§4 The limit case

§4.1 The 1st limit case

Suppose that λ is an ω-point of the iteration, $\gamma = \gamma_\lambda$, and we wish to give γ^+ the cofinality ω. Let β be the supremum of the cardinals $< \beta_\lambda$. Then either $\beta = \beta_\lambda$, $2^{\underset{\smile}{\beta}} = \beta$, and $cf(\beta) = \omega_1$, or else we are doing the second construction (hence $2^{\underset{\smile}{\beta}} = \beta$ and $2^\beta = \beta^+$ by GCH) and $\beta_\lambda = \beta^+ \in A$, where $\beta_\lambda \leq \gamma^{(\omega_1)}$. We must find a completion $\mathbb{B}_\lambda$ of $\bigcup_{i<\lambda} \mathbb{B}_i$ which gives every regular $\tau \leq \beta$ the cofinality ω. If $cf(\beta) = \omega_1$, we want β^+ to remain a cardinal in $V^{\mathbb{B}_\lambda}$. Otherwise we want β^+ to have cofinality ω_1 in $V^{\mathbb{B}_\lambda}$. We know, of course, that $2^{\underset{\smile}{\gamma}} = \gamma$, that every $\eta < \gamma$ is collapsed to ω_1 by some $\mathbb{B}_i$ $(i<\lambda)$ and that some $\mathbb{B}_i$ $(i<\lambda)$ gives γ cofinality ω.

Set: $M = L^A_\beta$, where $L_\eta[A] = H_\eta$ whenever $\eta \leq \beta$ and $2^{\underset{\smile}{\eta}} = \eta$. Set $Q = \langle L^A_\gamma, \langle \mathbb{B}_i \mid i<\lambda \rangle \rangle$, $N = \langle H_{\beta^+}, Q, M, <, m \rangle$ where $<$ well orders H_{β^+}.

Imitating the definitions in the second successor case, we set:

Def $\Gamma_* =$ the set of $\langle Q, B, C \rangle$ s.t.

- $Q = \langle L^A_\gamma, \mathbb{B} \rangle$ models Zermelo
- $\mathbb{B} = \langle \mathbb{B}_i \mid i < \gamma_0 \rangle$ where $\gamma_0 \leq \gamma$ is a limit ordinal; $\mathbb{B}_i \in Q$ is a complete BA in Q, $\mathbb{B}_j$ is proper over $\mathbb{B}_i$ in Q whenever $i < j < \gamma_0$ and $i, j \notin Ac$ in Q; $\sup_{i<\gamma_0} d_i = \gamma$ where $d_i = d(\mathbb{B}_i) \geq \omega_1$ for $0 < i < \gamma_0$ and $\mathbb{B}_0 = 2$.
- $B \subset \bigcup_i \mathbb{B}_i$ s.t. $B_i = B \cap \mathbb{B}_i$ is $\mathbb{B}_i$ - generic over Q for $i < \gamma_0$
- $C \subset \gamma_0$, $\sup C = \gamma_0$, $\mathrm{otp}(C) = \omega$.

(Set $c_n =$ the n-th element of C).

Def For $u = \langle Q_u, B^u, C^u \rangle$, $v = \langle Q_v, B^v, C^v \rangle$ set. $\pi : u \triangleleft_* v \xleftrightarrow{\mathrm{Df}} (\pi : Q_u \prec Q_v \wedge$
$\wedge\ \pi'' B^u \subset B^v \wedge \pi'' C^u = C^v)$

Def $u \triangleleft_* v \xleftrightarrow{\mathrm{Df}} \bigvee \pi\ \ \pi : u \triangleleft_* v$.

Fact 1 Let $v = \langle Q_v, B^v, C^v \rangle \in \Gamma_*$, $\alpha \leq d =_{\mathrm{Df}} \omega_1^{Q_v}$. There is at most one pair $\langle u, \pi \rangle$ s.t. $\pi : u \triangleleft_* v$ and $\alpha = d_u$.

proof of Fact 1.

Set $Q^B = Q[B] = \bigcup_{i<\gamma_0^Q} Q[B_i]$ for $\langle Q, B, C\rangle \in \Gamma_*$. Let $\pi : u \triangleleft_* v$ s.t. $\alpha = \alpha_u$. Then, since $\pi`` B_i^u \subset B_{\pi(i)}^v$ for $i < \gamma_0^u$, π extends to a unique $\pi_i : Q_u[B_i^u] \prec Q_v[B_{\pi(i)}^v]$ s.t. $\pi_i(B_i^u) = B_{\pi(i)}^v$. Set: $\pi^* = \bigcup_i \pi_i$.

Then $\pi^* : Q^u[B^u] \to_{\Sigma_0} Q^v[B^v]$ cofinally. Set $f_i^u =$ the $Q^u[B^u]$ - least f s.t. f maps $\alpha = \omega_1^{Q^u}$ onto $L_{\alpha_{c_i}}^{A^u}$. Let f_i^v be defined similarly for $i < \gamma_0^v$.

Then $\pi^*(f_i^u) = f_i^v$ and $\pi^*(f_i^u(\xi)) = \pi(f_i^u(\xi)) = f_i^v(\xi)$ for $i < \omega$, $\xi <$.

Hence $\mathrm{rng}(\pi) = \{f_i^v(\xi) \mid i<\omega, \xi<\alpha\}$.

QED (Fact 1)

Hence:

__Fact 2__ Let $u, v \in \Gamma_*$. There is at most one π s.t. $\pi : u \triangleleft_* v$.

__Def__ Let $u \triangleleft_* v$. $\pi_{uv} =: $ that π s.t. $\pi : u \triangleleft_* v$.

We easily get:

Fact 3 Let $u \lhd_* v \lhd_* w$. Then $u \lhd_* w$ and $\pi_{uw} = \pi_{vw} \cdot \pi_{uv}$.

(i.e. $\langle \Gamma_*, \Pi_* \rangle$ is a commutative system where $\Pi_* = \langle \pi_{u,v} \mid u \lhd_* v \rangle$.)

Fact 4 $\langle \Gamma_*, \Pi_* \rangle$ is continuous – i.e. if $I \subset \Gamma_*$, $R \subset I^2$ s.t. R is directed (i.e. $\bigwedge u v \in I \bigvee w \in I \; u, v R w$) and $u \lhd_* v \lhd_* w_0$ whenever $u R v$, then there is a unique w_1 s.t. $u \lhd_* w_1 \lhd_* w_0$ for $u \in I$ and $\langle w_1, \langle \pi_{u,w_1} \mid u \in I \rangle \rangle$ is the direct limit of $\langle I, \langle \pi_{uv} \mid u R v \rangle \rangle$.

(Note It follows that $\pi_{u w_0} = \pi_{w_1 w_0} \cdot \pi_{u w_1}$ for $u \in I$.)

Hence:

Fact 5 $\{ d_u \mid u \underset{\neq}{\lhd}_* w \}$ is closed in $d_w + 1$

<u>Def</u> M is a <u>smooth</u> <u>model</u> iff

- $M = L^A_\beta$ models ZFC^- or Zermelo
- $2^\omega = \omega_1$ in M (in particular, ω_1^M exists)

<u>Def</u> Q is a <u>smooth segment</u> of M iff
$Q = \langle L^A_\gamma, R_1, \ldots, R_n \rangle$ where

- $M = L^A_\beta$ is a smooth model
- $\gamma > \omega$ is a limit cardinal in M and
  $L_\gamma[A] = H^M_\gamma$
- $R_1, \ldots, R_n \in Q$

<u>Def</u> Γ = the set of $\langle Q, M, B \rangle$ s.t.

- Q is a smooth segment of M
- $\langle Q, \mathbb{B}, C \rangle \in \Gamma_*$ for a C s.t.
  $C \in M[B_i]$ for some $i < \gamma_0^Q$,

(<u>Note</u> Each B_i is $\mathbb{B}_i$ - generic over M,
where $Q = \langle L^A_\gamma, \mathbb{B} \rangle$.)

<u>Def</u> Let $u, v \in \Gamma$, $u = \langle Q_u, M_u, B^u \rangle$, $v = \langle Q_v, M_v, B^v \rangle$.
$\pi : u \lhd v$ iff

- $\pi \upharpoonright Q_u : \langle Q_u, C, B^u \rangle \lhd_* \langle Q_v, \pi``C, B^v \rangle$
  for a $C \in M_u[B^u_i]$, $i < \omega$.
- There is M_{uv} s.t. $\langle M_{uv}, \pi \rangle$ = the liftup of
  $\langle M_u, \pi \upharpoonright Q_u \rangle$.

<u>Fact 6</u> Let $u, v \in \Gamma$. There is at most one π s.t. $\pi : u \lhd v$.

proof.

Let $\pi : u \lhd v$. We show that π is unique. Let i be least s.t. γ_0^u is ω-cofinal in M_u. Let C = the least $C \in M_u[B_i^u]$ s.t. $C \subset \gamma_0^u = \sup C$ and $\mathrm{otp}(C) = \omega$. Let $i' = \pi(i)$. Then $\pi '' B_i^u \subset B_{i'}^v$, where $\pi(|B_i^u|) = |B_{i'}^v|$. Hence there is $\pi' \supset \pi$ s.t. $\pi' : M_u[B_i^u] \prec M_v[B_{i'}^v]$ and $\pi(B_i^u) = B_{i'}^v$. Let $C' = \pi'(C)$. Then C', i' are defined from $\langle Q_v, M_v, B^v \rangle$ as C, i were defined from $\langle Q_u, M_u, B^u \rangle$. Set:
$\bar{u} = \langle Q_u, B^u, C \rangle$, $\bar{v} = \langle Q_v, B^v, C' \rangle$. Then $\bar{u}$ is defined from u + $\bar{v}$ is defined from $\bar{v}$ by the same def. But clearly

$\pi \upharpoonright Q_u : \bar{u} \lhd_* \bar{v}$; hence $\pi \upharpoonright Q_u = \pi_{\bar{u}\bar{v}}$.

Thus $\pi \upharpoonright Q_u$ depends only on the pair u, v. But then so does π, since π depends only on $\pi \upharpoonright Q_u$. QED (Fact 6)

<u>Def</u> $u \lhd v \longleftrightarrow_{Df} \bigvee \pi \; \pi : u \lhd v$

For $u \lhd v$ we set:

π_{uv} = that π s.t. $\pi : u \lhd v$

We easily get:

Fact 7 $\langle \Gamma, \Pi \rangle$ is a continuous commutative system, where $\Pi = \langle \pi_{uv} \mid u \triangleleft v \rangle$

Hence:

Fact 8 $\{\alpha_u \mid u \underset{\neq}{\triangleleft} v\}$ is closed in α_v

Fact 9 Let $u, v \triangleleft w$, $\mathrm{rng}(\pi_{uw}) \subset \mathrm{rng}(\pi_{v,w})$. Then $u \triangleleft v$ and $\pi_{vw}\pi_{uv} = \pi_{uw}$.

Let Q, M, N be as defined above. Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \gamma_v \rangle$ where $\gamma_0 = \lambda$ be the iteration up to λ. Suppose $B \subset \bigcup_i \mathbb{B}_i$ s.t. $B_i = B \cap \mathbb{B}_i$ is $\mathbb{B}_i$-generic for $i < \lambda$. Then $\langle Q, M, B \rangle \in \Gamma$.

As a preliminary to defining $\mathbb{B}_\lambda$ we define:

Let $\mathcal{L}$ be the language on N with:

<u>Predicate</u> $\in$; <u>Constants</u> $\underline{x}$ $(x \in N)$, $\mathring{B}$

<u>Axioms</u> ZFC^-, $\bigwedge v (v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$,

$H_{\omega_1} = \underline{H_{\omega_1}}$, $\langle \underline{Q}, \underline{M}, \mathring{B} \rangle \in \Gamma$ and:

(*) For each $\xi < \underline{\beta}$ there are u, π s.t. $u \in H_{\omega_1}$, $\pi : u \lhd \langle \underline{Q}, \underline{M}, \mathring{B} \rangle \wedge \xi \in \mathrm{rng}(\pi) \wedge \Psi$,

where $\Psi = \begin{cases} \sup \pi'' \beta_u = \underline{\beta} & \text{if } \beta \text{ is regular,} \\ u = u & \text{if not} \end{cases}$

(<u>Note</u> $\sup \pi'' \beta = \underline{\beta}$ says the same as $M_u, \langle \underline{Q}, \underline{M}, \mathring{B} \rangle = \underline{M}$, recalling

$\langle M_{uv}, \pi \rangle =$ the lift up of $\langle M_u, \pi \upharpoonright Q_u \rangle$ for $u \lhd v$.)

If B is $\mathbb{B}_i$ - generic for some $i < \gamma_0$, we can, in $V[B]$, define: $\mathbb{B}/B = \langle \mathbb{B}_{i+j}/B \mid j < \gamma_0 - i \rangle$

$Q^B = \langle Q[B], B \rangle = \langle L_{\gamma}^{A,B}, \mathbb{B}/B \rangle$

$M^B = \langle M[B], B \rangle = L_{\beta}^{A,B}$

$N^B = \langle N[B], B \rangle = \langle H_{\beta^+}^{V[B]}, \in, Q^B, M^B, <, \dots \rangle$

where $<$ well orders N^B.

We can then define a language $\mathcal{L}_B$ over N^B as before in $V[B]$ with Q^B, M^B in place of Q, M.

