t. $f : \delta \leftrightarrow a$ for a $\delta < \gamma$. We may suppose w.l.o.g. that $f = \mathring{f}^B$ where $\mathring{f} \in M_{\tilde{u}}$ and $\Vdash_{\mathbb{B}_u} \mathring{f}$ is a function defined on $\check{\delta}$.

Arguing in $M_{\tilde{u}}$ choose for each $\nu < \delta$ a maximal antichain A_ν in in the set $\{ b \in \mathbb{B}_u \mid \bigvee x \; b \Vdash \mathring{f}(\check{\nu}) = \check{x} \}$

For each $b \in A_\nu$ let $x_{\nu b}$ be that x s.t. $b \Vdash \mathring{f}(\check{\nu}) = \check{x}$. Set $X = \{x_{\nu b} \mid \nu < \bar{\alpha} \wedge b \in A_\nu\}$. Clearly $\bar{\bar{X}} < \gamma$ in $M_{\tilde{u}}$, since $\mathbb{B}_u$ satisfies the γ-chain condition and γ is regular in $M_{\tilde{u}}$. But $x \in \tilde{\pi}(X)$

QED (Claim)

Finally we note that, if $\tilde{\pi}^*$ is the unique extension of $\tilde{\pi}$ s.t. $\tilde{\pi}^* : M_{\tilde{u}}^{B^{\tilde{u}}} < M_v^{B^v}$, then $\tilde{\pi}^* = \pi$, since π has its defining properties. In particular, then $\tilde{\pi}^* {}'' C_{\tilde{u}} = C_v$.

The uniqueness of the pair $\langle \tilde{u}, \tilde{\pi} \rangle$ is evident. QED (Lemma 5.2)

Finally we prove:

<u>Lemma 5.3</u> Let $\pi : u \lhd' v$ where $u = \langle M_u, B^u, C^u \rangle$, $v = \langle M_v, B^v, C^v \rangle \in \Gamma'$, where Let $\pi : \langle M_u, \bar{A} \rangle \prec \langle M_v, A \rangle$, where $\langle M_v, A \rangle$ models ZFC^- or Zermelo. Then $\pi^* : \langle M_u^{B^u}, \bar{A} \rangle \prec \langle M_v^{B^v}, A \rangle$.

proof. B^v is $\mathbb{B}^v$-generic over $\langle M_v, A \rangle$, B^u is $\mathbb{B}^u$-generic over $\langle M_u, \bar{A} \rangle$, and $\pi'' B^u \subset B^v$. Hence there is a unique $\pi^+ \supset \pi$ s.t.

$\pi^+ : \langle M_u^{B^u}, \bar{A} \rangle \prec \langle M_v^{B^v}, A \rangle$. But then $\pi^+ = \pi^* =$ the unique $\pi^* \supset \pi$ s.t. $\pi^* : M_u^{B^u} \prec M_v^{B^v}$.

QED (5.3)

Lemma 5.4 Let $u = \langle M_u, B^u, C^u \rangle$, $v = \langle M_v, B^v, C^v \rangle \in \Gamma'$. Let $\tilde{\pi} : \langle Q_u^{B^u}, C^u \rangle \triangleleft_* \langle Q_v^{B^v}, C^v \rangle$ s.t. $\tilde{\pi}(B^u \cap x) = B^v \cap \tilde{\pi}(x)$ for $x \in Q_u$. Let $\pi : M_u \prec M_v$ s.t. $\pi \upharpoonright Q_u = \tilde{\pi} \upharpoonright Q_u$ and $\langle M_{uv}, \bar{\pi} \rangle =$ the liftup of $\langle M_u, \pi \upharpoonright Q_u \rangle$, where $\bar{\pi} : M_u \to M_{uv}$ cofinally. Then $\pi : \langle Q_u, B^u, C^u \rangle \triangleleft' \langle Q_v, B^v, C^v \rangle$. Moreover, $\pi^* \upharpoonright Q_u^{B^u} = \tilde{\pi}$, where $\pi^* \supset \pi$ s.t. $\pi^* : M^{B^u} \prec M^{B^v}$.

proof. It sufficies to show that $\pi^* \upharpoonright Q_u^{B^u} = \tilde{\pi}$. For each $x \in Q_u^{B^u}$ there is a pair $\langle \alpha, r \rangle \in Q_u^{B^u}$ s.t. $r \subset \alpha^2$ and $\langle \alpha, r \rangle \simeq \langle TC(\{x\}), \in \rangle$, so it sufficies to show $\pi^*(r) = \tilde{\pi}(r)$ for $r \in Q_u^{B^u}$ s.t. $r \subset \gamma^2$. Let $r = \dot{r}^{B^u}$, where $\dot{r} \subset \alpha^2$, $\alpha < \gamma_u$. For each $z \in \alpha^2$ let A_z be a maximal antichain in $\{b \in \mathbb{B}^u \mid b \Vdash \check{z} \in \dot{r} \vee b \Vdash \check{z} \notin \dot{r}\}$. Then $A_z \in Q$ and $A = \{\langle b, z \rangle \mid z \in \alpha^2 \wedge b \in A_z\} \in Q$ since $Q = H_{\gamma_u}^{M_u}$ and $\mathbb{B}^u$ satisfies γ_u-cc in M_u. Set $a = \operatorname{rng}(A)$. Then $a \in Q$ and $\pi^*(a \cap B^u) = \tilde{\pi}(a \cap B^u) = \pi(a) \cap B^v$.

But $\pi^*(\sigma) = \tilde{\pi}^*(\{z \mid \bigvee b \in a \cap B^u \ \langle b, z\rangle \in A\}) =$
$= \{z \mid \bigvee b \in \tilde{\pi}(a) \cap B^u \ \langle b, z\rangle \in \tilde{\pi}(A)\} = \tilde{\pi}(\sigma)$.

QED (5.4)

Working in V, define $Q = H_\gamma$, $M = L_\beta^{A, \mathbb{B}_\mu}$ and $N = \langle H_{\beta^+}, N, <, \dots\rangle$ as before. We shall define an infinitary language $\mathcal{L}'$ on N. Using $\mathcal{L}'$ we shall then define an $\mathcal{L}$-forcing $\mathbb{P}' = \mathbb{P}_{\mathcal{L}'}$ s.t. $BA(\mathbb{P})$ is isomorphic to $\mathbb{B}_\mu * \mathring{\mathbb{B}}$, hence to $\mathbb{B}_{\mu+1}$.

$\mathcal{L}'$ is the infinitary language on N with:

<u>Predicate</u> $\in$, <u>Constants</u> $\underline{x}$ $(x \in N)$, $\mathring{B}$, $\mathring{C}$

<u>Axioms</u> ZFC$^-$, $\bigwedge v(v \in \underline{x} \leftrightarrow \bigvee\!\!\bigvee_{z \in x} v = \underline{z})$ for $x \in N$, $\langle \underline{M}, \mathring{B}, \mathring{C}\rangle \in \Gamma'$, and

$H_{\omega_1} = H_{\omega_1}$, $\langle \underline{M}, \mathring{B}, \mathring{C}\rangle \in \Gamma'$, and

$(*)$ $\bigwedge x \in \underline{M} \ \bigvee u \in H_{\omega_1} \ \bigvee \pi\ (\pi : u \triangleleft' \langle \underline{M}, \mathring{B}, \mathring{C}\rangle \wedge$
$\wedge\, x \in \operatorname{rng}(\pi) \wedge \Psi(\pi))$ (where: $\Psi(\pi) = \bigcup \operatorname{rng}(\pi) = \underline{M}$ if β is regular, $\pi = \pi$ if not.)

We now pause to introduce formally a convention which we have already employed tacitly. Let $\mathcal{L}$ be an infinitary language on an admissible set N. All of our languages have what we shall call the "special constants" $\underline{x}$ $(x \in N)$, and the axioms include:

$\bigwedge v(v \in \underline{x} \leftrightarrow \bigvee\!\!\bigvee_{z \in x} v = \underline{z})$ for $x \in N$

Now let $\mathfrak{A}$ be a solid model s.t. $N \subset$ wfcore($\mathfrak{A}$) and $\mathfrak{A}$ interprets the predicates and non special constants of $\mathcal{L}$. We say "$\mathfrak{A}$ models $\mathcal{L}$" to mean that $\mathfrak{A}$ becomes a model of the axioms $\mathcal{L}_1$ if we enhance it by giving the special constants the interpretation $\underline{x}^{\mathfrak{A}} = x$.