If $B \subset \bigcup_i \mathbb{B}_i$ s.t. $B_i = B \cap \mathbb{B}_i$ is $\mathbb{B}_i$ - generic for $i < \gamma_0$, we set:

$B/B_i = \{b/B_i \mid b \in B\}$, where $b \to b/B_i$ is the canonical projection of $\bigcup_i \mathbb{B}_i$ onto $\bigcup_j \mathbb{B}_j/B_i$.

(We shall follow the convention in §3 of saying that $\mathfrak{A} = \langle |\mathfrak{A}|, \in^{\mathfrak{A}}, B^{\mathfrak{A}} \rangle$ is a solid model of $\mathcal{L}$ iff $\mathfrak{A}$ is solid, $N \subset$ wfcore$(\mathfrak{A})$, and $\mathfrak{A}$ becomes a model of $\mathcal{L}$ if we interpret $\underline{x}$ by x for $x \in N$.)

We then get:

<u>Lemma ⓪</u>

(a) If $\mathfrak{A} = \langle |\mathfrak{A}|, \in^{\mathfrak{A}}, B \rangle$ is a solid model of $\mathcal{L}$, then $B \subset \bigcup_i B_i$ and $B_i = B \cap \mathbb{B}_i$ is $\mathbb{B}_i$ - generic over V for $i < \gamma_0$.

(b) Let $B \subset \bigcup_i \mathbb{B}_i$ s.t. $B_i = B \cap \mathbb{B}_i$ is $\mathbb{B}_i$ - generic for $i < \gamma_0$. Let $i_0 < \gamma_0$. Then $\mathfrak{A} = \langle |\mathfrak{A}|, \in^{\mathfrak{A}}, B \rangle$ is a solid model of $\mathcal{L}$ iff $\mathfrak{A}_{i_0} = \langle |\mathfrak{A}|, \in^{\mathfrak{A}}, B/B_{i_0} \rangle$ is a solid model of $\mathcal{L}_{B_{i_0}}$.

The proof is straightforward.

Lemma 1 $\mathcal{L}$ is consistent.

proof.
Since δ_0 is an ω-point, there is an $i < \delta_0$ s.t. $\Vdash_{\mathbb{B}_i} \check{\delta}_0$ is ω-cofinal. By Lemma 0 it suffices to show that if B is $\mathbb{B}_i$-generic, then $\mathcal{L}_B$ is consistent (this being a statement in $V[B]$). Hence we may assume w.l.o.g. that δ_0 is ω-cofinal in V. Let $C = \{c_i \mid i < \omega\}$ where $\langle c_i \mid i < \omega \rangle$ is monotone and cofinal in δ_0. Set:
$\mathbb{B}'$ = the inverse limit of $\langle \mathbb{B}_{c_n} \mid n < \omega \rangle$.
Then $\mathbb{B}'$ is subcomplete by the ω-case of the iteration lemma for subcomplete algebras. Let B be $\mathbb{B}'$-generic. Set $B_\xi = B \cap \mathbb{B}_\xi$ for $\xi < \delta_0$. ~~In~~ $V[B]$ let $\sigma : \bar{N}[\bar{B}] \prec N[B]$, $\sigma(\bar{B}) = B$, $\sigma(\bar{C}) = C$; where $\bar{N}$ is countable & transitive. Let $\sigma(\bar{Q}) = Q$. Clearly we have:

(1) $\sigma \upharpoonright \bar{Q} : \langle \bar{Q}, \bar{B}, \bar{C} \rangle \triangleleft_* \langle Q, B, C \rangle$.

Set: $\langle \tilde{N}, \tilde{\sigma} \rangle$ = the liftup of $\langle \bar{N}, \sigma \upharpoonright \bar{Q} \rangle$

Then $\tilde{\sigma} : \bar{N} \prec \tilde{N}$ cofinally and there is a unique $\tilde{k} : \tilde{N} \prec N$ s.t. $\tilde{k} \upharpoonright Q = \mathrm{id}$ and $k\tilde{\sigma} = \sigma$. Let $\tilde{\mathcal{L}}$ be defined over $\tilde{N}$ the way $\mathcal{L}$ was defined over N (with $\tilde{M} = k^{-1}(M)$ taking the place of M). Then $k : \langle \tilde{N}, \tilde{\mathcal{L}} \rangle \prec \langle N, \mathcal{L} \rangle$ and it suffices to prove:

<u>Claim</u> $\tilde{\mathcal{L}}$ is consistent.

We show, in fact, that $\langle H_{\omega_2}^{V[B]}, B \rangle$ models $\tilde{\mathcal{L}}$. The proof is a virtual repetition of the corresponding step in the proof of §3 Lemma 1. The details are left to the reader.

QED (Lemma 1)

The same proof obviously yields:

<u>Cor 1.1</u> Let B be $\mathbb{B}_i$-generic ($i < \delta_0$). Then $\mathcal{L}_B$ is consistent.

Exactly as in §3 we get:

<u>Lemma 2</u> Let $\mathfrak{A}$ be a solid model of $\mathcal{L}$. Let $\langle A_n \mid n < \omega \rangle \in \mathfrak{A}$ s.t. $A_n \subset M$ for $n < \omega$. There is $u = \langle Q_u, M_u, B_u \rangle \in \Gamma \cap H_{\omega_1}$ s.t.

- $u \lhd \langle Q, M, B \rangle$
- $\pi : \langle M_u, \bar{A}_n \rangle \prec \langle M, A_n \rangle$ for $n < \omega$

where $\pi = \pi_{u, \langle Q, M, B \rangle}$, $\bar{A}_n = \pi^{-1} {}'' A_n$.

Cor 2.1 Let B be $\mathbb{B}_i$-generic ($i<\aleph_0$). Lemma 2 holds of $\mathcal{L}_B$ in $V[B]$ (with M^B in place of M).

We now define the conditions $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$.

Def $\tilde{\mathbb{P}}$ = the set of $\langle p_0, p_1\rangle$ s.t.

- $p_0 = \langle Q_p, M_p, B^p\rangle \in \Gamma \cap H_{\omega_1}$
- $p_1 = F^p$ is a countable set of pairs $\langle a,\bar a\rangle$ s.t. $\bar a \subset M_p$, $a \subset M$.

Def For $p\in\tilde{\mathbb{P}}$ let φ_p be the conjunction of

- $p_0 \lhd \langle \underline{Q}, \underline{M}, \mathring{B}\rangle$
- $\pi_p : \langle \underline{M}_p, \underline{a}\rangle \prec \langle \underline{M}, \underline{a}\rangle$ for $\langle a,\bar a\rangle\in F^p$

  where $\pi_p = \pi_{p_0, \langle \underline{Q}, \underline{M}, \mathring{B}\rangle}$.

- $\sup \pi_p '' \mathrm{On}_{\underline{M}_p} = \underline{\beta}$ if β is regular

Def $\mathbb{P} = \mathbb{P}_{\mathcal{L}} = \{p\in\tilde{\mathbb{P}} \mid \mathcal{L}(p) \text{ is consistent}\}$ where $\mathcal{L}(p) = \mathcal{L} + \varphi_p$. For $p,q\in\mathbb{P}$ set: $p\le q$ iff the following hold:

- $R^q \subset R^p$ where $R^p = \mathrm{rng}(F^p)$, $D^p = \mathrm{dom}(F^p)$
- $q_0 \lhd p_0$
- $\pi_{q_0 p_0} : \langle M_q, \bar a\rangle \prec \langle M_p, a'\rangle$ whenever $\langle a,\bar a\rangle\in F^q$, $\langle a,a'\rangle\in F^p$.

Exactly as before:

Lemma 3.1 Let $p, q \in \mathbb{P}$. Then $p \leq q$ iff

- $R^q \subset R^p$
- $\mathcal{L}(p) \vdash (\mathcal{L}(q) \wedge rng(\pi_q) \subset rng(\pi_p))$

Def $\pi_{qp} = \pi_{q_0 p_0}$ for $p \leq q$.

Lemma 3.2 Let $p \in \mathbb{P}$. Then

- $(F^p)^{-1}$ is a function
- If R^p is closed under set difference, then $F^p : D^p \longleftrightarrow R^p$
- $\pi^p =_{Df} F^p \upharpoonright M_p$ is injective into M.

Lemma 3.3 $\mathbb{P} \neq \emptyset$. Moreover, if $q, p \in \mathbb{P}$ and $\mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent, there is $r \leq p, q$. Moreover, for any countable $R \subset \mathbb{P}(M)$ we can choose r s.t. $R \subset R^r$.

Cor 3.4 p, q are compatible in $\mathbb{P} \longleftrightarrow$ $\longleftrightarrow \mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent

Cor 3.5 Let $p \in \mathbb{P}$. Let $R \subset \mathbb{P}(M)$ be countable. There is $q \leq p$ s.t. $R \subset R^q$.

Cor 3.6 Let $p \in \mathbb{P}$. Let $u \subset M$ be countable. There is $q \leq p$ s.t. $u \subset rng(\pi^q)$.

<u>Lemma 3.7</u> Let $p \in \mathbb{P}$, $u \subset M_p$, u finite. There is $q \leq p$ s.t. $q_0 = p_0$, $u \subset \mathrm{rng}(\pi q)$.

<u>Lemma 3.8</u> Let G be $\mathbb{P}$-generic. Then

(a) $\langle \langle Q_p, M_p, B^p \rangle \mid p \in G \rangle$, $\langle \pi_{pq} \mid q \leq p$ in $G \rangle$ is a directed system with limit:
$\langle Q, M, B^G \rangle$, $\langle \pi_p^G \mid p \in G \rangle$.
(Moreover: $\pi_p^G = \bigcup \{ \pi q \mid q \leq p, |q| = |p|, q \in G \}$)

(b) $p_0 \lhd \langle Q, M, B^G \rangle$ with $\pi_p = \pi_{p_0}^G$, $\langle Q, M, B^G \rangle$

(c) $\pi_p^G : \langle M_p, \bar{a} \rangle \prec \langle M, a \rangle$ for $\langle a, \bar{a} \rangle \in F^p$.

The proofs are exactly as in §3.

Imitating the proof of §3 Lemma 6.13 (Case 2), we get the reviability lemma:

<u>Lemma 4.1</u> Let $p \in \mathbb{P}$. Let $\bar{B}$ be s.t.

- $\bar{B} \subset \bigcup_{i<\gamma_0^p} \mathbb{B}_i^p$ and $\bar{B}_i = \bar{B} \cap \mathbb{B}_i^p$ is $\mathbb{B}_i^p$ - generic over Q_p for $i < \gamma_0^p$
- $Q_p[\bar{B}] = Q_p[B^p]$

Then $p' \in \mathbb{P}$ where $p'_0 = \langle Q_p, M_p, \bar{B}' \rangle$, $p'_1 = p_1$.

proof (sketch)

Let $\mathfrak{A} = \langle |\mathfrak{A}|, \dot{B}^{\mathfrak{A}} \rangle$ be a solid model of $\mathcal{L}(p)$. Then $Q[B] = Q[\dot{B}^{\mathfrak{A}}]$

Set $B = \bigcup_{i<\gamma_0^p} \pi_p(\bar{B}_i)$.

and $B \subset \bigcup_{i<\gamma_0} \mathbb{B}_i$ s.t. $B_i = B \cap \mathbb{B}_i$ is $\mathbb{B}_i$ - generic over Q (hence over V) for $i < \gamma_0$.

Set $\mathfrak{A}' = \langle |\mathfrak{A}|, B \rangle$.

<u>Claim</u> $\mathfrak{A}'$ models $\mathcal{L}(p')$.

We first show that $\mathfrak{A}'$ models $\mathcal{L}$. All axioms are trivial except (*). We verify (*).

Let $i < \gamma_0^p$ s.t. $\Vdash^{M_p}_{\mathbb{B}_i} \check{\gamma}_0^p$ is ω-cofinal.

Then, since $\pi_p^{\mathfrak{A}}{}''\bar{B}_i \subset B_{\pi(i)}$, $\pi_p^{\mathfrak{A}}$ extends to $\pi^*: M_p^{\bar{B}_i} \prec M^{B_{\pi(i)}}$. Let $\bar{C} \in M_p^{\bar{B}_i}$ s.t. $\bar{C} \subset \gamma_0^p = \sup(\bar{C})$, $\mathrm{otp}(\bar{C}) = \omega$. Let $\pi^*(\bar{C}) = C$. It follows easily that $\pi_p \upharpoonright Q_p : \langle Q_p, \bar{B}, \bar{C} \rangle \triangleleft_* \langle Q, B, C \rangle$.

But then $\pi_p : \langle Q_p, M_p, \bar{B} \rangle \lhd \langle Q, M, B \rangle$.

Now let $\xi < \beta$. By Lemma 2 there is $\pi_1 : \mathfrak{u} \lhd \langle Q, M, B^{\mathfrak{Ol}} \rangle$ s.t. $\mathfrak{u} \in H_{\omega_1}$, $\pi_1 \in \mathfrak{Ol}$ and $\mathrm{rng}(\pi_p) \cup \{\xi\} \subset \mathrm{rng}(\pi_1)$. Set $\pi_0 = \pi_1^{-1} \pi_p$. It follows that

$$\pi_0 : P_0 \lhd \mathfrak{u}, \ \pi_1 : \mathfrak{u} \lhd \langle Q, M, B^{\mathfrak{Ol}} \rangle$$

and $\pi_p = \pi_1 \circ \pi_0$.