This convention was often used tacitly in [LF] and was employed here in the formulation of Cor 3.10, where we wrote: "$\langle H_\theta^{V[C]}, c\rangle$ models $\mathcal{L}(p)$".

. . . . . .

We now prove:

__Lemma 6.1__ $\mathcal{L}'$ is consistent.

proof. Let B be $\mathbb{B}_\mu$ - generic and let $\mathfrak{A} = \langle |\mathfrak{A}|, c^{\mathfrak{A}}\rangle$ be a solid model of $\mathcal{L}_B$. Then $\mathfrak{A}' = \langle |\mathfrak{A}|, B, c^{\mathfrak{A}}\rangle$ models $\mathcal{L}'$, by Lemma 5.2. QED (6.1)

We also note:

__Lemma 6.2__ Let $\mathfrak{A} = \langle |\mathfrak{A}|, B^{\mathfrak{A}}, c^{\mathfrak{A}}\rangle$ be a solid model of $\mathcal{L}'$. Then $B^{\mathfrak{A}}$ is $\mathbb{B}_\mu$ - generic over V, $N^B \subset$ wfcore($\mathfrak{A}$) and $\langle |\mathfrak{A}|, c^{\mathfrak{A}}\rangle$ models $\mathcal{L}_B$.

We obviously have the analogue of Lemma 2:

Lemma 6.3 Let $\mathfrak{A}$ be a model of $\mathcal{L}$; Let $B = \dot{B}^{\mathfrak{A}}$, $C = \dot{C}^{\mathfrak{A}}$. Let $\langle A_n \mid n < \omega \rangle \in \mathfrak{A}$ s.t. $A_n \subset MB$ for $n < \omega$. Then the conclusion of Lemma 2 holds.

prf. $\langle |\mathfrak{A}|, C \rangle$ models $\mathcal{L}_B$

We now define $\mathbb{P}' = \mathbb{P}_{\mathcal{L}'}$.

Def $\tilde{\mathbb{P}}$ = the set of $p = \langle p_0, p_1 \rangle$ s.t.

- $p_0 = \langle M_p, B^p, C^p \rangle \in \Gamma' \cap H_{\omega_1}$
- $p_1 = F^p$ is an at most countable set of pairs $\langle a, \bar{a} \rangle$ s.t. $\bar{a} \subset M_p$, $a \subset M$,

Def For $p \in \tilde{\mathbb{P}}$ let φ'_p be the conjunction of

- $p_0 \lhd' \langle \underline{M}, \mathring{B}, \mathring{C} \rangle$ Let $\pi_p =_{H} \pi_{p_0, \langle \underline{M}, \mathring{B}, \mathring{C} \rangle}$.
- $\pi_p : \underline{\langle M_p, \bar{a} \rangle} \prec \underline{\langle M, a \rangle}$ for $\langle a, \bar{a} \rangle \in F^p$
- $\pi_p : M_p \to \underline{M}$ cofinally if β is regular

Def $\mathcal{L}'(p) = \mathcal{L}' + \varphi'_p$

$$\mathbb{P}' = \mathbb{P}_{\mathcal{L}'} = \{ p \mid \mathcal{L}'(p) \text{ is consistent} \}$$

Set $R^p = \text{rng}(F^p)$, $D^p = \text{dom}(F^p)$

Def For $p, q \in \mathbb{P}'$ set:

$$p \le q \longleftrightarrow (q_0 \lhd' p_0 \wedge R^q \subset R^p \wedge$$
$$\wedge \; \pi_{q_0 p_0} : \langle M_q, \bar{a} \rangle \prec \langle M_p, a' \rangle$$
whenever $\langle a, \bar{a} \rangle \in F^q$, $\langle a, a' \rangle \in F^p$

As before we get:

$$p \leq q \longleftrightarrow (R^q \subset R^p \wedge \mathcal{L}'(p) \vdash (\mathcal{L}'(q) \wedge$$
$$\wedge \operatorname{rng}(\pi_q) \subset \operatorname{rng}(\pi_p))).$$

We set:

Def $\pi_{qp} = \pi_{q_0 p_0}$ for $p \leq q$.

For $p \in \mathbb{P}'$ define:

Def $p^* = \langle p_0^*, p_1^* \rangle$ where

$$p_0^* = \langle M_p^{B^p}, c^p \rangle, \quad p_1^* = p_1.$$

Using Lemmas 5.1 – 5.3 we easily get:

Lemma 6.4 Let $p \in \mathbb{P}'$

(a) If $\mathfrak{A} = \langle |\mathfrak{A}|, B, C \rangle$ is a solid model of $\mathcal{L}'(p)$, then $\langle |\mathfrak{A}|, C \rangle$ is a solid model of $\mathcal{L}_B(p^*)$

(b) If $\mathfrak{A} = \langle |\mathfrak{A}|, C \rangle$ is a solid model of $\mathcal{L}_B(p^*)$, then $\langle |\mathfrak{A}|, B, C \rangle$ is a solid model of $\mathcal{L}'(p)$.

The proof of Lemma 3.2 goes through as before, as do the proofs of the extension lemmas 3.4 – 3.8. In particular:

Lemma 6.5 p is compatible with q in $\mathbb{P}'$ iff $\mathcal{L}'(p) \cup \mathcal{L}'(q)$ is consistent.

Using the extension lemmas we conclude just as before:

<u>Lemma 6.6</u> Let G be $\mathbb{P}'$-generic. Then

(a) $\langle p_0 \mid p \in G \rangle$, $\langle \pi_{p,q} \mid q \leq p,\ q, p \in G \rangle$ is a directed system with the direct limit:

$\langle M, B^G, C^G \rangle$, $\langle \pi_p^G \mid p \in G \rangle$

(Moreover $\pi_p^G = \bigcup \{ \pi^q \mid q_0 = p_0 \wedge q \leq p \wedge q \in G \}$)

(b) $\pi_p^G : p_0 \lhd' \langle M, B^G, C^G \rangle$ for $p \in G$

(c) $\pi_p^G : \langle M_p, \bar{a} \rangle \prec \langle M, a \rangle$ for $\langle a, \bar{a} \rangle \in F^p$.

<u>Def</u> $\mathbb{B}' = BA(\mathbb{P}')$

We define an embedding $k' : \mathbb{B}_\mu \to \mathbb{B}'$ by $k'(b) = [\![\check{b} \in B^{\dot{G}}]\!]$, where $\dot{G}$ = the canonical $\mathbb{P}'$-generic name.

<u>Lemma 6.7</u> k' is an complete injective homomorphism.

prf: k' is clearly a complete homomorphism: e.g. $k'(\bigcap_i b_i) = [\![\bigcap_i b_i \in B^{\dot{G}}]\!] = \bigcap_i [\![\check{b}_i \in B^{\dot{G}}]\!] = \bigcap_i k'(b_i)$. Injectivity follows from:

<u>Claim</u> $k'(b) = 0 \to b = 0$.

Suppose not. Let $b \in \mathbb{B}$ where B is $\mathbb{B}_\mu$-generic. Let $\mathfrak{A} = \langle |\mathfrak{A}|, c \rangle$ be a solid model of . Then $\mathfrak{A}' = \langle |\mathfrak{A}|, B, C \rangle$ is a solid model of $\mathcal{L}'$. By Lemma there is p s.t. $\mathfrak{A}' \models \mathcal{L}'(p)$ and $b \in \operatorname{rng}(\pi^p)$. Let $\pi^p(\bar{b}) = b$. Then $p \Vdash \check{b} \in B^{\dot{G}}$, $\bar{b} \in B^p$. It follows that $p \Vdash \check{b} \in B^{\dot{G}}$, Hence $0 \neq [p] \subset [\![\check{b} \in B^{\dot{G}}]\!] = k'(b) = 0$. Contr! QED (6.7)

We now consider the factor algebra $\mathbb{B}'/k'``B$, where $B$ is $\mathbb{B}_\mu$-generic. For greater perspicuity we write $\mathbb{B}'/B$ for $\mathbb{B}'/k'``B$ and b/B for $b/k'``B$ when $b \in \mathbb{B}'$. Remembering the definition of p^* ($p \in \mathbb{P}'$) we prove:

<u>Lemma 6.8</u> Let B be $\mathbb{B}_\mu$-generic. Let $p, q \in \mathbb{P}'$. $[p]/B$, $[q]/B$ are compatible in $\mathbb{B}'/B$ iff p^*, q^* are compatible in $\mathbb{P}_B$.

proof.

($\leftarrow$) Suppose not. Then $[p] \cap [q]/B = 0$. Hence $[p] \cap [q] \cap k'(b) = 0$ for a $b \in B$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}_B(p^*) \cup \mathcal{L}_B(q^*)$. Then $\mathfrak{A}' = \langle |\mathfrak{A}|, B, C^{\mathfrak{A}} \rangle$ is a solid model of $\mathcal{L}'(p) \cup \mathcal{L}'(q)$ by Lemma 6.4. Hence by Lemma 6.3 there is $r \le p, q$ s.t. $\mathfrak{A} \models \mathcal{L}'(r)$ and $b \in rng(\pi^r)$. Let $\pi^r(\bar{b}) = b$. Then $\bar{b} \in B^r$. Thus $r \le p, q$ and $r \Vdash \check{b} \in B^{\dot{G}}$. Hence $[r] \subset [\![\check{b} \in B^{\dot{G}}]\!] = k'(b) = 0$. Contr!

($\rightarrow$) Suppose not. Then $\mathcal{L}_B(p^*) \cup \mathcal{L}_B(q^*)$ is inconsistent. Since B is $\mathbb{B}_\mu$-generic there is $b \in B$ s.t.

(1) $b \Vdash_{\mathbb{B}_\mu} \mathcal{L}_{\dot{B}}(\check{p}^*) \cup \mathcal{L}_{\dot{B}}(\check{q}^*)$ is inconsistent.

Now let $\tilde{G}$ be $\mathbb{B}'/\mathbb{B}$ - generic s.t. $[p]/\mathbb{B}, [q]/\mathbb{B} \in \tilde{G}$. Set:

$$G = \{p \in \mathbb{P}' \mid [p]/\mathbb{B} \in \tilde{G}\}.$$

Then G is $\mathbb{B}'$-generic, $p, q \in G$, and $B = B^G$. By genericity there is $r \leq p, q$ in G s.t. $b \in \mathrm{rng}(\pi^r)$. Let $\pi^r(\bar{b}) = b$. Then $\bar{b} \in B^r$ since $b \in B$. Now let $\mathfrak{M}' = \langle |\mathfrak{M}'|, B', C\rangle$ be a solid model of $\mathcal{L}'(r)$. Then $b \in B'$ since $\pi^r(\bar{b}) = b$ and $\bar{b} \in B^r$. But then $\mathfrak{M} = \langle |\mathfrak{M}'|, C\rangle$ is a solid model of $\mathcal{L}_{B'}(p^*) \cup \mathcal{L}_{B'}(q^*)$, where $b \in B'$ and B' is $\mathbb{B}_\mu$ - generic. Contradiction! by (1). QED (6.8)

<u>Lemma 6.9</u> Let B be $\mathbb{B}_\mu$ - generic. Then $\{[p^*] \mid [p]/\mathbb{B} \neq 0 \text{ in } \mathbb{B}'/\mathbb{B}\}$ is dense in $\mathbb{B}_B = BA(\mathbb{P}_B)$.

proof.

We first note that $\{[p^*] \mid [p]/\mathbb{B} \neq 0 \text{ in } \mathbb{B}'/\mathbb{B}\}$ is the same as the set: $\{[p] \mid p \in \hat{\mathbb{P}}_B\}$ where $\hat{\mathbb{P}}_B$ is the set of $p \in \mathbb{P}_B$ s.t. $F^p \in V$.

(To see this, let $p = \langle p_0, p_1\rangle \in \hat{\mathbb{P}}_B$ and $p_0 = \langle \tilde{M}^p, C^p\rangle$ with $\tilde{M}^p = L_{\gamma_p}^{A^p, B^p, B^p}$. Set $p_0' = \langle M^{p\prime}, B^p, C^p\rangle$ with $M^{p\prime} = L_{\gamma_p}^{A^p, B^p}$. $p_1' = p_1$. Then if $\mathfrak{M}$ is a solid model of $\mathcal{L}_B(p)$, it follows that $\mathfrak{A}' = \langle |\mathfrak{A}|, B, C\rangle$ models $\mathcal{L}(p')$, where $\mathfrak{A} = \langle |\mathfrak{A}|, C\rangle$ (using Lemma 5.2). Hence $p' \in \mathbb{P}'$, $p = p'^*$, and $[p']/B \neq 0$ by Lemma 6.8 (taking $p = q = p'$ in the statement of Lemma 6.8).)

Hence it suffices to show:

<u>Claim</u> Let $q \in \mathbb{P}_B$. There is $p \in \hat{\mathbb{P}}_B$ s.t. $[p] \subset [q]$.

proof.

Let $A = \langle a_i \mid i < \omega\rangle \in V[B]$ enumerate R^q, Let $D \subset \beta$ s.t. A is $\langle M^B, D\rangle$-definable, where $D \in V[B]$. Let $D = \dot{D}^B$ and set:

$$E = \{\langle \nu, b\rangle \mid b \in \mathbb{B}_\mu \wedge \nu < \beta \wedge b \Vdash \check{\nu} \in \dot{D}\}$$

Then $D = \{\nu \mid \bigvee b \in B\ \langle \nu, b\rangle \in E\}$. Hence A is $\langle M^B, E\rangle$-definable. Set:

$$N^* = \langle H_\theta, N^B, M^B, <, B, E, A, \ldots\rangle \quad \text{in } V[B]$$

where $\theta > (2^\beta)^+$ is a cardinal. Let $p \leq q$ in $\mathbb{P}_B$ s.t. p conforms to N^*. Set:

$$\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{N}, \bar{M}, <, \bar{B}, \bar{E}, \bar{A}, \ldots\rangle.$$

Then $\bar{M} = M_p = L^{A^p, \mathbb{B}^p, B^p}_{\beta_p}$, $\bar{B} = B^p$, and $\bar{A}$ is $\langle \bar{M}, \bar{E}\rangle$-definable by the same definition. Now form p', p'' by:

$$p'_0 = p_0, \quad p'_1 = \{\langle a, \bar{a}\rangle \in F^p \mid a \in R^q\}$$

$$p''_0 = p_0, \quad p''_1 = \{\langle E, \bar{E}\rangle\} \text{ where } \langle E, \bar{E}\rangle \in F^p.$$

Then $p' \leq q$ in $\mathbb{P}_B$ and $p'' \in \hat{\mathbb{P}}_B$. We show:

<u>Claim</u> $[p''] \subset [p']$ in $BA(\mathbb{P}_B)$

Let $G \ni p''$ be $\mathbb{P}_B$ - generic. We show:

<u>Claim</u> $p' \in G$

Since $p'_0 = p''_0$ we have:

$\pi : p'_0 \triangleleft \langle M^B, C^G \rangle$ where $\pi = \pi^G_{p''}$.

It remains to show:

$\pi : \langle M_{p'}, \bar{a} \rangle \prec \langle M^B, a \rangle$ for $\langle a, \bar{a} \rangle \in F^{p'}$.

$a = A(i)$ is $\langle M^B, E \rangle$ - definable and $\bar{a} = \bar{A}(i)$ is $\langle M_{p'}, \bar{E} \rangle$ definable by the same definition. But $\pi : \langle M_{p'}, \bar{E} \rangle \prec \langle M^B, E \rangle$.