If we set $B' = \bigcup_{i < \delta_0^p} \pi_0(\bar{B}_i) = \bigcup_{i < \delta_0} \pi_1^{-1}(B_i)$, we have $\pi_0 : P_0' \lhd \mathfrak{u}' \wedge \pi_1 : \mathfrak{u}' \lhd \langle Q, M, B \rangle$, where $\mathfrak{u}' = \langle Q_u, M_u, B' \rangle$. (Recall that $P_0' = \langle Q_p, M_p, \bar{B} \rangle$. But $\xi \in \mathrm{rng}(\pi_1)$, which proves (*). Hence $\mathfrak{Ol}' \models \mathcal{L}$. Since $\pi_p^{\mathfrak{Ol}} : P_0' \lhd \langle Q, M, B \rangle$ and $P_1' = P_1$, we see that $\mathfrak{Ol}$ models $\mathcal{L}(p')$.

QED(4.1)

It turns out, however, that the $Q_p[\bar{B}] = Q_p[B^p]$ is ~~too~~ restrictive for our purposes. We formulate another re-viability principle which evades the restriction:

Lemma 4.2 Let θ be big enough to verify the proudness of $\mathbb{B}_j$ over $\mathbb{B}_i$ for all $i < j < \gamma_0$ s.t. $i, j \in Ac$. Suppose moreover that $\theta > 2^{\beta}$. Let $N^* = \langle H_\theta, N, <, \mathbb{P}, \mathbb{B}, m \rangle$. Let p conform to N^*. Let $\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{N}, <, \bar{\mathbb{P}}, \bar{\mathbb{B}}, m \rangle$. (Hence $\bar{\mathbb{B}} = \mathbb{B}^p$.) Let $\bar{B}' \subset \bigcup_i \bar{\mathbb{B}}_i$ s.t. $\bar{B}_i' = \bar{B}' \cap \bar{\mathbb{B}}_i$ is $\bar{\mathbb{B}}_i$ - generic over $\bar{N}$ for $i < \gamma_0^p$. Then $p' \in \mathbb{P}$ where

$$p_0' = \langle Q_p, M_p, \bar{B}' \rangle \;, \quad p_1' = p_1$$

proof

Let $\mathfrak{A} = \langle |\mathfrak{A}|, \dot{B}^{\mathfrak{A}} \rangle$ be a solid model of $\mathcal{L}(p)$. Set: $B = \dot{B}^{\mathfrak{A}}$. Then $\pi = \pi_p^{\mathfrak{A}}$ extends to $\pi^* : \bar{N}^* \prec N^*$ s.t. $\pi \cup F^p \subset \pi^*$. We assume that $\mathfrak{A}$ (and hence π^*) lies in a generic extension of V. Note that B_i is $\mathbb{B}_i$ - generic over V for $i < \gamma_0$. Let $\langle \bar{\delta}_i \mid i < \omega \rangle$ be a monotone cofinal sequence in $\bar{\gamma}_0 = \gamma_0^p$ s.t. $\bar{\delta}_i \in Ac$ in $\bar{N}^*$ for $i < \omega$. Set $\delta_i = \pi(\bar{\delta}_i)$. Then $\langle \delta_i \mid i < \omega \rangle$ is monotone and cofinal in γ_0 and $\delta_i \in Ac$ for $i < \omega$.

But then $\mathbb{B}_{\delta_i}$ is proved over $\mathbb{B}_{\delta_h}$ for $h < i < \omega$, where B_{δ_i} is π^*-conforming. Thus we can successively form B'_{δ_i} s.t. B'_{δ_i} is $\mathbb{B}_{\delta_i}$-generic, $\pi^{*}{}''\bar{B}'_{\delta_i} \subset B'_{\delta_i}$, and $V[B'_{\delta_i}] = V[B_{\delta_i}]$.

Set: $B' = \bigcup_i B'_{\delta_i}$. Then $B'_\xi = B' \cap \mathbb{B}_\xi$ is $\mathbb{B}_\xi$-generic for $\xi < \delta_0$. Let $\bar{C} \in M_p[\bar{B}'_{i_0}]$ s.t. $\bar{C} \subset \bar{\delta}_0 = \sup \bar{C}$ and $\sup \bar{C} = \omega$. π^* extends uniquely to $\tilde{\pi} : \bar{N}^*[\bar{B}'_{i_0}] \prec N^*[B'_{\pi(i_0)}]$ s.t. $\tilde{\pi}(\bar{B}'_{i_0}) = B'_{\pi(i_0)}$, since $\pi^{*}{}''\bar{B}'_{i_0} \subset B'_{\pi(i_0)}$.

Let $\tilde{\pi}(\bar{C}) = C$. Then $\pi''\bar{C} = C$.

It follows easily that:

(1) $\pi \restriction \bar{Q}_p : \langle Q_p, \bar{B}', \bar{C} \rangle \lhd_* \langle Q, B', C \rangle$.

Hence:

(2) $\pi : \langle Q_p, M_p, \bar{B}' \rangle \lhd \langle Q, M, B' \rangle$,

where $p'_0 = \langle Q_p, M_p, \bar{B}' \rangle$.

Now set: $\mathfrak{U}' = \langle |\mathfrak{U}|, B' \rangle$. It follows as in Lemma 4.1 that $\mathfrak{U}' \models \mathcal{L}(p')$. QED (4.2)

By a virtual repetition of the proof of §3 Lemma 3.9 using 4.2, we get:

Lemma 4.3 $\mathbb{P}$ adds no reals

But then as before:

Lemma 4.4 Let $\theta \geq 2^{\beta}$ be regular. If $G \ni p$ is $\mathbb{P}$-generic and $B = B^G$, then $\langle H_\theta^{V[G]}, B \rangle$ models $\mathcal{L}(p)$.

Def Let $B \subset \bigcup_{i<\delta_0} \mathbb{B}_i$ s.t. $B_i = B \cap \mathbb{B}_i$ is $\mathbb{B}_i$-generic for $i < \delta_0$.

G^B = the set of $p \in \mathbb{P}$ s.t. $p_0 \triangleleft \langle Q, M, B \rangle$ and, letting $\pi = \pi_{p_0, \langle Q, M, B\rangle}$ we have:

$\pi : \langle M_p, \bar{a} \rangle \prec \langle M, a \rangle$ for all $\langle a, \bar{a} \rangle \in F^p$.

Exactly as in §3 Lemma 3.11 we have:

Lemma 4.5 Let G be $\mathbb{P}$-generic. Then $G = G^B$ where $B = B^G$.

Hence as in §3 Lemma 3.12:

Lemma 4.6 Let G be $\mathbb{P}$-generic. Then $\overline{\overline{\beta}} \leq \omega_1$ in $V[G]$

Def We define a homomorphism of $\bigvee_i \mathbb{B}_i$ into $BA(\mathbb{P})$ by:

$$k(b) = [\![\check{b} \in \mathring{B}^{\mathring{G}}]\!]_{\mathbb{P}}$$

Lemma 4.7 k is injective

prf.

Suppose not. Then $k(b) = 0$ for a $b \in \bigvee_i \mathbb{B}_i$ s.t. $b \neq 0$. But $\mathcal{L} + \underline{b} \in \mathring{B}$ is consistent. (To see this choose B s.t. $b \in B$ in the proof of Lemma 1.) Let $\mathfrak{A}$ be a solid model of $\mathcal{L} + \underline{b} \in \mathring{B}$. Then $b \in B^{\mathfrak{A}}$. By Lemma 2 there is $u \in \Gamma \wedge H_{\omega_1}$ s.t. $u \lhd \langle Q, M, B \rangle$ and $\pi(\bar{b}) = b$ for some $\bar{b}$ where $\pi = \pi_{u, \langle Q, M, B \rangle}$. Define p by:

$p_0 = u$, $p_1 = \{\langle b, \bar{b} \rangle\}$. Then $\mathfrak{A}$ models $\mathcal{L}(p)$. Hence $p \in \mathbb{P}$. But if $G \ni p$ is $\mathbb{P}$-generic, then $b \in B^G$, hence

$0 \neq [p] \subset [\![\check{b} \in \mathring{B}^{\mathring{G}}]\!] = \sigma(b)$.

Contr!

QED (4.7)

We now show that $BA(\mathbb{P})$ is, in fact, generated by $\{k(b) \mid b \in \bigcup_{i<\gamma_0} \mathbb{B}_i\}$.

As a preliminary we prove:

<u>Lemma 4.8</u> Let G be $\mathbb{P}$-generic, $B = B^G$. Then $G \in V[B]$.

proof.

(1) Let $u \in H_{\omega_1} \cap \Gamma$, $\pi : u \lhd \langle Q, M, B \rangle$.

Then $u \in V[B]$.

proof.

Let ξ be least s.t. $\Vdash_{\mathbb{B}_\xi} \mathrm{cf}(\check{\gamma}_0) = \omega$. Then $\xi \in \mathrm{rng}(\pi)$ since $\pi : M_u \prec M$, $\pi(\mathbb{B}^u) = \mathbb{B}$.

Let $\pi(\bar{\xi}) = \xi$. Let $\bar{C} \in M_u$, $\bar{C} \subset \gamma_0^u = \sup(\bar{C})$, $\mathrm{otp}(\bar{C}) = \omega$. π extends uniquely to $\pi^* : M_u[B^u_{\bar{\xi}}] \prec M[B_\xi]$ s.t. $\pi^*(B^u_{\bar{\xi}}) = B_\xi$, since $\pi'' B^u_{\bar{\xi}} \subset B_\xi$. Let $C = \pi^*(\bar{C})$.

Then $C = \pi'' \bar{C}$ and

$\pi \restriction Q_u : \langle Q_u, B^u, \bar{C} \rangle \lhd_* \langle Q, B, C \rangle$,

where $\langle Q, B, C \rangle \in V[B]$. Hence

$\mathrm{rng}(\pi \restriction Q_u) =$ the smallest $X \prec \langle Q, B, C \rangle$ s.t. $\omega_1^{Q_u} \cup C \subset X$.

Hence $\pi \restriction Q_u \in V[B]$. But then

$\pi \in V[B]$, since, letting $\pi : M_n < \tilde{M}$ cofinally we have:

$\langle \tilde{M}, \pi \rangle =$ the liftup of $\langle M_n, \pi \upharpoonright a_n \rangle$.

QED (1)

But then by Lemma 4.5 we have

$p \in G \longleftrightarrow V[B] \models p \in G^B$.

Hence $G \in V[B]$. QED (4.8)

As a corollary of the proof:

Lemma 4.9 $BA(\mathbb{P})$ is generated by $\{h(b) \mid b \in \bigcup_{i<\gamma_0} \mathbb{B}_i\}$.

proof.

Let $\mathring{B} \in V^{\mathbb{P}}$ s.t. $\Vdash_{\mathbb{P}} \mathring{B} = B^{\mathring{G}}$.

Standard methods show:

$[\![\varphi(\check{x}_1, \ldots, \check{x}_n, \mathring{B}) \check{V}[\mathring{B}]]\!]_{\mathbb{P}} \in \mathbb{B}^*$,

for all $x_1, \ldots, x_n \in V$ + all ZF-Formulae φ, where $\mathbb{B}^*$ is the complete subalgebra of $BA(\mathbb{P})$ generated by $\{h(b) \mid b \in \bigcup_{i<\gamma_0} \mathbb{B}_i\}$.

By the proof of Lemma 4.8, however:

$\Vdash (\check{p} \in \mathring{G} \leftrightarrow (\check{p} \in G^{\mathring{B}})_{V[\mathring{B}]})$; hence

$[p] = [\![\check{p} \in \mathring{G}]\!] = [\![(\check{p} \in G^{\mathring{B}})_{V[\mathring{B}]}]\!] \in \mathbb{B}^*$, where

$BA(\mathbb{P})$ is generated by $\{[p] \mid p \in \mathbb{P}\}$.

QED (4.9)

We now estimate the size of $BA(\mathbb{P})$:

<u>Lemma 4.10</u> $\overline{\overline{BA(\mathbb{P})}} \leq \beta_\lambda^+$.

proof:

If $\beta \neq \beta_\lambda$, then $\beta_\lambda = \beta^+ = 2^\beta$, since GCH then holds below κ. But $\overline{\overline{\mathbb{P}}} \leq 2^\beta$ and hence the result follows. Now let $\beta = \beta_\lambda$. Then $cf(\beta) = \omega_1$. It suffices to show:

<u>Sublemma 4.10.1</u> $BA(\mathbb{P})$ has a dense subset of size β.

proof (sketch)

The proof is virtually identical to that of § 3 Sublemma 3.15.1. Set $H = H_{(2^\beta)^+}$. Then $\langle H[G], B\rangle$ models $\mathcal{L}$ whenever G is $\mathbb{P}$-generic and $B = \mathbb{B}^G$. We can give every $\mathcal{L}$-sentence φ an interpretation $[\![\varphi]\!] \in BA(\mathbb{P})$ in $H^{\mathbb{P}}$, interpreting $\mathring{B}$ by $\ddot{B}$ where $\ddot{B} \in H^{\mathbb{P}}$, $\Vdash^H_{\mathbb{P}} \ddot{B} = B^{\mathring{G}}$, $\mathring{G}$ being the canonical generic name, $\underline{\kappa}$ is interpreted by $\check{x}$. It then suffices to show:

Claim For each $p \in \mathbb{P}$ there is an $\mathcal{L}$-statement $\varphi \in M$ s.t. $[\![\varphi]\!] \neq 0$ and $[\![\varphi]\!] \subset [p]$ in $BA(\mathbb{P})$.