QED (6.9)

Set: $\mathbb{B}_B = BA(\mathbb{P}_B)$. Set:

$A' = \{[p]/B \mid p \in \mathbb{P}' \wedge [p]/B \neq 0\}$. Then

A' is dense in $\mathbb{B}'/B$. But

$A = \{[p^*] \mid [p]/B \in A'\}$ is dense in $\mathbb{B}_B$

By Lemma 6.8 we have:

$$[p]/B \cap [q]/B = 0 \text{ in } \mathbb{B}'/B \longleftrightarrow [p^*] \cap [q^*] = 0 \text{ in } \mathbb{B}_B$$

But for $a, b \in A$ we have (in $\mathbb{B}_B$):

$$a \subset b \longleftrightarrow \bigwedge c \in A\, (c \cap b = 0 \rightarrow c \cap a = 0)$$

since A is dense in $\mathbb{B}_B$. Similarly for A', $\mathbb{B}'/B$. Hence:

$[p]/B \subset [q]/B$ in $\mathbb{B}'/B$ $\longleftrightarrow$
$[p^*] \subset [q^*]$ in $\mathbb{B}_B$. Hence:

Cor 6.9.1 There is $\sigma_B : \mathbb{B}'/B \overset{\sim}{\longleftrightarrow} \mathbb{B}_B$ uniquely defined by: $\sigma_B([p]/B) = [p^*]$.

But $\sigma_B = \mathring{\sigma}^B$ where:

$\Vdash_{\mathbb{B}_\mu} \mathring{\sigma} : \check{\mathbb{B}}'/\mathring{B} \overset{\sim}{\longleftrightarrow} \mathring{\mathbb{B}}$.

and $\Vdash_{\mathbb{B}_\mu} \mathring{\mathbb{B}} = BA(\mathbb{P}_{\mathring{\mathbb{B}}})$.

$\sigma(a) =$ that a' s.t. $\Vdash_{\mathbb{B}_\mu} a' = \mathring{\sigma}(\check{a}/\mathring{B})$,

then $\sigma : \mathbb{B}' \overset{\sim}{\longrightarrow} \mathbb{B}_\mu * \mathring{\mathbb{B}}$ is an injective homomorphism. But σ is onto since if $a \in \mathbb{B}_\mu * \dot{\mathbb{B}}$ and $\Vdash_{\mathbb{B}_\mu} t = \mathring{\sigma}^{-1}(a)$, then $\Vdash t \in \check{\mathbb{B}}'/\mathring{B}$ and hence there is a unique b s.t. $\Vdash \check{b}/\mathring{B} = t$. Hence $\Vdash a = \dot{\sigma}(\check{b}/\dot{B})$.

We note finally that $\sigma k' = k$, where $k : \mathbb{B}_\mu \longrightarrow \mathbb{B}_\mu * \mathring{\mathbb{B}}$ is the natural injection. We let $c = k'(b) = [\![\check{b} \in B^{\dot{G}}]\!]_{\mathbb{P}'}$. We then have:

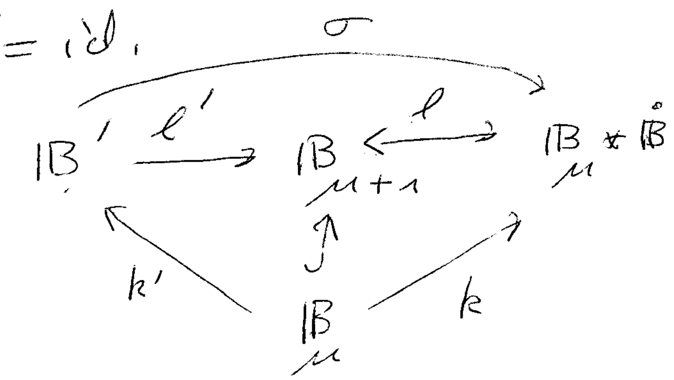
$\Vdash_{\mathbb{B}_\mu} \check{c}/\mathring{B} = \begin{cases} 1 \text{ if } \check{b} \in \mathring{B} \\ 0 \text{ if } \check{b} \notin \mathring{B} \end{cases}$ in $\check{\mathbb{B}}'/\mathring{B}$

$\Vdash_{\mathbb{B}_\mu} \dot{\sigma}(\check{c}/\mathring{B}) = \begin{cases} 1 \text{ if } \check{b} \in \mathring{B} \\ 0 \text{ if } \check{b} \notin \mathring{B} \end{cases}$ in $\mathring{\mathbb{B}} = k(b)$,

since this is the way $k : \mathbb{B}_\mu \longrightarrow \mathbb{B}_\mu * \mathring{\mathbb{B}}$ was defined. Hence $\sigma(k'(b)) = k(b)$.

Setting $l' = l \circ \sigma$ we have:

<u>Lemma 6.10</u> There is $l' : \mathbb{B}' \overset{\sim}{\longleftrightarrow} \mathbb{B}_{\mu+1}$ s.t. $l'k' = id$.



Thus, having established that $\mathbb{B}'$ is a representation of $\mathbb{B}_{\mu+1}$, we examine its properties more closely. We note, of course, that by the last result $\mathbb{B}'$ adds no reals and is, in fact, $\mathbb{P}'$-subcomplete. Hence, if G is $\mathbb{P}'$-generic we know that $\langle H_\theta, B^G, c^G \rangle$ models $\mathcal{L}'$ whenever $\theta > 2^\beta$ is regular. (The only problematical axiom was $H_{\omega_1} = \underline{H}_{\omega_1}$, which is now established.)

<u>Def</u> Let B be $\mathbb{B}_\mu$ - generic and $C \prec Q^B$ be countable.

$G^{B,C} =_{Df}$ the set of $p \in \mathbb{P}'$ s.t. there is π s.t.

- $\pi : p_0 \triangleleft' \langle M, B, C \rangle$
- $\pi : \langle M_p, \bar{a} \rangle \prec \langle M, a \rangle$ whenever $\langle a, \bar{a} \rangle \in F^p$.

<u>Lemma 6.11</u> Let G be $\mathbb{P}'$ - generic. Let $B = B^G$, $C = C^G$. Then $G = G^{B,C}$.

proof

Suppose not. Then there is $p \in G^{B,C} \setminus G$. But then there is $q \in G$ s.t. p, q are incompatible. Hence $p, q \in G^{B,C}$ & hence $\langle H_\theta, B, C \rangle \models \mathcal{L}(p) \cup \mathcal{L}(q)$ for $\theta = 2^{\beta+}$. Contr. QED (6.11)

The following lemma is sometimes useful in dealing with the case that $\mu \notin Ac$.

Set $\tilde{\mathbb{B}}_\lambda = \bigsqcup_{i < \lambda} (\mathbb{B}_i \setminus \{0\})$ for $\lambda \leq \mu$.

Hence $\tilde{\mathbb{B}}_\lambda$ is dense in $\mathbb{B}_\lambda$ whenever $\lambda \notin A$ or $cf(\lambda) = \omega_1$, since $\mathbb{B}_\lambda$ is then the direct limit of $\langle \mathbb{B}_i \mid i < \lambda \rangle$.

<u>Lemma 6.12</u> Let $\mu \notin A$. (Hence $\mu = \gamma$ is strongly inaccessible in V.) Let $B \subset \mathbb{B}_\mu$ s.t. $B \cap \mathbb{B}_i$ is $\mathbb{B}_i$ - generic for $i < \mu$. Then B is $\mathbb{B}_\mu$ - generic.

proof. $\tilde{\mathbb{B}}_\mu = \bigcup_{i<\mu} \mathbb{B}_i$ is dense in $\mathbb{B}_\mu$ and $\mathbb{B}_\mu$ satisfies μ-cc. Hence $\tilde{\mathbb{B}}_\mu = \mathbb{B}_\mu$. Let Δ be dense in $\mathbb{B}_\mu$. Then $\{\lambda < \mu \mid \Delta \cap \tilde{\mathbb{B}}_\lambda$ is dense in $\tilde{\mathbb{B}}_\lambda\}$ is club in μ. But $\{\lambda < \mu \mid \tilde{\mathbb{B}}_\lambda$ is dense in $\mathbb{B}_\lambda\}$ is stationary in μ, since it contains all $\lambda < \mu$ s.t. $cf(\lambda) = \omega_1$. Hence $\Delta \cap \tilde{\mathbb{B}}_\lambda$ is dense in $\mathbb{B}_\lambda$ for a $\lambda < \gamma$. Hence $\Delta \cap B \cap \mathbb{B}_\lambda \neq 0$ by genericity. QED (6.12)

$\mathbb{P}'$ satisfies a rather strong form of revisability:

Lemma 6.13 Let $p \in \mathbb{P}$, Let B', C' be s.t.

- B' is $\mathbb{B}^p$-generic over M_p
- $Q_p^{B'} = Q_p^{B^p}$
- $C' < Q_p^{B'}$ cofinally.

Then $p' \in \mathbb{P}$, where $p_0' = \langle M_p, B', C' \rangle$, $p_1' = p_1$.

prf. S
Let $\mathfrak{A}$ be a solid model of $\mathcal{L}'(p)$. Let $\pi = \pi_p^{\mathfrak{A}}$
and $\pi^* =$ that $\pi^* \supset \pi$ s.t. $\pi^* : M_p^{B^p} \prec M^{B^{\mathfrak{A}}}$.

Set $B = \bigcup_{x \in Q_p} \pi^*(x \cap B')$, $C = \pi^{*}{}''C'$.

Let $\mathfrak{A}$ be the result of replacing $B^{\mathfrak{A}}, C^{\mathfrak{A}}$ with B, C.

Claim $\mathfrak{A} \models \mathcal{L}(p')$

(1) B is $\mathbb{B}_\mu$-generic over V

prf.

Case 1 $\mu \in Ac$

Then $\mathbb{B}_\mu$ has a dense set of size $d(\mathbb{B}_\mu) < \delta$, since $d(\mathbb{B}_\mu) = \omega_1$ in $V[B^{\mathfrak{A}}]$. ($B^{\mathfrak{A}}$ is $\mathbb{B}_\mu$-generic over V, since it is $\mathbb{B}_\mu$-generic over $N = H_{\beta^+}^V$.) But then $\mathbb{B}^p$ has a dense set of size $< \delta_p$ in M_p. Let $D_p \in Q_p$ be dense in $\mathbb{B}^p$. Then $B' \cap D_p$ is D_p-generic over Q_p. But $\pi(D_p) = D$ where D is dense in $\mathbb{B}_\mu$ and $\pi^*(B' \cap D_p) = B \cap D$, where $\pi^* : Q_p^{B^p} \prec Q^{B^{\mathfrak{A}}}$. Since $B' \cap D_p$ is D_p-generic over Q_p, $B \cap D$ is D-generic

$Q = H_\gamma$, hence over V, Hence B is $\mathbb{B}_\mu$-generic over V. QED (Case 1)

Case 2 $\mu \notin Ac$.

Then $\mu = \gamma$ is strongly inaccessible. It follows that the set D of $\lambda < \mu$ s.t. every $u \in V_\lambda$, $u \subset \mathbb{B}_\mu$ is predense in $\mathbb{B}_\mu$ iff it is predense in $\mathbb{B}_\mu \cap V_\lambda$, is club in μ. D is M-definable. Hence there are

arbitrarily large $\delta \in D \cap \mathrm{rng}(\pi_p^M)$.

Let Δ be dense in $\mathbb{B}_\mu$ and let $A \subset \Delta$ be a max. antichain. Then $A \in V_\delta$ for a $\bar\delta \in D$ s.t. $\pi_p^M(\bar\delta) = \delta$, by μ-cc. Then $\pi^*(B' \cap V_{\bar\delta}^Q) = B \cap V_\delta$ and;

$M_p \models$ Every $u \in V_{\bar\delta}$ is predense in $\mathbb{B}^p$ iff in $V_{\bar\delta} \cap \mathbb{B}^p$

Since B' is $\mathbb{B}^p$-generic it follows that $M_p^B \models (u \cap (B' \cap V_{\bar\delta}) \neq \emptyset$ whenever u is predense in $B' \cap V_{\bar\delta}$)

by genericity. But then the same holds of $B \cap V_\delta$. Hence $B \cap V_\delta \cap A \neq \emptyset$. Hence $B \cap \Delta \neq \emptyset$. QED (1)

(2) $Q^B = Q^{B\prime\prime}$

pf. of (2)

Set $Q^B(\xi) = L_\xi[A, \mathbb{B}_\mu, B]$ for $\xi \leq \delta$.

Similarly for $Q^{B'\Omega}(\xi)$.

Let $Q_p^{B'}(\xi)$, $Q_p^{B^p}(\xi)$ have the same definition in $A^p, \mathbb{B}^p$ for $\xi \leq \delta_p$. Then

$\pi^*(Q_p^{B'}(\xi)) = Q^B(\pi(\xi))$, $\pi^*(Q_p^{B^p}(\xi)) = Q^{B'\Omega}(\pi(\xi))$,

(c) Let $\xi < \delta$, $\xi = \pi(\bar{\xi})$. There is $\bar{\varsigma} > \bar{\xi}$ s.t. $Q_p^{B'}(\bar{\xi}) \in Q_p^{B^p}(\bar{\varsigma})$, since $Q_p^{B'} = Q_p^{B^p}$. Hence $Q^B(\xi) \in Q^{B'\Omega}(\varsigma)$ where $\pi(\bar{\varsigma}) = \varsigma$.

($\supset$) is entirely similar. QED (2)

But then $\pi^* \restriction Q_p^{B'} : Q_p^{B'} \prec Q^B$ and $\pi^* {}'' C' = C$. Hence:

(3) $\pi^* \restriction Q_p^{B'} : \langle Q_p^{B'}, C' \rangle \lhd \langle Q^B, C \rangle$.

By the definition of $\mathbb{B}$ we have;

(4) $\pi^*(B' \cap x) = B \cap \pi(x)$ for $x \in Q_p$.

Since $\langle \tilde{M}, \pi \rangle$ is the liftup of $\langle M_p, \pi \restriction Q \rangle$, where $\pi : M \to \tilde{M}$ cofinally, we conclude by Lemma 5.4 that:

(5) $\pi : \langle M_p, B', C' \rangle \lhd' \langle M, B, C \rangle$, where $\pi = \pi_p^{M\Omega}$.

Since $p_1' = p_1$ we, of course, have:

(6) $\pi : \langle M_p, \bar{a} \rangle \prec \langle M, a \rangle$ whenever $\langle a, \bar{a} \rangle \in F^{p'}$.

Thus we have shown:

(7) $\mathfrak{A}' \models \varphi_{p'}$.

It remains only to show:

<u>Claim</u> $\mathfrak{A}' \models \mathcal{L}'$.

All axioms except (*) are trivial. We verify (*). Let $x \in M$. By Lemma 6.3 there is $u = \langle \bar{M}, \bar{B}, \bar{C} \rangle \in H_{\omega_1}$ and $\pi \in \mathfrak{A}$ s.t. $\pi : u \triangleleft' \langle M, B^{\mathfrak{A}}, C^{\mathfrak{A}} \rangle$, and $x \in \text{rng}(\pi)$, and

$\pi : \langle \bar{M}, \bar{B}, \bar{C}, \bar{B}', \bar{C}' \rangle \prec \langle M, B^{\mathfrak{A}}, C^{\mathfrak{A}}, B, C \rangle$.

Using Lemma 5.4 it follows easily that:

$\pi : \langle \bar{M}, \bar{B}', \bar{C}' \rangle \triangleleft' \langle M, B, C \rangle$,

where $x \in \text{rng}(\pi)$. QED (Lemma 6.13)

As a corollary of the proof:

<u>Lemma 6.14</u> Let p, B', C', P' be as above. Let $q \leq p$. Set $B = \bigcup_{x \in M_p} \pi^*_{pq}(x \cap B')$, $C = \pi^*_{pq}{}''C'$. Then

(a) $Q_q^B = Q_q^{B^q}$ and B is $\mathbb{B}^q$-generic over M_q (hence $q' \in \mathbb{P}'$ where $q'_0 = \langle M_q, B, C \rangle$, $q'_1 = q_1$)

(b) $\pi_{p'q'} = \pi_{pq}$.