The proof is an almost literal repetition of the corresponding step in the proof of §3 Sublemma 3.15.1. We leave the details to the reader. QED (4.10)

Lemma 4.11 Let G be $\mathbb{P}$-generic. Then $cf(\tau) = \omega$ in $V[G]$ whenever $\tau \in [\gamma_\lambda, \beta]$ is regular in V.

proof.

Let $\tau = \pi_p^G(\bar{\tau})$. Then $\sup \pi_p^G{}''\bar{\tau} = \tau$, since $\pi_p^G : M_p \to \tilde{M}$ is γ_p-cofinal, where $\pi_p^G(\gamma_p) = \gamma = \gamma_\lambda$. QED (4.11)

Lemma 4.12 Let G be $\mathbb{P}$-generic. Then

(a) $\overline{\overline{\beta}}_\lambda = \omega_1 = cf(\beta_\lambda)$ in $V[G]$

(b) $\beta_\lambda^+ = \omega_2^{V[G]}$

proof.

$\overline{\overline{\beta}} \leq \omega_1$ since $\beta \subset \bigcup_{p \in G} rng(\pi_p^G)$,

where $\pi_p^G = \pi_{p_0, \langle Q, M, B^G \rangle}$ depends only on $p_0 \in H_{\omega_1}$.

If $\beta = \beta_\lambda$, then $cf(\beta) = \omega_1$ in V, hence in $V[G]$. Moreover, β^+ remains a cardinal in $V[G]$, since $BA(\mathbb{P})$ has a dense subset of size β.

Now let $\beta_\lambda = \beta^+$. Then $cf(\beta) = \omega$ in $V[G]$, since either β is regular or $cf(\beta) = \omega$ in V. We also know $\beta^+ = 2^\beta$, since GCH holds below κ. By [LF] §4 Lemma 4.1 (Fact 11 of §3 in this paper) we conclude that $\overline{\overline{\beta_\lambda}} = \omega_1$ in $V[G]$. But β_λ^+ remains a cardinal in $V[G]$ since $\overline{\overline{\mathbb{P}}} \leq \beta^+ = \beta_\lambda$. Hence $cf(\beta_\lambda) = \omega_1$ in $V[G]$, since otherwise another application of §3 Fact 11 would give: $\overline{\overline{\beta_\lambda^+}} \leq \omega_1$ in $V[G]$.

QED (Lemma 4.12)

We choose σ, $\mathbb{B}_\lambda$ (recall $\lambda = \delta_0$) s.t.

$$\begin{tikzcd} \sigma : BA(\mathbb{P}) \arrow[r, leftrightarrow, "\sim"] & \mathbb{B}_\lambda \\ \bigcup_i \mathbb{B}_i \arrow[u, "k"] \arrow[ur, hook] & \end{tikzcd}$$

and $\mathbb{B}_\lambda \subset H_{\beta^+}$. $\mathbb{B}_\lambda$ is the completion of $\bigcup_i \mathbb{B}_i$ which we sought. However, we must still verify many of its properties.

Now let $i_0 < \delta_0$ + let B be $\mathbb{B}_{i_0}$ - generic.
Set: $\mathbb{B}^+ = \mathbb{B}^\frown \langle \mathbb{B}_\lambda \rangle = \langle \mathbb{B}_i \mid i \leq \lambda \rangle$.
We shall have to prove that the properties we have shown to hold of $\mathbb{B}^+$ also hold of $\mathbb{B}^+/B = \langle \mathbb{B}_{i_0+h}/B \mid h \leq \delta_0 - i_0 \rangle$ in $V[B]$. We know by the induction hypothesis that the salient properties of $\mathbb{B}$ hold of $\mathbb{B}/B = \langle \mathbb{B}_{i_0+h}/B \mid h < \delta_0 - i_0 \rangle$ in

$V[\mathbb{B}]$. But in $V[\mathbb{B}]$ we can use the language $\mathcal{L}_B$ over N^B to construct conditions $\mathbb{P}_B$ exactly as $\mathbb{P}$ was constructed from $\mathcal{L}$. If we have established properties of $\mathbb{B}, \mathbb{P}$, we can repeat our proofs to verify the same properties of $\mathbb{B}/B$, $\mathbb{P}_B$ in $V[B]$. Now let

$k_B : \left(\bigcup_{h<\gamma_0 - i_0} \mathbb{B}_{i_0+h}/B\right) \to BA(\mathbb{P}_B)$ be defined in $V[B]$ as k was defined in V. We then define σ^B, $\mathbb{B}^B_{\lambda-i_0}$ in $V[B]$ like σ, $\mathbb{B}_\lambda$ in V. We have:

$$\begin{tikzcd}
BA(\mathbb{P}_B) \arrow[r, leftrightarrow, "\sim", "\sigma^B"'] & \mathbb{B}^B_{\lambda-i_0} \\
\bigcup_i \mathbb{B}_i/B \arrow[u, "k_B"] \arrow[ur, hook] &
\end{tikzcd}$$

It suffices to prove:

<u>Lemma 5.1</u> There is $\mu : BA(\mathbb{P})/B \overset{\sim}{\longleftrightarrow} \mathbb{B}^B_{\lambda-i_0}$ s.t. $\mu(b/B) = b/B$ for $b \in \bigcup_i \mathbb{B}_i$.

proof

This is equivalent to:

<u>Claim</u> There is $\delta : BA(\mathbb{P})/k(B) \xleftrightarrow{\sim} BA(\mathbb{P}_B)$ s.t. $\delta(k(b)/k(B)) = k_B(b/B)$ for $b \in \bigcup_i \mathbb{B}_i$, where $k(B) =_{df} k''\mathbb{B}$.

We can then set: $\mu = \sigma^B \circ \delta$.

$$\begin{tikzcd} BA(\mathbb{P})/k(B) \arrow[r, "\delta"] & BA(\mathbb{P}_B) \arrow[r, "\sigma^B"] & \mathbb{B}^B_{\lambda - i_0} \\ & \bigcup_i \mathbb{B}_i/B \arrow[ul, bend left] \arrow[u, "k_B"] \arrow[ur] & \end{tikzcd}$$

The proof stretches over several lemmas and closely follows that of §3 Cor 6.9.1. $\mathbb{P}_B$ plays the same role as the $\mathbb{P}_B$ of that proof. The role of $\mathbb{P}'$ is played by:

$$\mathbb{P}' =_{df} \{p \in \mathbb{P} \mid i_0 \in \operatorname{rng}(\pi^p)\},$$

Note that $\mathbb{P}'$ is dense in $\mathbb{P}$.

As before, we assign to each $p \in \mathbb{P}'$ a potential element p^* of $\mathbb{P}_B$:

$p_0^* = \langle Q_p^{B^p}, M_p^{B^p}, B^p/B^p_{i_p} \rangle$, $p_1^* = p_1$,

where $\pi^p(i_p) = i_0$ and

$B^p/B = \{b/B \mid b \in B^p\}$ whenever B is $\mathbb{B}^p_{i_0}$ - generic over M_p.

Lemma 5.1.1 Let B be $\mathbb{B}_{i_0}$ - generic. Let $p, q \in \mathbb{P}$. $[p]/k(B)$, $[q]/k(B)$ are compatible in $\mathbb{B}A(\mathbb{P})/k(B)$ iff p^*, q^* are compatible in $\mathbb{P}_B$.

proof.

($\Leftarrow$) Suppose not. Then $[p] \wedge [q]/k(B) = 0$. Hence $[p] \wedge [q] \wedge k(b) = 0$ for a $b \in B$. Let $\mathfrak{A}^* = \langle |\mathfrak{A}^*|, B^* \rangle$ be a solid model of $\mathcal{L}_B(p^*) \cup \mathcal{L}_B(q^*)$. Then $\mathfrak{A} = \langle |\mathfrak{A}^*|, B' \rangle$ where a solid model of $\mathcal{L}(p) \cup \mathcal{L}(q)$ where $B' = B * B^* = \{ b \in \bigcup_i B_i \mid b/B \in B' \}$, by Lemma 0. Hence there is $r \leq p, q$ s.t. $\mathfrak{A} \models \mathcal{L}'(r)$ and $b \in rng(\pi^r)$. Let $\pi^r(\bar{b}) = b$. Then $\bar{b} \in B^r$. Hence $r \Vdash \check{b} \in B^{\dot{G}}$. Hence $[r] \subset [\![\check{b} \in B^{\dot{G}}]\!] = k(b)$, where $[r] \wedge k(b) = 0$. Contr!

($\Rightarrow$) Suppose not. Then $\mathcal{L}_B(p^*) \cup \mathcal{L}_B(q^*)$ is inconsistent. Hence there is $b \in B$ s.t.

(1) $b \Vdash_{\mathbb{B}_{i_0}} (\mathcal{L}_{\dot{B}}(\check{p}^*) \vee \mathcal{L}_{\dot{B}}(\check{q}^*)$ is inconsistent).

Let $\tilde{G}$ be $\mathbb{B}A(\mathbb{P})/k(B)$ - generic s.t. $[p]/k(B)$, $[q]/k(B) \in \tilde{G}$. Set

$$G = \{ p \in \mathbb{P} \mid [p]/k(B) \in \tilde{G} \}$$

Then G is $\mathbb{P}$ - generic with $p, q \in G$ and $B = B_{i_0}^G$. Let $r \in G$ s.t. $r \leq p, q$ and $b \in rng(\pi^r)$. Let $\pi^r(\bar{b}) = b$

Then $\bar{b} \in B^{r}$. Let $\mathfrak{A}' = \langle |\mathfrak{A}'|, B' \rangle$ be a solid model of $\mathcal{L}(r)$. Then $\mathfrak{A} = \langle |\mathfrak{A}'|, B'/B \rangle$ is a solid model of $\mathcal{L}_B(p^*) \cup \mathcal{L}_B(q^*)$, when $b \in B'$ and B' is $\mathbb{B}_{i_0}$ - generic, contradicting (1). Contr!
QED (5.1.1)

As a corollary:

<u>Cor 5.1.2</u> $[p]/k(B) \neq 0 \leftrightarrow p^* \in \mathbb{P}_B$
for $p \in \mathbb{P}'$ and $\mathbb{B}_{i_0}$ - generic B.

<u>Lemma 5.1.3</u> $\{[p^*] \mid [p]/k(B) \neq 0\}$
is dense in $BA(\mathbb{P}_B)$.

proof. (We follow the proof of §3 Lemma 6.4)
(We follow the proof of Lemma 0,
By the same argument as before, using Lemma 0,
$\{p^* \mid [p]/k(B) \neq 0\}$ is the same as the set $\hat{\mathbb{P}}$ of $p \in \mathbb{P}_B$ s.t. $i_0 \in rng(\pi^p)$ and $F^p \in V$. We show:

<u>Claim</u> Let $q \in \mathbb{P}_B$. There is $p \in \hat{\mathbb{P}}$ s.t. $[p] \subset [q]$.

Let $A = \langle a_i \mid i < \omega \rangle \in V[B]$ enumerate R^q as before. Let $D \subset \beta$ s.t. A is $\langle M^B, D \rangle$ - definable, where $D \in M[B]$. Let $D = \mathring{D}^B$ and set, as before,
$E = \{\langle \nu, b \rangle \mid b \in \mathbb{B}_{i_0} \wedge \nu < \beta \wedge b \Vdash_{\mathbb{B}_{i_0}} \check{\nu} \in \mathring{D}\}$

Then A is $\langle M^B, E\rangle$-definable as before.
In $V[B]$ define:
$N^* = \langle H_\theta, N^B, M^B, Q^B, <, B, E, A, \dots\rangle$, where $\theta > (2^B)^+$
is a cardinal and $<$ well orders N^*. Let
$p \leq q$ s.t. p conforms to N^*. Again set:
$\bar{N} \cong \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{N}, \bar{M}, \bar{Q}, <, \bar{B}, \bar{E}, \bar{A}, \dots\rangle$,
Then $\bar{M} = M_p$, $\bar{Q} = Q_p$, $\bar{B} = B^p$ and $\bar{A}$ is $\langle \bar{M}, \bar{E}\rangle$-
-definable by the same definition.
Form p', p'' by:
$p'_0 = p_0$, $p'_1 = \{\langle a, \bar{a}\rangle \in F^p \mid a \in R^q\}$
$p''_0 = p_0$, $p''_1 = \{\langle E, \bar{E}\rangle\}$ where $\langle E, \bar{E}\rangle \in F^p$.
Then $p' \leq q$ in $\mathbb{P}_B$ and $p'' \in \hat{\mathbb{P}}_B$. We show:

<u>Claim</u> $[p''] \subset [p']$ in $BA(\mathbb{P}_B)$.

prf. Let $G \ni p''$ be $\mathbb{P}_B$-generic. It suffices to show:

<u>Claim</u> $p' \in G$.

Since $p'_0 = p''_0$ we have:
$\pi : p'_0 \lhd \langle Q^B, M^B, B^G\rangle$ where $\pi = \pi^G_{p''}$.
We need only show:
$\pi : \langle M_{p'}, \bar{a}\rangle \prec \langle M^B, a\rangle$ for $\langle a, \bar{a}\rangle \in F^{p'}$.
as before, $a = A(i)$ is $\langle M^B, E\rangle$-definable and
$\bar{a} = \bar{A}(i)$ is $\langle M_{p'}, \bar{E}\rangle$-definable by the same definition, where $\pi : \langle M_p, \bar{E}\rangle \prec \langle M^B, E\rangle$.

QED (5.1.3)

Lemma 5.1.4 $[p]/k(B) \subset [q]/k(B)$ in $BA(\mathbb{P})/B$ iff $[p^*] \subset [q^*]$ in $BA(\mathbb{P}_B)$ for $p, q \in \mathbb{P}'$.

proof.

Set $A = \{[p]/k(B) \mid p \in \mathbb{P}'\} \setminus \{0\}$

$A' = \{[p^*] \mid p \in A\} = \{[p^*] \mid p \in \mathbb{P}'\} \setminus \{0\}$

Since A is dense in $BA(\mathbb{P})$ we have:

$a \subset b \longleftrightarrow \bigwedge c \in A\ (c \cap b = 0 \rightarrow c \cap a = 0)$

for $a, b \in BA(\mathbb{P})/k(B)$, in particular, this holds for $a, b \in \{[p]/k(B) \mid p \in \mathbb{P}'\}$.

Similarly:

$a \subset b \longleftrightarrow \bigwedge c \in A'\ (c \cap a = 0 \rightarrow c \cap b = 0)$

for $a, b \in BA(\mathbb{P}_A)$ (in particular for $a, b \in \{[p^*] \mid p \in \mathbb{P}'\}$).

But:

$[p]/k(B) \cap [q]/k(B) = 0 \longleftrightarrow$

$\longleftrightarrow [p^*] \cap [q^*] = 0$

for $p, q \in \mathbb{P}'$ by Lemma 5.1.1.

The conclusion is immediate.

QED (5.14)

Cor 5.1.5 There is a unique

$\delta : BA(\mathbb{P})/k(B) \overset{\sim}{\longleftrightarrow} BA(\mathbb{P}_B)$ s.t.

$\delta([p]/k(B)) = [p^*]$ for $p \in \mathbb{P}'$.

We complete the proof of 5,1 by showing:

<u>Lemma 5,1,6</u> $\delta(k(b)/k(B)) = k_B(b/B)$ for $b \in \bigcup_i B_i$.

proof.

It suffices to show that if F is a $BA(\mathbb{P})/k(B)$ - generic filter and $F^* = \delta''F$, then
$k(b)/k(B) \in F \longleftrightarrow k_B(b/B) \in F^*$

Set: $G = \{p \in \mathbb{P} \mid [p]/k(B) \in F\}$. Then G is $\mathbb{P}$-generic. Hence there is $p \in \mathbb{P}' \cap G$ s.t. $b \in rng(\pi^p)$. Let $\pi^p(\bar{b}) = b$. Then $\bar{b} \in B^p \longleftrightarrow b \in B^G \longleftrightarrow$
$\longleftrightarrow k(b)/k(B) = [\![\check{b} \in B^{\dot{G}}]\!]/k(B) \in F$, since
$F = \{a/k(B) \mid a \in BA(\mathbb{P}) \wedge a \cap G \neq 0\}$.
But $\bar{b} \in B^p \longleftrightarrow \bar{b}/B^p_{i_p} \in B^p/B^p_{i_p} = B^{p*}$.
Set $G^* = \{p \in \mathbb{P}_B \mid [p] \in F^*\}$. Then G^* is $\mathbb{P}_B$ - generic. But $\pi^{G^*}_{p*}(B^p_{i_p}) = B$, since $\pi^{G^*}_{p*} : M_p^{B^p_{i_p}} \prec M^B$. Hence
$\pi^{G^*}_{p*}(\bar{b}/B^p_{i_p}) = b/B$. If
$k(b)/k(B) \in F$, then $\bar{b} \in B^p$ and
$\bar{b}/B^p_{i_p} \in B^{p*}$. Hence $b/B = \pi^{G^*}_{p*}(\bar{b}/B^p_{i_p}) \in$
$\in B^{G^*}$. Hence $k_B(b/B) =$
$= [\![\check{b}/B \in B^{\dot{G}}]\!] \in F^*$.
Similarly, if $k(b)/k(B) \notin F$, then
$k_B(b/B) \notin F^*$. QED (Lemma 5,1)

Lemma 6 $\mathbb{B}_\lambda$ is subcomplete.

proof.
There is $i_0 < \lambda$ s.t. $\Vdash_{\mathbb{B}_{i_0}} cf(\check{\lambda}) = \omega$. Since
$$\mathbb{B}_\lambda \simeq \mathbb{B}_{i_0} * \mathring{\mathbb{B}} \text{ where } \Vdash_{\mathbb{B}_{i_0}} \mathring{\mathbb{B}} = \check{\mathbb{B}}_\lambda / \dot{B}$$
($\dot{B}$ being the canonical generic name), and $\mathbb{B}_{i_0}$ is subcomplete, it suffices to show that $\mathbb{B}_\lambda \setminus B$ is subcomplete in $V[B]$ whenever B is $\mathbb{B}_{i_0}$-generic. But $\mathbb{B}_\lambda \setminus B \simeq BA(\mathbb{P}_B)$, so it suffices to show that $\mathbb{P}_B$ is subcomplete in $V[B]$. But then it suffices to prove the subcompleteness of $\mathbb{P}$ under the assumption $i_0 = 0$, since the same proof would apply to $\mathbb{P}_B$ in $V[B]$.

Let $W = L_\tau^A$ where $2^\beta < \theta < \tau$, τ is regular, $H_\theta \subset W$, and W verifies the prondness of $\mathbb{B}_\xi$ over $\mathbb{B}_\zeta$ for $\zeta < \xi < \delta_0$. Let $\sigma : \overline{W} \prec W$ s.t. $\overline{W}$ is countable and full. Let $\sigma(\bar{\theta}, \overline{\mathbb{P}}, \overline{\mathbb{Q}}, \overline{M}, \overline{N}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{P}, \mathbb{Q}, M, N, s, \lambda_i$ $(i = 1, \ldots, n)$ where $\mathbb{P} \in H_{\lambda_i}$ (hence $N \in H_{\lambda_i}$) and $\lambda_i < \theta$ for $i = 1, \ldots, n$. Let $\overline{G}$ be $\overline{\mathbb{P}}$-generic over $\overline{W}$.

Claim There is $q \in \mathbb{P}$ s.t. whenever $G \ni q$ is $\mathbb{P}$-generic, there is $\sigma_0 \in V[G]$ with:

(a) $\sigma_0 : \overline{W} \prec W$

(b) $\sigma_0(\overline{\theta}, \overline{\mathbb{P}}, \overline{\mathbb{Q}}, \overline{M}, \overline{N}, \overline{s}, \overline{\lambda}_i) = \theta, \mathbb{P}, \mathbb{Q}, M, N, s, \lambda_i$ $(i = 1, \ldots, n)$

(c) $\sup \sigma_0{}''\overline{\lambda}_i = \sup \sigma''\overline{\lambda}_i$ $(i = 0, \ldots, n)$

where $\overline{\lambda}_0 = On \cap \overline{W}$

(d) $\sigma_0{}''\overline{G} \subset G$.

Let $\langle \overline{c}_i \mid i < \omega \rangle \in \overline{M}$ ~~be~~ monotone and cofinal in $\overline{\delta}_0 = \sigma^{-1}(\delta_0)$. Set $c_i = \sigma(\overline{c}_i)$. Then $\langle c_i \mid i < \omega \rangle$ is monotone and cofinal in δ_0. Let $\sigma(\overline{\mathbb{B}}) = \mathbb{B}$, where $\mathbb{B} = \langle \mathbb{B}_i \mid i < \delta_0 \rangle$. Then $\sigma(\overline{\mathbb{B}}_{\overline{c}_n}) = \mathbb{B}_{c_n}$ for $n < \omega$. Let $\tilde{\mathbb{B}} \supset \bigcup_{i<n} \mathbb{B}_i$ be the inverse limit of $\langle \mathbb{B}_{c_n} \mid n < \omega \rangle$. Modifying slightly the proof of the ω-case of the iteration theorem for subcompleteness, we get:

Fact Let $\overline{B} \subset \bigcup_n \overline{\mathbb{B}}_{\overline{c}_n}$ s.t. $\overline{B}_n = \overline{B} \cap \overline{\mathbb{B}}_{\overline{c}_n}$ is $\overline{\mathbb{B}}_{c_n}$-generic over $\overline{W}$ for $n < \omega$. There is a $b \in \tilde{\mathbb{B}} \setminus \{0\}$ s.t. whenever $\tilde{B} \ni b$ is $\tilde{\mathbb{B}}$-generic, then there is $\tilde{\sigma} \in V[\tilde{B}]$ s.t.

(a) $\tilde{\sigma} : \overline{W} \prec W$

(b) $\tilde{\sigma}(\overline{\theta}, \ldots, \overline{\lambda}_i) = \theta, \ldots, \lambda_i$ $(i = 1, \ldots, n)$

(c) $\sup \tilde{\sigma}''\overline{\lambda}_i = \sup \sigma''\overline{\lambda}_i$ $(i = 0, \ldots, n)$

(d) $\tilde{\sigma}''\overline{B}_n \subset \tilde{B}_n = \tilde{B} \cap \mathbb{B}_{c_n}$ for $n < \omega$.

Making use of this fact we get:

<u>Sublemma 6.1</u> Let δ be least s.t. $L_\delta(W)$ is admissible. The following language $\mathcal{L}^*$ on $L_\delta(W)$ is consistent:

<u>Predicate</u>: $\in$, <u>Constants</u> $\underline{x}$ $(x \in L_\delta(W))$, $\dot{\sigma}$, $\dot{B}$

<u>Axioms</u>: ZFC^-, $\bigwedge v\,(v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$,

$\dot{\sigma}: \bar{W} \prec W$, $\dot{\sigma} \upharpoonright \bar{M}: \langle \underline{\bar{Q}}, \underline{\bar{M}}, \underline{\bar{B}} \rangle \lhd \langle \underline{Q}, \underline{M}, \dot{B} \rangle$,

$\dot{\sigma}(\underline{\bar{\theta}}, \underline{\bar{\mathbb{P}}}, \underline{\bar{Q}}, \underline{\bar{M}}, \underline{\bar{N}}, \underline{\bar{\lambda}}, \underline{\bar{\lambda}_i}) = \underline{\theta}, \underline{\mathbb{P}}, \underline{Q}, \underline{M}, \underline{\lambda}, \lambda_i$ $(i = 1, \ldots, n)$

$\sup \dot{\sigma}\text{"}\underline{\bar{\lambda}_i} = \underline{\sup \sigma\text{"}\bar{\lambda}_i}$ $(i = 0, \ldots, n)$

[<u>Note</u> These axioms do not say that $\omega_1 = \underline{\omega_1}$.]

proof (sketch) We follow the proof of §3 Lemma 4.1.

We first let $\mathcal{L}_0$ be like $\mathcal{L}^*$ except that the axiom $\sup \dot{\sigma}\text{"}\bar{\lambda}_i = \underline{\sup \sigma\text{"}\bar{\lambda}_i}$ is replaced by: $\sup \dot{\sigma}\text{"}\bar{\lambda}_i = \lambda_i$ $(i = 0, \ldots, n)$, where $\lambda_0 = \tau$, Let $\bar{H} = H^{\bar{W}}_{\bar{\gamma}}$ + Let $\tilde{\sigma}$, $\tilde{B}$ be as in the above Fact. ~~Let $\tilde{\sigma} \upharpoonright \bar{H} \prec H'$ cofinally~~, where $\sigma(\bar{\gamma}) = \gamma$. and let $\langle W', \sigma' \rangle$ be the liftup of $\langle \bar{W}, \tilde{\sigma} \upharpoonright \bar{H} \rangle$. Note that

$\tilde{\sigma} \upharpoonright \bar{Q}: \langle \bar{Q}, \bar{B}, \bar{C} \rangle \lhd_* \langle Q, B, C \rangle$,

where $\bar{C} = \{\bar{c}_i \mid i < \omega\}$, $C = \{c_i \mid i < \omega\}$,

Hence $\sigma' \upharpoonright \bar{M}: \langle \bar{Q}, \bar{M}, \bar{B} \rangle \lhd \langle Q, M', B \rangle$, where $\sigma'(\bar{M}) = M'$. There is a canonical $k: W' \prec W$ s.t. $k\sigma' = \sigma$ and $k \upharpoonright H' = \mathrm{id}$. Let $\mathcal{L}_0'$ be defined on $L_\delta(W')$ the way $\mathcal{L}_0$ was defined on $L_\delta(W)$, where

δ' is least s.t. $L_{\delta'}(W')$ is admissible. (More precisely, $\mathcal{L}_0$ is defined in W and parameters $\vec{p} \in W$; $\mathcal{L}_0'$ has the same def. in W' and $k^{-1}(\vec{p})$.) Then $\langle H^{V[\tilde{B}]}_{\omega_2}, \sigma' \rangle$ models $\mathcal{L}_0'$. Hence $\mathcal{L}_0'$ is consistent. Assume w.l.o.g. $\lambda_0 > \lambda_1 > \dots > \lambda_n$ and let:

$\sigma \restriction H^{\bar{W}}_{\lambda_n} : H^{\bar{W}}_{\lambda_n} < H''$ cofinally.