Proof (sketch).

Repeat the proof of (1) – (5) mutatis mutandis. QED (6.14)

Lemma 7 Let G be $\mathbb{P}'$ – generic, $B = B^G$, $C = C^G$. Let B' be $\mathbb{B}_\mu$ – generic s.t. $Q^{B'} = Q^B$. Let $C' < Q^B$ be countable and cofinal in Q^B. Assume that B', C' lie in a generic extension of $V[G]$ which adds no reals. Set: $G' = G^{B', C'}$. Then G' is $\mathbb{P}'$ – generic and $V[G'] = V[G]$.

proof.

There is a $p \in G$ s.t. $C' \subset rng(\pi^*)$ and $B' \cap x \in rng(\pi^*)$ for all $x \in C$, where $\pi = \pi_p^G$ and π^* is the unique $\pi^* \supset \pi$ s.t. $\pi^* : M_p^{B^p} < M^B$. Set:

$$B^{p\prime} = \bigcup_{x \in C} \pi^{*-1}(B' \cap x), \quad C^{p\prime} = \pi^{*-1}{}''C'.$$

Then:

(1) $B', C' \in V[G]$ since
$$B' = \bigcup_{x \in C^p} \pi^*(B^{p\prime} \cap x), \quad C' = \pi^*{}''C^{p\prime}$$

(2) $\pi^*(B^{p\prime} \cap x) = B' \cap \pi(x)$ for $x \in M_p$

(3) $B^{p\prime}$ is $\mathbb{B}^p$ – generic over M_p

proof.
Let $\Delta \in M_p$ be dense in $\mathbb{B}^p$. Then $\pi(\Delta)$ is dense in $\mathbb{B}_\mu$. Hence $\pi(\Delta) \cap B' \cap \pi(x) \neq \emptyset$ for some $x \in C^p$, since $B' \cap \pi(\Delta) \neq \emptyset$. Hence $\Delta \cap B^{p\prime} \cap x \neq \emptyset$. QED (3)

(4) $Q_p^{B^{p'}} = Q_p^{B^p}$

proof.

Set $Q'_p(\xi) =_{df} L_\xi[A^p, \mathbb{B}^p, B^{p'}]$ for $\xi < \delta_p$,

$Q_p(\xi) =_{df} L_\xi[A^p, \mathbb{B}^p, B^p]$

Then $\pi^*(Q'_p(\xi)) = Q'(\pi(\xi)) = L_{\pi(\xi)}[A, \mathbb{B}_\mu, B']$

and $\pi^*(Q_p(\xi)) = Q(\pi(\xi)) = L_{\pi(\xi)}[A, \mathbb{B}_\mu, B]$.

Since $Q^{B'} = Q^B$, there is $\zeta < \delta_p$

s.t. $Q'(\pi(\xi)) \in Q(\pi(\zeta))$. Hence $Q_p^{B^{p'}} \subset Q_p^{B^p}$.

Similarly $Q_p^{B^p} \subset Q^{B^{p'}}$. QED

By Lemma 6.13 we conclude:

(5) $p' \in \mathbb{P}'$ where $p'_0 = \langle M_p, B^{p'}, C^{p'} \rangle$, $p'_1 = p_1$.

Now let $q \leq p$. Set:

$B^{q'} = \bigcup_{x \in Q_p} \pi^*_{pq}(B^{p'} \cap x)$, $C^{q'} = \pi^*_{pq}{}''C^{p'}$,

$q'_0 = \langle M_q, B^{q'}, C^{q'} \rangle$, $q'_1 = q_1$

By Lemma 6.14 we have:

(6) (a) $B^{q'}$ is $\mathbb{B}^q$-generic over M_q and

$Q_q^{B^{q'}} = Q_q^{B^q}$

(b) $q' \in \mathbb{P}'$

(c) $\pi_{p'q'} = \pi_{pq}$.

If $q \leq r \leq p$ we could, of course, define q' from r' the way we defined it from p'. It is easily seen that we get the same thing. Hence:

(7) $q \leq r \leq p \rightarrow q' \leq r' \leq p'$.

Set $\Delta_0 = \{q \mid q \leq p\}$, $\Delta_1 = \{q \mid q \leq p'\}$.

Set $\sigma(q) = q'$ for $q \in \Delta_0$. Then $\sigma'' \Delta_0 \subset \Delta_1$ and $r \leq q \rightarrow \sigma(r) \leq \sigma(q)$ for $r, q \in \Delta_0$.

Now let $q \leq p'$. We can reverse the above operation σ by setting

$$B^{\tilde{q}} = \bigcup_{x \in Q^{p'}} \pi^*_{p'q}(B^p_x), \quad C^{\tilde{q}} = \pi^*_{p'q}{}'' C^p,$$

$\tilde{q} = \sigma^{-1}(q)$ is defined by:

$$\tilde{q}_0 = \langle M_q, B^{\tilde{q}}, C^{\tilde{q}} \rangle, \quad \tilde{q}_1 = q_1.$$

Repeating the above proofs we get:

$\sigma^{-1}(q) \in \Delta_1$, $r \leq q \rightarrow \sigma^{-1}(r) \leq \sigma^{-1}(q)$.

Moreover $\sigma^{-1}\sigma(q) = q$ for $q \in \Delta_0$ and $\sigma\sigma^{-1}(q) = q$ for $q \in \Delta_1$.

Hence:

(8) $\sigma : \langle \Delta_0, \leq \rangle \overset{\sim}{\longleftrightarrow} \langle \Delta_1, \leq \rangle$.

At $q \in \Delta_0 \cap G$ it then follows easily that $\sigma(q) \in G' = G^{B',C'}$. Similarly $q \in \Delta_1 \cap G' \to \sigma^{-1}(q) \in G = G^{B,C}$.

Hence:

(9) $\sigma''(\Delta_0 \cap G) = \Delta_1 \cap G'$.

Hence

(10) $V[G] = V[G']$, since

$$G' = \{ r \mid \bigvee q \in \Delta_0 \cap G \ \sigma(q) \leq r \}$$

$$G = \{ r \mid \bigvee q \in \Delta_1 \cap G' \ \sigma^{-1}(q) \leq r \}.$$

Finally:

(11) G' is $\mathbb{P}'$-generic.

proof

Let Δ be dense in $\mathbb{P}'$. Set $\Delta' = \{ q \in \Delta_0 \mid \sigma(q) \in \Delta \}$. Then Δ' is dense above p in $\mathbb{P}'$. Hence $G \cap \Delta' \neq \emptyset$. Hence $G' \cap \Delta \cap \Delta_1 = \sigma'' G \cap \Delta' \neq \emptyset$.

QED (Lemma 7)

Setting: $\mathbb{B}' = BA(\mathbb{P}')$, we note that F is $\mathbb{B}'$-generic iff $G = \{p \mid [p] \in F\}$ is $\mathbb{P}'$-generic and

$$F = F_G =_{Df} \{b \in \mathbb{B}' \mid G \cap b \neq \emptyset\}.$$

The proof of Lemma 7 actually gives:

<u>Cor 7.1</u> Let G, B, C, B', C', G' be as above. There is $\sigma^* \in V$ s.t. $\sigma^*: \mathbb{B}' \overset{\sim}{\longleftrightarrow} \mathbb{B}'$ and $F_{G'} = \sigma'' F_G$.

proof. (Assume w.l.o.g. $G' \neq G$)
Set: $\Delta_2 = \{r \mid r$ is incompatible with p and $p'\}$.
Then $\Delta_0 \cup \Delta_1 \cup \Delta_2$ is dense in $\mathbb{P}'$.
Since $G \neq G'$, we have $B \neq B'$ or $C \neq C'$. Thus
Hence $B^p \neq B^{p'}$ or $C^p \neq C^{p'}$. Thus p, p' are incompatible and $\Delta_0, \Delta_1, \Delta_2$ are mutually disjoint.
Set $\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2$. We define:

$$\sigma': \langle \Delta, \leq \rangle \overset{\sim}{\longleftrightarrow} \langle \Delta, \leq \rangle$$

by: $$\sigma'(q) = \begin{cases} \sigma(q) & \text{if } q \in \Delta_0 \\ \sigma^{-1}(q) & \text{if } q \in \Delta_1 \\ q & \text{otherwise} \end{cases}.$$

Since $f: \mathbb{B}' \overset{\sim}{\longleftrightarrow} BA(\langle \Delta, \leq \rangle)$ where $f(b) = b \cap \Delta$, we have:

σ' induces $\sigma^*: \mathbb{B}' \stackrel{\sim}{\longleftrightarrow} \mathbb{B}'$ defined by: $\sigma^*([p]) = [\sigma'(p)]$ for $p \in \Delta'$.