Let $\sigma'' : \bar{W} < W''$ be the liftup of $\bar{W}$ by $\sigma \restriction H^{\bar{W}}_{\lambda_n}$. Let $k'' : W'' < W$ s.t. $k'' \restriction H'' = \mathrm{id}$ and $k''\sigma'' = \sigma$. Then there is $k' : W' < W''$ s.t. $k'\sigma' = \sigma''$ and $k' \restriction H' = \mathrm{id}$. Moreover $k = k''k'$. Let δ'' be least s.t. $L_{\delta''}(W'')$ is admissible and let $\mathcal{L}_0''$ be defined over $L_{\delta''}(W'')$ as $\mathcal{L}_0$ was defined over $L_\delta(W)$. The statement that $\mathcal{L}_0'$ is consistent is $\Pi_1(L_{\delta'}(W'))$ in parameters $\vec{p}' = k^{-1}(\vec{p})$ and W'. Hence the statement that $\mathcal{L}_0''$ is consistent is $\Pi_1(L_{\delta''}(W''))$ in W'' and $\vec{p}'' = k''^{-1}(\vec{p}) = k'(\vec{p}')$. Hence $\mathcal{L}_0''$ is consistent by the fullness of $\mathcal{L}_0''$. Let $\Omega = \bar{W}$. Note that $k''(M) = M$. Let $\langle |\Omega|, \sigma'', B'' \rangle$ be a solid model of $\mathcal{L}_0''$ lying in some generic extension $V[G]$ of $V[\tilde{B}]$.

Then $\langle H_\mu^{V[G]}, k''\circ\sigma'', B''\rangle$ models $\mathcal{L}^*$,

QED (Lemma 6.1)

Now set: $N^* = \langle H_\Omega, W, Q, M, N, \mathbb{B}, \mathbb{P}, \sigma, \lambda_1, \ldots, \lambda_n, \ldots\rangle$, where $\Omega > \bar{\tau}$ is a cardinal. Let p conform to N^*. Set $\bar{N}^* = \bar{N}^*(p, N^*) = \langle H', W', \ldots, \mathbb{B}', \mathbb{P}', \ldots\rangle$. Let $\mathcal{L}'$ be defined in $\bar{N}^*$ like $\mathcal{L}$ is in N^*. Let $\mathfrak{A} \in H_{\omega_1}$ be a solid model of $\mathcal{L}'$.

Set: $\sigma^* = \dot{\sigma}^{\mathfrak{A}}$, $B' = \dot{B}^{\mathfrak{A}}$. Define $q \in \mathbb{P}$ by:

$$q_0 = \langle Q_p, M_p, B'\rangle, \quad q_1 = p_1.$$

(Then $q \in \mathbb{P}$ by Lemma 4.2) We show that this q satisfies the Claim.

Let $G \ni q$ be $\mathbb{P}$-generic. Then

$$\langle \bar{Q}, \bar{M}, \bar{B}\rangle \lhd q_0 \text{ with } \pi_{\langle \bar{Q}, \bar{M}, \bar{B}\rangle, q_0} = \sigma^* \restriction \bar{M}$$

(since $Q_p = Q'$, $M_p = M'$). But $q_0 \lhd \langle Q, M, B\rangle$ where $B = \mathbb{B}^G$. Moreover $\pi_{q_0, \langle Q, M, B\rangle} = \pi_q^G$.

Hence $\langle \bar{Q}, \bar{M}, \bar{B}\rangle \lhd \langle Q, M, B\rangle$ with:

$$\pi_{\langle \bar{Q}, \bar{M}, \bar{B}\rangle, \langle Q, M, B\rangle} = \pi_q^G \circ \sigma^* \restriction \bar{M},$$

Let π^* be the unique $\pi^* \supset \pi_q^G \cup F_q$ with $\pi^*: \bar{N}^* \prec N^*$. Set: $\sigma_0 = \pi^* \sigma^* \restriction \bar{W}$.

Then $\pi_{\langle \bar{Q}, \bar{M}, \bar{B}\rangle, \langle Q, M, B\rangle} = \sigma_0 \restriction \bar{M}$,

This σ_0 witnesses the Claim.

(a) – (c) are straightforward,

We verify (d): $\sigma_0 '' \bar{G} \subset G$.

Let $\bar{s} \in \bar{G}$, $s = \sigma_0(\bar{s})$. Then $\bar{s}_0 = s_0$ since $\sigma_0 \upharpoonright H^{\bar{M}}_{\omega_1} = \mathrm{id}$. Thus $s_0 \lhd \langle \bar{Q}, \bar{M}, \bar{B} \rangle$ with $\pi_{s_0, \langle \bar{Q}, \bar{M}, \bar{B} \rangle} = \pi^{\bar{G}}_{\bar{s}_0}$. Set $\pi = \sigma_0 \pi^{\bar{G}}_{s_0}$.

<u>Claim</u>: $s \in G$ with $\pi^G_s = \pi$.

We know that $G = G^B$, so it is enough to show:

$\pi : \langle M_s, \bar{a} \rangle \prec \langle M, a \rangle$ whenever $\langle a, \bar{a} \rangle \in F^s$.

Let $\langle a, \bar{a} \rangle \in F^s$. Then $a = \sigma_0(a')$ where $\langle a', \bar{a} \rangle \in F^{\bar{s}}$. Thus $\pi^{\bar{G}}_{\bar{s}} : \langle M_s, \bar{a} \rangle \prec \langle \bar{M}, a' \rangle$ and $\sigma_0 \upharpoonright \bar{M} : \langle M, a' \rangle \prec \langle M, a \rangle$, since $\sigma_0(\langle \bar{M}, a' \rangle) = \langle M, a \rangle$. QED (Lemma 6)

Lemma 7.1 Let G be $\mathbb{P}$-generic, $B = B^G$. Let $B' \subset \bigcup_{i<\gamma_0} \mathbb{B}_i$ s.t. $B'_i = B' \cap \mathbb{B}_i$ is $\mathbb{B}_i$-generic for $i < \gamma_0$. Let $Q[B] = Q[B']$. Assume moreover that B' lies in a generic extension of $V[G]$ which adds no reals. Then

(a) $G' = G^{B'}$ is $\mathbb{P}$-generic.

(b) $V[G] = V[G']$

proof.

We imitate the proof of §3 Lemma 7. Let $i_0 < \gamma_0$ s.t. $\Vdash_{\mathbb{B}_{i_0}} cf(\check{\lambda}) = \omega$. Let $C \in M[B_{i_0}]$ s.t. $C \subset \gamma_0 = \sup(C)$, $otp(C) = \omega$. Let $\langle c_i \mid i < \omega \rangle$ enumerate C. There is $p \in G$ s.t. $B'_{c_i} \in rng(\pi^*)$ for $i < \omega$, where π^* is the unique $\pi^* \supset \pi^G_p \upharpoonright Q_p$ s.t. $\pi^* : Q_p[B^p] < Q[B]$ and $\pi^*(B^p_{\bar{c}_i}) = B^p_{c_i}$ for $i < \omega$. Set:

$$B^{p\prime} = \bigcup_{i<\omega} \pi^{*-1}(B'_{c_i}).$$

Then:

(1) $B' \in V[G]$ since $B' = \bigcup_{i<\omega} \pi^*(B^{p\prime}_{\bar{c}_i})$

(2) $\pi^*(B^{p\prime}_\xi) = B'_{\pi(\xi)}$ for $\xi < \gamma_0^p$

(3) $B^{p\prime}_\xi$ is $\mathbb{B}^p$-generic over M_p for $\xi < \gamma_0^p$

proof. (straightforward)

(4) $Q_p[B^{p\prime}] = Q_p[B^p]$,

prf straightforward

Hence by the 1st revisability principle:

(5) $p' \in \mathbb{P}'$ where $p'_0 = \langle Q_p, M_p, B^{p\prime} \rangle$, $p'_1 = p_1$

Now let $q \leq p$. Let π^*_{pq} be the unique extension of $\pi_{pq} \upharpoonright Q$ s.t. $\pi^*_{pq} : Q_p[B^p] \prec Q_q[B^q]$ and $\pi^*_{pq}(B^p_\xi) = B^q_{\pi_{pq}(\xi)}$ for $\xi < \aleph_0^p$. Set:

$B^{q'} = \bigcup_{i<\omega} \pi^*_{pq}(B^{p'}_{\bar{c}_i})$, $q'_0 = \langle Q_q, M_q, B^{q'} \rangle$, $q'_1 = q_1$

We easily get:

(6) (a) $B^{q'}_i$ is $\mathbb{B}^q_i$ - generic over M_q for $i < \aleph_0^q$

(b) $q' \in \mathbb{P}'$

(c) $\pi_{p'q'} = \pi_{pq}$,

proof straightforward, using the fact that π_{pq} extends to $\pi^i_{pq} : Q_p[B^p_i] \prec Q_q[B^{q'}_{\pi_{pq}(i)}]$ s.t. $\pi^i_{pq}(B^{p'}_i) = B^{q'}_{\pi_{pq}(i)}$ for $i < \aleph_0^p$. QED (6)

If $q \leq r \leq p$, we could define q' from r' the way we defined it from p'. Hence:

(7) $q \leq r \leq p \rightarrow q' \leq r' \leq p'$;

moreover $\pi_{q'r'} = \pi_{qr}$.

Set: $\Delta_0 = \{q \mid q \leq p\}$, $\Delta_1 = \{q \mid q \leq p'\}$.

Set $\sigma(q) = q'$. Arguing exactly as in §3 Lemma 7 we get:

(8) $\sigma : \langle \Delta_0, \leq \rangle \overset{\sim}{\longleftrightarrow} \langle \Delta_1, \leq \rangle$.

But then $q \in \Delta_0 \cap G \rightarrow \sigma(q) \in G' = G^{B'}$ and $q \in \Delta_1 \cap G' \rightarrow \sigma^{-1}(q) \in G = G^B$. Hence:

(9) $\sigma''(\Delta_0 \cap G) = \Delta_1 \cap G'$.

As before we conclude:

(10) $V[G] = V[G']$.

(11) G' is $\mathbb{P}$-generic,

since if Δ is dense in $\mathbb{P}$, then $\Delta' =$

$= \{q \in \Delta_0 \mid \sigma(q) \in \Delta\}$ is dense above p in $\mathbb{P}$,

Hence $G \cap \Delta' \neq 0$. Hence $G' \cap \Delta \cap \Delta_1 =$

$= \sigma'' G \cap \Delta' \neq \emptyset$. QED (Lemma 7.1)

Carrying this proof a step further we get:

<u>Cor 7.2</u> Let G, B, G', B' be as above. There is $\sigma^* \in V$ s.t. $\sigma^* : BA(\mathbb{P}) \overset{\sim}{\longleftrightarrow} BA(\mathbb{P})$ and $F_{G'} = \sigma^{*}{}'' F_G$ (where $F_G = \{b \in BA(\mathbb{P}) \mid b \cap G \neq \emptyset\}$ is the generic ultrafilter given by G).

proof (assume w.l.o.g. $G \neq G'$).

As in §3 Cor 7.1 set

$\Delta_2 = \{r \mid r$ is incompatible with p and $p'\}$.

Then $B \neq B'$ since $G \neq G'$. Hence $B^p \neq B^{p'}$.

Hence $\Delta_0, \Delta_1, \Delta_2$ are mutually disjoint.

Set $\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2$. Define

$\sigma' : \langle \Delta, \leq \rangle \longleftrightarrow \langle \Delta, \leq \rangle$ by:

$$\sigma'(q) = \begin{cases} \sigma(q) & \text{if } q \in \Delta_0 \\ \sigma^{-1}(q) & \text{if } q \in \Delta_1 \\ q & \text{otherwise} \end{cases}$$

Since Δ is dense in $\mathbb{P}$, σ' induces a unique $\sigma^* : BA(\mathbb{P}) \overset{\sim}{\longleftrightarrow} BA(\mathbb{P})$ s.t. $\sigma^*([p]) = [\sigma'(p)]$ for $p \in \Delta$. QED (7.2)

This can obviously be rewritten as:

<u>Cor 7.3</u> Let G be $\mathbb{B}_\lambda$-generic and $B = G \cap \bigcup_{i<\lambda} \mathbb{B}_i$. Let $B' \subset \bigcup_{i<\lambda} \mathbb{B}_i$ s.t. $B'_i = B' \cap \mathbb{B}_i$ is $\mathbb{B}_i$-generic for $i<\lambda$ and $H_\gamma[B'] = H_\gamma[B]$. There is an automorphism $\pi \in V$ of $\mathbb{B}_\lambda$ s.t. $\pi''B = B'$. (Hence $G' = \pi''G$ is $\mathbb{B}_\lambda$-generic and $V[G'] = V[G]$ with $B' = G \cap \bigcup_{i<\lambda} \mathbb{B}_i$.)

Lemma 7.4 $\mathbb{B}_\lambda$ is symmetrically proud over $\mathbb{B}_\xi$ whenever $\xi \in \lambda \cap Ac$.

proof.

This obviously follows by:

Main Claim Let $\theta > 2^{\mathbb{B}}$ be big enough to verify the proudness of $\mathbb{B}_\xi$ for all $\xi < \lambda$ s.t. $\xi \in Ac$. Let G be $\mathbb{P}$-generic and π-conforming, where $\pi : \overline{W} \prec W = H_\theta$, and $\overline{W}$ is countable and transitive. Set $B = B^G$. Let $\xi \in \lambda \cap Ac$ and:

$\pi(\bar{\xi}, \overline{\mathbb{P}}, \langle \overline{\mathbb{B}}_i \mid i < \bar{\lambda} \rangle) = \xi, \mathbb{P}, \langle \mathbb{B}_i \mid i < \lambda \rangle$.

Suppose that $\overline{B}''$ is $\mathbb{B}_{\bar{\xi}}$-generic over $\overline{W}$ and B'' is $\mathbb{B}_\xi$-generic s.t. $V[B''] = V[B_\xi]$ (where $B_i = B \cap \mathbb{B}_i$) and $\pi''\overline{B}'' \subset B''$. Let $\overline{G}'$ be $\overline{\mathbb{P}}$-generic over $\overline{W}$ s.t. $\overline{B}'' = B^{\overline{G}'} \cap \mathbb{B}_{\bar{\xi}}$. Set $\overline{B}' = B^{\overline{G}'}$. There is G' s.t.

- G' is $\mathbb{P}$-generic
- $B' \supset B''$, where $B' = B^{G'}$
- $\pi''\overline{G}' \subset G'$
- There is $\sigma : BA(\mathbb{P}) \xrightarrow{\sim} BA(\mathbb{P})$ s.t. $\sigma''F_G = F_{G'}$.

proof.

Since $cf(\lambda) = \omega$ in $V[G]$ and π takes $\bar{\lambda}$ cofinally to λ, we can pick $\langle \bar{c}_i \mid i < \omega \rangle$ monotone and cofinal in $\bar{\lambda}$. Set $c_i = \pi(\bar{c}_i)$. Then $\langle c_i \mid i < \omega \rangle$ is monotone and cofinal in λ. We may suppose w.l.o.g. that $c_0 = \xi$ and $c_i \in Ac$ for $i < \omega$. Thus $\mathbb{B}_i$ is proud over $\mathbb{B}_h$ for $h < i < \omega$. Set: $\bar{B}'_i = \bar{B} \cap \mathbb{B}_{\bar{c}_i}$ for $i < \omega$. Using proudness, we successively pick B'_i $(i < \omega)$ s.t.

- $B'_0 = B''$
- $B'_{i+1} \supset B'_i$ is $\mathbb{B}_{c_{i+1}}$ - generic
- $V[B'_i] = V[B_i]$ where $B_i = B \cap \mathbb{B}_{c_i}$
- $\pi `` \bar{B}'_i \subset B'_i$

Set $B' = \bigcup_i B'_i$. Since $Q = H_\gamma$ in V and $\mathbb{B}_{c_i} \in Q$, we have: $Q[B'_i] = Q[B_i]$. Hence $Q[B'] = \bigcup_i Q[B'_i] = \bigcup_i Q[B_i] = Q[B]$. By Lemmas 7.1 + 7.2 the conclusion then holds for $G' = G^{B'}$. All verifications are trivial except for:

$\pi''\bar{G}' \subset G'$,

which we now prove. We first note:

(1) $\pi\upharpoonright\bar{Q} : \langle \bar{Q}, \bar{B}', \bar{C}\rangle \triangleleft_* \langle Q, B', C\rangle$

for any $\bar{C} \in \bar{M}$ s.t. $\bar{C} \subset \bar{\gamma}_0 = \sup \bar{C}$ and $\mathrm{otp}(\bar{C}) = \omega$.

(Here $\pi(\bar{Q}, \bar{M}) = Q, M$)

But then:

(2) $\pi\upharpoonright\bar{M} : \langle \bar{Q}, \bar{M}, \bar{B}'\rangle \triangleleft \langle Q, M, B'\rangle$

proof.

Let $\pi\upharpoonright\bar{M} : \bar{M} \to_{\Sigma_0} \tilde{M}$ cofinally. We must show that $\langle \tilde{M}, \pi\upharpoonright\bar{M}\rangle$ is the liftup of $\langle \bar{M}, \pi\upharpoonright\bar{Q}\rangle$. In other words, we must show that $\pi\upharpoonright\bar{M} : M \to \tilde{M}$ is $\bar{\gamma}$-cofinal, where $\pi(\bar{\gamma}) = \gamma$. Let $\bar{G} = \pi^{-1}{}''G$. Then $\bar{G}$ is $\bar{\mathbb{P}}$-generic over $\bar{W}$. Hence π extends uniquely to $\pi^* : \bar{W}[\bar{G}] \prec W[G]$ s.t. $\pi^*(\bar{G}) = G$. But then, for any $x \in \tilde{M}$ there is $\xi \in \bar{M}$ s.t. $x \in \pi(L^{\bar{A}}_{\xi})$ (where $\bar{M} = L^{\bar{A}}_{\beta}$). Hence there is $\bar{\Omega} \in \bar{G}$ s.t. $\xi \in \mathrm{rng}(\pi^{\bar{G}}_{\bar{\Omega}})$. But $\pi^{\bar{G}}_{\bar{\Omega}} : \Omega_0 \triangleleft \langle \bar{Q}, \bar{M}, \bar{B}\rangle$, where $\bar{B} = B^{\bar{G}}$. Let $\pi^{\bar{G}}_{\bar{\Omega}} : M_{\bar{\Omega}} \to \tilde{M}_{\bar{\Omega}}$ cofinally. Then the map is $\gamma_0^{\bar{\Omega}}$-cofinal — i.e.

$$\bigwedge z \in \tilde{M}_{\bar{\Omega}} \bigvee u \in M_{\bar{\Omega}} \, (\bar{u} \le \gamma_0^{\bar{\Omega}} \text{ in } M_{\bar{\Omega}} \wedge z \in \pi^{\bar{G}}_{\bar{\Omega}}(y)),$$

Let $\tilde{M}_\nu = \pi(\tilde{\bar{M}}_{\bar\nu})$, $\tilde{\pi} = \pi^*(\pi^{\bar G}_{\bar\nu}) = \pi \circ \pi^{\bar G}_{\bar\nu}$ (since $\pi^*(M_{\bar\nu}) = M_{\bar\nu}$). Then $\tilde{M}_\nu$ is a segment of $M = \pi(\bar M)$ and $\pi(\xi) \in rng(\tilde\pi) \subset \tilde{M}_\nu$. Hence $x \in \pi(L^{\bar A}_\xi) \subset$ $\subset \tilde{M}_\nu$. But:

$\bigwedge z \in \tilde{M}_\nu \bigvee u \in M_{\bar\nu} (\bar u \leq \gamma_0^{\bar\nu}$ in $M_{\bar\nu} \wedge z \in \tilde\pi(u))$

Hence $x \in \tilde\pi(u) = \pi(u^*)$, where $u^* = \pi^{\bar G}_{\bar\nu}(u) \in \bar M$ and $\bar{\bar{u}}^* \leq \bar\gamma_0$ in $\bar M$.

QED(2)

Now let $\bar\nu \in \bar G'$; $\nu = \pi(\bar\nu)$

<u>Claim</u> $\nu \in G'$.

We again have $\nu_0 = \pi(\bar\nu_0) = \bar\nu_0$, hence:

(3) $\nu_0 \lhd \langle \bar Q, \bar M, \bar B'\rangle \lhd \langle Q, M, B'\rangle$

with $\pi_{\nu_0, \langle Q, M, B'\rangle} = \pi \circ \pi^{\bar G'}_{\bar\nu}$.

Set: $\tilde\pi = \pi \circ \pi^{\bar G'}_{\bar\nu}$. We claim that $\nu \in G'$ with $\tilde\pi = \pi^{G'}_\nu$. Clearly $\nu \in \mathbb{P}$, since $\bar\nu \in \mathbb{P}$. Since $G' = G^{B'}$, it suffices to show:

(4) $\tilde\pi : \langle M_\nu, \bar a\rangle \prec \langle M, a\rangle$ whenever $\langle a, \bar a\rangle \in F^\nu$.

Then $F^{\Omega} = \pi(F^{\bar\Omega})$ and $a = \pi(a')$, where $\langle a', \bar a\rangle \in F^{\bar\Omega}$. Hence

$$\pi_{\bar\Omega}^{\bar G'} : \langle M_\Omega, \bar a\rangle \prec \langle \bar M, a'\rangle, \text{ and}$$

$$\pi \restriction \bar M : \langle \bar M, a'\rangle \prec \langle M, a\rangle,$$

since $\pi(\langle \bar M, a'\rangle) = \langle M, a\rangle$.

QED (Lemma 7.4)

<u>Lemma 8</u> $\mathbb{B}_\lambda$ is symmetrical over $\mathbb{B} = \langle \mathbb{B}_i \mid i < \lambda\rangle$

proof.

This will follow from:

<u>Claim</u> Let $\sigma : \bigcup_i \mathbb{B}_i \overset{\sim}{\longleftrightarrow} \bigcup_i \mathbb{B}_i$ s.t. for sufficiently large i, $\sigma \restriction \mathbb{B}_i : \mathbb{B}_i \overset{\sim}{\longleftrightarrow} \mathbb{B}_i$ for sufficiently large i. There is $\sigma' : BA(\mathbb{P}) \overset{\sim}{\longleftrightarrow} BA(\mathbb{P})$ s.t.

$\sigma' k = k\sigma$

proof.

Set: $N^* = \langle H_\theta, N, \sigma, <, \ldots\rangle$ where $\theta > 2^\beta$,

$$\Delta = \{p \in \mathbb{P} \mid p \text{ conforms to } N^*\}.$$

Then Δ is dense in $\mathbb{P}$. For $p \in \Delta$ set:

$$N_p^* = \bar N^*(p, N^*) = \langle H_p, N_p, \sigma_p, <, \ldots\rangle$$

where $N_p = \langle \tilde H_p, Q_p, M_p, <, \ldots\rangle$.

Then $p_0 = \langle Q_p, M_p, B^p\rangle$. For $p \in \Delta$

define $p' \in \mathbb{P}$ by:

$p'_0 = \langle Q_p, M_p, \sigma_p `` B^p \rangle$, $p'_1 = p_1$

Then $Q_p[B^p] = Q_p[\sigma_p `` B^p]$ and $p' \in \mathbb{P}$ by the first revivability lemma. But then $p' \in \Delta$ since $p'_1 = p_1$. It follows easily that $p \leq q \leftrightarrow p' \leq q'$ for $p, q \in \Delta$.

Moreover, if p^* is defined as p' was defined with σ_p^{-1} in place of σ_p, then $p \in \Delta \rightarrow p^* \in \Delta$,

$p \leq q \leftrightarrow p^* \leq q^*$ for $p, q \in \Delta$,

$p'^* = p$ and $p^{*\prime} = p$ for $p \in \Delta$. Hence $p \mapsto p'$ is an automorphism of $\langle \Delta, \leq_{\mathbb{P}} \rangle$. Hence there is a unique automorphism σ' of $BA(\mathbb{P})$ s.t. $\sigma'([p]) = [p']$ for $p \in \mathbb{P}$. We must show:

<u>Claim</u> $\sigma' k = k \sigma$

Recalling that $k(b) = [\![\check{b} \in \dot{B^G}]\!]$ for $b \in \bigcup_i \mathbb{B}_i$, this becomes:

$$\sigma'([\![\check{b} \in B^{\dot{G}}]\!]) = [\![\check{\sigma(b)} \in B^{\dot{G}}]\!]$$

It suffices to show:

<u>Claim</u> Let G be $\mathbb{P}$-generic. Then

$$G \cap \sigma'(\llbracket \check{b} \in \dot{B}^{\check{G}} \rrbracket) \neq 0 \longleftrightarrow \sigma(b) \in B^G.$$

prf.

($\rightarrow$) $D = \{p \in \Delta \mid b \in \mathrm{rng}(\pi^p)\}$ is dense in $\llbracket \check{b} \in \dot{B}^{\check{G}} \rrbracket$. Hence $p' \in G$ for a $p \in D$. Thus $Q_{p'} = Q_p$, $M_{p'} = M_p$, $F^{p'} = F^p$ and $B^{p'} = \sigma_p{}'' B^p$. Hence $\pi^{p'} = \pi^p$ and $N^*_{p'} = N^*_p$. Hence $\sigma_{p'} = \sigma_p$. Let $\pi^p(\bar{b}) = b$. Then $\sigma_{p'}(\bar{b}) = \sigma_p(\bar{b}) \in B_{p'}$. But then $\pi^G_{p'}(\bar{b}) = b$ and $\pi^G_{p'}(\sigma_{p'}(\bar{b})) = \sigma \pi^G_{p'}(\bar{b}) = \sigma(b) \in B^G$, since $\pi^G_{p'}$ extends to $\pi^* : N^*_{p'} < N^*$ (hence $\pi^*(\sigma_{p'}) = \sigma$).

QED($\rightarrow$)

($\leftarrow$) Let $\sigma(b) \in B^G$. Then there is $p \in G \cap \Delta$ s.t. $\sigma(b) \in \mathrm{rng}(\pi^p)$. Note that $\sigma_p : \bigcup_i \mathbb{B}^p_i \overset{\sim}{\longleftrightarrow} \bigcup_i \mathbb{B}^p_i$, since $\pi^*(\sigma_p) = \sigma$, where π^* extends π^G_p, $\pi^* : N^*_p < N^*$. Hence there is $\bar{b} \in \bigcup_i \mathbb{B}^p_i$ s.t. $\pi^p(\sigma_p(\bar{b})) = \sigma(b)$. Hence $\sigma_p(\bar{b}) \in B_p$. Let $q \in \Delta$ s.t. $p = q'$. Then $B_p = \sigma_q{}'' B_q$ and $\bar{b} \in B_q$, since $\sigma_q(\bar{b}) \in \sigma_q{}'' B_q$. Hence $q \in \llbracket \check{b} \in \dot{B}^{\check{G}} \rrbracket$ and $p = q' \in \sigma'(\llbracket b \in \dot{B}^{\check{G}} \rrbracket) \cap G$.