QED (7.1)

If we run the proof with $C' = C^G$, we get:

<u>Cor 7.2</u> Let G be $\mathbb{B}_{\mu+1}$ - generic and let $B = G \cap \mathbb{B}_\mu$. Let B' be $\mathbb{B}_\mu$ - generic s.t. $H_\gamma[B'] = H_\gamma[B]$, where $\gamma = \gamma_{\mu+1}$. Then there is $\pi \in V$ s.t. π is an automorphism of $\mathbb{B}_{\mu+1}$ and $\pi''B = B'$. (Hence $G' = \pi''G$ is $\mathbb{B}_{\mu+1}$ - generic and $V[G'] = V[G]$.)

Lemma 8 $\mathbb{B}_{\mu+1}$ is symmetrically proud over $\mathbb{B}_\xi$ whenever $\xi \le \mu$ s.t. $\xi \in Ac$.

proof

This reduces to:

Main Claim Let $\theta > 2^\beta$ be big enough to verify the proudness of $\mathbb{B}_i$ for all $i \le \mu$ s.t. $i \in Ac$. Let G be $\mathbb{P}'$-generic and let $\pi \in V[G]$ s.t. G is π-conforming and $\pi : \bar{W} \prec W = H_\theta$, where $\bar{W}$ is countable and transitive. Set $B = B^G \cap \mathbb{B}_\xi$. Let:

$$\pi(\bar{\xi}, \bar{\mathbb{P}}', \langle \bar{\mathbb{B}}_i \mid i \le \bar{\mu} \rangle) = \xi, \mathbb{P}', \langle \mathbb{B}_i \mid i \le \mu \rangle.$$

Suppose that $\bar{B}'$ is $\bar{\mathbb{B}}_{\bar{\xi}}$-generic over $\bar{W}$ and B' is $\mathbb{B}_\xi$-generic s.t. $V[B'] = V[B]$ and $\pi '' \bar{B}' \subset B'$.

Let $\bar{G}'$ be $\bar{\mathbb{P}}'$-generic over $\bar{W}$ s.t. $\bar{B}' = B^{\bar{G}'} \cap \bar{\mathbb{B}}_{\bar{\xi}}$. There is G' s.t.

- G' is $\mathbb{P}'$-generic
- $B' = B^{G'} \cap \mathbb{B}_\xi$
- $\pi '' \bar{G}' \subset G'$
- There is $\sigma : \mathbb{B}' \overset{\sim}{\longleftrightarrow} \mathbb{B}'$ s.t.

  $\sigma '' F_G = F_{G'}$

(where $\mathbb{B}' = BA(\mathbb{P}')$)

proof.

<u>Case 1</u> $\mu \in Ac$

We can assume w.l.o.g. that $\xi = \mu$. Since $V[B'] = V[B]$ we conclude: $Q^{B'} = Q^B$, using the fact that $\mathbb{B}_\mu$ satisfies γ-c.c. Set $\bar{C}' = c^{\bar{G}'}$, $C' = \pi^{*}{}''\bar{C}'$, where $\pi^* \supset \tilde{\pi}$ s.t. $\pi^* : W[\bar{B}'] \prec W[B']$ and $\pi^*(\bar{B}') = B'$. Set $G' = G^{B', C'}$. Then by Lemma 7.1 we have:

(1) G' is $\mathbb{P}'$-generic and $B' = B^{G'}$

(2) There is $\sigma : \mathbb{B}' \overset{\sim}{\longleftrightarrow} \mathbb{B}'$ s.t. $\sigma'' F_G = F_{G'}$.

We need only prove:

(3) $\pi''\bar{G}' \subset G'$.

proof.

Let $\bar{p} \in \bar{G}'$, $p = \pi(\bar{p})$. Since $\pi_p^{\bar{G}'} : P_0 \lhd' \langle \bar{M}, \bar{B}', \bar{C}' \rangle$ and $\pi(P_0) = P_0$, $\pi^*(\langle \bar{M}, \bar{B}', \bar{C}' \rangle) = \langle M; B', C' \rangle$, we have:

$\tilde{\pi} : P_0 \lhd' \langle M, B', C' \rangle$, where $\tilde{\pi} = \pi^*(\pi_p^{\bar{G}'}) = \pi \circ \pi_p^{\bar{G}'}$. It remains only to show:

$\tilde{\pi} : \langle M_p, \bar{a} \rangle \prec \langle M, a \rangle$ whenever $\langle a, \bar{a} \rangle \in F^{\pi(p)} = \pi(F^p)$,

Let $\pi(a') = a$. Then $\langle a', \bar{a} \rangle \in F^p$ and $\pi_p^{\bar{G}'} : \langle M_p, \bar{a} \rangle \prec \langle \bar{M}, a' \rangle$.

Hence $\tilde{\pi} = \pi \circ \pi_p^{\bar{G}'} : \langle M_p, \bar{a} \rangle \prec \langle M, a \rangle$
since $\pi(\langle \bar{M}, \bar{a}' \rangle) = \langle M, a \rangle$.

QED (Case 1)

Case 2 Case 1 fails

Then $\mu = \gamma$ is strongly inaccessible. Let $\langle \gamma_i \mid i < \omega \rangle \in V[G]$ be monotone and cofinal in γ s.t. $\gamma_0 = \xi$ and $\gamma_i \in C^G$ for $i < \omega$. Let $\pi(\bar{\gamma}_i) = \gamma_i$. We may also assume w.l.o.g. that $\gamma_i \in Ac$. Hence $\mathbb{B}_{\gamma_{i+1}}$ is proud over $\mathbb{B}_{\gamma_i}$ for $i < \omega$. Set:

$$\tilde{B} = B^G, \quad \tilde{B}_i = \tilde{B} \cap \mathbb{B}_{\gamma_i}, \quad \tilde{C} = C^G$$
$$\bar{B}'' = B^{\bar{G}'}, \quad \bar{B}_i'' = \bar{B}'' \cap \bar{\mathbb{B}}_{\bar{\gamma}_i}, \quad \bar{C}'' = C^{\bar{G}'}.$$

(Hence $\tilde{B}_0 = B$, $\bar{B}_0'' = \bar{B}$). By proudness we may successively choose B_i'' $(i < \omega)$ s.t. $B_0'' = B'$, $B_{i+1}'' \supset B_i''$, B_i'' is $\mathbb{B}_{\gamma_i}$-generic, $\pi''\bar{B}_i'' \subset B_i''$, and $V[B_i''] = V[\tilde{B}_i]$. Set:

$$B'' = \{ a \in \mathbb{B}_\mu \mid \bigvee b \in \bigcup_{i<\omega} B_i'' \; b \subset a \},$$

Then $B'' \cap \mathbb{B}_\eta$ is $\mathbb{B}_\eta$-generic for $\eta < \mu$. Hence by Lemma 6.12:

(4) B'' is $\mathbb{B}_\mu$-generic.

We then set: $G' = G^{\mathbb{B}'', C''}$, where $C'' = \pi^{*}{}''\bar{C}''$. The rest of the proof is exactly as in Case 1.

QED (Lemma 8)

Finally we prove:

<u>Lemma 9</u> $\mathbb{B}_{\mu+1}$ is symmetrical over $\mathbb{B}_{\mu}$.

proof.

This reduces to:

<u>Main Claim</u> Let $\sigma : \mathbb{B}_{\mu} \overset{\sim}{\longleftrightarrow} \mathbb{B}_{\mu}$. There $\sigma^{*} : \mathbb{B}' \overset{\sim}{\longleftrightarrow} \mathbb{B}'$ s.t. $\sigma^{*}k' = k'\sigma$.

proof.

Let $N^{*} = \langle H_{\theta}, N, \sigma, <, \mathbb{B}_{\mu} \ldots \rangle$, $\theta > 2^{\beta}$.

Set $\Delta = \{p \mid p$ conforms to $N^{*}\}$.

Then Δ is dense in $\mathbb{B}'$. For $p \in \Delta$ define $p' = \tilde{\sigma}(p)$ by:

Set: $\bar{N}^{*} = \bar{N}^{*}(p, N^{*}) = \langle \bar{H}, \bar{N}, \bar{\sigma}, <, \bar{\mathbb{B}} \ldots \rangle$.

Hence $\mathbb{B}^{p}$ is $\bar{\mathbb{B}} = \mathbb{B}^{p}$-generic over $\bar{M} = M_{p}$ and $\bar{\sigma} : \bar{\mathbb{B}} \overset{\sim}{\longleftrightarrow} \bar{\mathbb{B}}$.

Hence $Q^{B^{p}} = Q^{\bar{\sigma}'' B^{p}}$, since $\bar{N}^{*}[B^{p}] = \bar{N}^{*}[\bar{\sigma}'' B^{p}]$ and:

$Q^{BP} = H_{\omega_2}^{\bar{N}^*[BP]}$. Set:

$p_0' = \langle M_p, \bar{\sigma}''B^p, C^p \rangle$, $p_1' = p_1$.

Then $p' \in \mathbb{P}'$. It follows easily by earlier lemma that

$$p \leq q \longleftrightarrow \tilde{\sigma}(p) \leq \tilde{\sigma}(q)$$

and $\pi_{pq} = \pi_{\tilde{\sigma}(p), \tilde{\sigma}(q)}$ for $p, q \in \Delta$,

Moreover, if we set: $\tilde{\sigma}^{-1}(p) = p''$ where

$$p_0'' = \langle M_p, \bar{\sigma}^{-1}{}''B^p, C^p \rangle, \; p_1'' = p_1,$$

Then $\tilde{\sigma}^{-1}\sigma(p) = \sigma\sigma^{-1}(p) = p$,

Thus $\sigma : \langle \Delta, \leq \rangle \overset{\sim}{\longleftrightarrow} \langle \Delta, \leq \rangle$. Hence

$\tilde{\sigma}$ induces $\sigma^* : \mathbb{B}' \overset{\sim}{\longleftrightarrow} \mathbb{B}'$ s.t.

$\sigma^*([p]) = [\tilde{\sigma}(p)]$. It remains only to show:

<u>Claim</u> $\sigma^* k' = k'\sigma$.

pf. Let $b \in \mathbb{B}_u$.

Set: $\Delta^* = \{p \in \Delta \mid b, \sigma(b) \in \mathrm{rng}(\pi^p)\}$.

Then Δ^* is dense in Δ and

$\tilde{\sigma}''\Delta^* = \Delta^*$ since π^p depends only on p_1, hence $\pi^p = \pi^{\tilde{\sigma}(p)}$. Then for $p \in \Delta^*$ we have, letting $\pi^p(\bar{b}) = b$, (hence $\pi^p(\bar{\sigma}(\bar{b})) = \sigma(b)$):

$\sigma^*([p]) \subset \sigma^* k'(b) \longleftrightarrow [p] \subset k'(b) = [\![\check{b} \in \mathbb{B}^{\check{G}}]\!]$

$\longleftrightarrow \bar{b} \in \mathbb{B}^p \longleftrightarrow \bar{\sigma}(\bar{b}) \in \bar{\sigma}''\mathbb{B}^p = \mathbb{B}^{\tilde{\sigma}(p)} \longleftrightarrow$

$\longleftrightarrow [\tilde{\sigma}(p)] \subset k'(\sigma(b))$

$\qquad\quad \| $

$\longleftrightarrow \sigma^*([p]) \subset k'(\sigma(b))$.

Since $\{\sigma^*([p]) \mid p \in \Delta^*\}$ is dense in $\mathbb{B}'$,
we conclude: $\sigma^* k'(b) = k'(\sigma(b))$

QED (Lemma 9)

Lemma 10 $\langle \mathbb{B}_i \mid i \leq \mu+1 \rangle$ satisfies (a) - (h) of § 2.3.

proof.

(h) is straightforward.

We prove (a) for $i = \mu+1$. Let B be $\mathbb{B}_\mu$-generic + G be $\mathbb{P}_B$-generic.

If $\gamma = \gamma_{\mu+1} \leq \tau < \beta_{\mu+1}$ and τ is regular in V, then τ remains regular in V[B], since $\gamma = \omega_2$ in V[B] and $\mathbb{B}_\mu \subset H_\gamma$. But then $cf(\tau) = \omega$ in V[B][G] by Lemma 3.13

If $\beta = \beta_{\mu+1}$, then $cf(\beta) = \omega_1$ in V, hence in V[B][G], since no new reals are added. But then $\bar{\bar{\beta}} = \omega_1$ in V[B][G] by Lemma 3.12 and $\beta^+ = \omega_1$ in V[B][G] by Lemma 3.15.

Otherwise $\beta_{\mu+1} = \beta^+ \in A_0$, where $2^\beta = \beta^+$.

Hence $\bar{\bar{\beta}}_{\mu+1} = cf(\beta_{\mu+1}) = \omega_1$ and $\beta^+_{\mu+1} = \omega_2$ in V[B][G] by Lemma 3.14.1 and the remark following it.

QED (a)

(b) follows for $i = \mu+1$ by Lemma 4.

(c), (d) are vacuous for $i = \mu+1$,
(e) holds by Lemma 8 and (f) by Lemma 9.
It remains to prove (g). We imitate the proof of (g) in the first successor case.
Let $h \le \mu$ and set $\tilde{\mathbb{B}}_i = \mathbb{B}_i / B$ for $h \le i \le \mu+1$,
where B is $\mathbb{B}_h$-generic. We know:
$\mathbb{B}_{\mu+1}/B' \simeq BA(\mathbb{P}_{B'})$ whenever B' is $\mathbb{B}_\mu$-generic. Hence if $\tilde{B}$ is $\tilde{\mathbb{B}}_\mu$-generic
we have:
$\tilde{\mathbb{B}}_{\mu+1}/\tilde{B} = (\mathbb{B}_{\mu+1}/B)/\tilde{B} = \mathbb{B}_{\mu+1}/B' = BA(\mathbb{P}_{B'})$
where $B' = B * \tilde{B} = \{b \in \mathbb{B}_\mu \mid b/B \in \tilde{\mathbb{B}}_\mu\}$ is
$\mathbb{B}_\mu$-generic. Hence:
$\Vdash_{\tilde{\mathbb{B}}_\mu} \tilde{\mathbb{B}}_{\mu+1}/\dot{B} \simeq BA(\mathbb{P}_{\check{B} * \dot{B}})$, $\dot{B}$ being the canonical generic name. Exactly as before we construct in $V[B]$ a
$\sigma : \tilde{\mathbb{B}}_{\mu+1} \overset{\sim}{\longleftrightarrow} \tilde{\mathbb{B}}_\mu * \ddot{\mathbb{B}}$, where
$\Vdash_{\tilde{\mathbb{B}}_\mu} \ddot{\mathbb{B}} = BA(\mathbb{P}_{\check{B} * \dot{B}})$, and observe
that $\sigma(b) = \tilde{k}(b)$ for $b \in \tilde{\mathbb{B}}_\mu$, where
$\tilde{k} : \tilde{\mathbb{B}}_\mu \longrightarrow \tilde{\mathbb{B}}_\mu * \ddot{\mathbb{B}}$ is the natural
projection. Hence $\langle \tilde{\mathbb{B}}_{h+i} \mid i \le \mu+1-h \rangle$
has the salient properties of $\langle \mathbb{B}_i \mid i \le \mu+1 \rangle$
and we can repeat all of our proofs in $V[B]$.

QED