QED (Lemma 8)

In order to satisfy the condition (f) of §2.3 we need:

<u>Lemma 9</u> Let $cf(\lambda) = \omega$ in V. Let $\langle c_i \mid i<\omega\rangle$ be cofinal and monotone in λ. Let $\langle b_i \mid i<\omega\rangle$ be a thread in $\langle \mathbb{B}_{c_i} \mid i<\omega\rangle$ (i.e. $h_{c_i, c_{i+1}}(b_{i+1}) \supset b_i$. Then $\bigcap_i b_i \neq 0$ in $\mathbb{B}_\lambda$.

proof.

This reduces to:

<u>Claim</u> Let $\langle c_i \mid i<\omega\rangle$, $\langle b_i \mid i<\omega\rangle$ be as above. Then $\bigcap_i k_i(b_i) \neq 0$ in $BA(\mathbb{P})$.

But $k_i(b_i) = [\![\check{b}_i \in B^{\dot{G}}]\!]$. We modify the proof that $\mathcal{L}$ is consistent to find a $p \in \mathbb{P}$ s.t. $p \in \bigcap_i [\![\check{b}_i \in B^{\dot{G}}]\!]$.

We first note that $\mathcal{L}' = \mathcal{L} + \bigwedge_i (\underline{b}_i \in \mathring{B})$ is consistent. To see this we rerun the consistent proof of $\mathcal{L}$, taking a $\mathbb{B}'$-generic B' s.t. $\bigcap_i b_i \in B'$, where $\mathbb{B}' \supseteq \bigcup_i \mathbb{B}_i$ is the inverse limit.

Now let $\mathfrak{A}$ be a solid model of $\mathcal{L}'$.

By Lemma 2 there is $u = \langle Q_u, M_u, B_u\rangle \in$ $\in \Gamma \wedge H_{\omega_1}$ s.t. $u \triangleleft \langle Q, M, B\rangle$ and $\pi(\bar{b}_i) = b_i$ for $i<\omega$, when $\pi = \pi_{u,\langle Q,M,B\rangle}$.

Set $p_0 = u$, $p_1 = \{\langle b_m, \bar{b}_m\rangle \mid m<\omega\}$, Then $p \in \mathbb{P}$ since $\mathfrak{A} \models \mathcal{L}(p)$. But then $\bar{b}_i \in B^p$, since $b_i \in B^{\mathfrak{A}}$. Hence, if $G \ni p$ is $\mathbb{P}$-generic, we have $b_i \in B^G$ for $i < \omega$. Hence $G \cap \bigcap_i k(b_i) = G \cap \bigcap_i [\![\check{b}_i \in B^{\mathring{G}}]\!] \neq \emptyset$. This shows that $[p] \subset \bigcap_i k(b_i)$. QED (Lemma 9)

Note Using Lemma 5.1 it follows easily that Lemma 9 holds of $\langle \mathbb{B}_{i_0+h} / B \mid h \leq \lambda - i_0 \rangle$ in $V[B]$ whenever $i_0 < \lambda$ and B is $\mathbb{B}_{i_0}$-generic.

We are now ready to show:

Lemma 10 $\langle \mathbb{B}_i \mid i \leq \lambda \rangle$ satisfies the conditions (a)–(h) of § 2.3.

proof

(h) is trivial. (a) follows by Lemma 4.11 and Lemma 4.12. (b) follows by Lemma 6. (c) holds vacuously. (d) holds by Lemma 9. (e) holds by Lemma 7.3. (f) follows by Lemma 8. (g) follows by Lemma 5.1

QED

This completes the first limit case.

§4.2 The 2nd limit case

We now consider the case that λ is an ω-point, $\gamma = \gamma_\lambda$, and $\gamma^+ \in A_0$. Hence γ^+ must acquire cardinality ω_1. We have $\beta_\lambda = \gamma^+$; hence we want: $\gamma^{++} = \omega_2$ in $V[G]$ if G is $\mathbb{B}_\lambda$-generic. Our construction closely parallels §4.1. Indeed, most of the proofs can be repeated almost verbatim. We shall therefore confine ourselves to detailing the differences in the two constructions. From now on let $\beta = \beta_\lambda = \gamma^+$. We Define Q, M, N exactly as before. We change the language $\mathcal{L}$ in exactly one respect: Replace (*) by the conjunction of:

(*) For each $\xi < \underset{\sim}{\beta}$ there are u, π s.t. $u \in H_{\omega_1}$, $\pi : u \lhd \langle \underset{\sim}{Q}, \underset{\sim}{M}, \dot{\beta} \rangle$, and $\xi \in \operatorname{rng}(\pi)$

(**) $\operatorname{cf}(\underset{\sim}{\beta}) > \omega$

$\lhd_*, \lhd, \Gamma_*, \Gamma$ are defined exactly as before. §3.1 Lemma 0 holds as before

We first show:

Lemma 1 $\mathcal{L}$ is consistent.

proof.

Just as before we may assume w.l.o.g. that $\lambda = \aleph_0$ is ω-cofinal in V. Again let $C = \{c_i \mid i < \omega\}$ where $\langle c_i \mid i < \omega \rangle \in V$ is monotone and cofinal in λ. Set:
$\mathbb{B}' =$ the inverse limit of $\langle \mathbb{B}_{c_i} \mid i < \omega \rangle$. $\mathbb{B}'$ is again subcomplete. Let B be $\mathbb{B}'$-generic and set:

$\mathfrak{A} = \langle H_{\beta^+}, B \rangle$. Claim $\mathfrak{A} \models \mathcal{L}$.

The only problematical axioms are $(*)$ and $(**)$. To see $(*)$, let $\xi \in M$. Let $\pi : \bar{M}[\bar{B}] \prec M[B]$ s.t. $\pi(\bar{B}) = B$ and $\xi, C \in \mathrm{rng}(\pi)$ and $\bar{M} \in H_{\omega_1}$. Let $\pi(\bar{C}) = C$, $\pi(\bar{Q}) = Q$. Then $\pi \upharpoonright \bar{Q} : \langle \bar{Q}, \bar{B}, \bar{C} \rangle \prec_* \langle Q, B, C \rangle$. Using the fact that every $x \in M$ has cardinality $\leq \aleph$ in M we get:
$\tilde{\pi}$ is $\bar{\aleph}$-cofinal, where $\tilde{\pi} : \bar{M} \to \tilde{M}$ cofinally and $\tilde{\pi}(\bar{\aleph}) = \aleph$. Hence
$\tilde{\pi} : \langle \bar{Q}, \bar{M}, \bar{B} \rangle \lhd \langle QM, B \rangle$ and $\xi \in \mathrm{rng}(\tilde{\pi})$.
This proves $(*)$.

We now prove (**). $\bar{\bar{\beta}} = \omega_1$ in $V[B]$ since $\beta \subset \bigcup \{ rng(\pi_{u,\langle M,B\rangle}) \mid u \in H_{\omega_1} \wedge u \triangleleft \langle Q, M, B\rangle \}$
Since GCH holds below κ, we have: $\bar{\bar{\mathbb{B}}}' \leq \beta$. Hence β^+ remains a cardinal in $V[B]$. But then $cf(\beta) = \omega_1$ in $V[B]$, since otherwise β^+ is collapsed by §4 Lemma 4.1 of [LF] (Fact 11 of §3.2 in this paper). QED (Lemma 1)
Repeating our proofs verbatim:

__Lemma 2__ Cor 1.1, Lemma 2, and Lemma 2.1 of §4.1 hold.

We then define $\tilde{\mathbb{P}}$, $\mathbb{P}$ exactly as before (except that in the def. of φ_p we omit the clause: $\sup \pi_p{}'' \beta_p = \underline{\beta}$, even though β is regular).

__Remark__ We don't know whether forcing with $\mathbb{P}$ is the same as forcing with the inverse limit of $\langle \mathbb{B}_i \mid i < \lambda \rangle$.

Repeating our previous proofs:

__Lemma 3__ Lemmas 3.1 – 3.8 of §4.1 hold, as do the revisability lemmas 4.1 and 4.2.
Lemma 4.3 ($\mathbb{P}$ adds no reals) goes through as well, but we shall need a stronger form of it. We first show:

<u>Lemma 3.1</u> Let G be $\mathbb{P}$-generic and $p \in G$. Then $\sup \pi_p^G{}''\beta_p < \beta$.

prf.
It suffices to observe that Δ is dense above p, where $\Delta = \{q \le p \mid \sup \pi_{pq}{}''\beta_p < \beta_q\}$. This follows by §4.1 Lemma 2. QED (3.1)

Our stronger version of §4.1 Lemma 4.3 now reads:

<u>Lemma 3.2</u> Let G be $\mathbb{P}$-generic. Let $f \in V[G]$ s.t. $f : \omega \to \beta$. There is $p \in G$ s.t. $f = \pi_p^G \circ \bar{f}$ for an $\bar{f} \in H_{\omega_1}^V$.

proof.
We can assume $f = \mathring{f}^G$, where $\Vdash_{\mathbb{P}} \mathring{f} : \omega \to \check{\beta}$. It suffices to show that the set Δ of $p \in \mathbb{P}$ s.t. $p \Vdash \mathring{f} = \pi_{\check{p}}^{\mathring{G}} \circ \check{\bar{f}}$ for an $\bar{f} \in H_{\omega_1}$ is dense in $\mathbb{P}$.
Let $r \in \mathbb{P}$. Set:
$N^* = \langle H_\theta, N, M, Q, \mathbb{P}, r, \quad \rangle$, where $\theta > (2^\beta)^+$ is big enough to verify the proudness of i over h for all $h < i < \lambda$ s.t. $h, i \in Ac$. Let p conform to N^*. Set:
$\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{N}, \bar{M}, \bar{Q}, \bar{\mathbb{P}}, r, \quad \rangle$.
(Hence $\bar{Q} = Q_p$, $\bar{M} = M_p$). Choose $\bar{G} \ni r$ s.t. $\bar{G}$ is $\bar{\mathbb{P}}$-generic over $\bar{N}^*$. Set $\bar{B} = B^{\bar{G}}$,

By the second revisability lemma we can define $q \in \mathbb{P}$ by:

$$q_0 = \langle \bar{Q}, \bar{m}, \bar{B} \rangle, \quad q_1 = p_1 \quad \text{where } \bar{B} = B^{G}.$$

~~Let~~ $\bar{f} = \mathring{\bar{f}}^{G}$. ~~It~~ suffices to prove:

__Claim__

(a) $q \Vdash \mathring{f} = \pi_q^{\dot{G}} \circ \check{\bar{f}}$, (b) r is compatible with q.

We first prove (a). Suppose not. ~~Let~~ $G \ni q$ s.t. $f \neq \pi_q^{G} \circ \bar{f}$, where $f = \mathring{f}^{G}$. Let $i < \omega$, $\bar{\eta} = \bar{f}(i)$, $\eta = \pi_q^{G}(\bar{\eta})$ s.t. $f(i) \neq \eta$. Then $\tilde{q} \in G$, where $\tilde{q}_0 = q_0$, $\tilde{q}_1 = q_1 \cup \{\langle \eta, \bar{\eta} \rangle\}$. Let $q' \leq \tilde{q}$, $q' \in G$ s.t. $q' \Vdash \mathring{f}(\check{i}) \neq \check{\eta}$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(q')$. Then $\mathfrak{A} \models \mathcal{L}(\tilde{q})$ and $\pi_q^{\mathfrak{A}}(\bar{\eta}) = \eta$, since $\pi^{\tilde{q}}(\bar{\eta}) = \eta$. But $\pi_q^{\mathfrak{A}}$ extends uniquely to $\pi^{*}: \bar{N}^{*} \prec N^{*}$ s.t. $\pi_q^{\mathfrak{A}} \cup F^{q} \subset \pi^{*}$. Let $\bar{s} \in \bar{G}$ s.t. $\bar{s} \Vdash_{\bar{\mathbb{P}}} \mathring{\bar{f}}(\check{i}) = \check{\eta}$. Then, letting $s = \pi^{*}(\bar{s})$, we have: $s \Vdash \mathring{f}(i) = \check{\eta}$. Hence s, q' are incompatible. We derive a contradiction by showing that they are compatible and in fact that $\mathfrak{A} \models \mathcal{L}(s)$ (hence $\mathcal{L}(q') \cup \mathcal{L}(s)$ is consistent).

$\pi^{\bar{G}}_{\bar{s}} : s_0 \lhd \langle \bar{Q}, \bar{M}, \bar{B} \rangle$, since $s_0 = \bar{s}_0$ and $\bar{s} \in \bar{G}$. But $\langle \bar{Q}, \bar{M}, \bar{B} \rangle = q_0$ and $\pi^{\circ \mathfrak{U}}_{q} : q_0 \lhd \langle Q, M, B \rangle$ where $B = \dot{B}^{\mathfrak{U}}$. Set $\pi = \pi^{\circ \mathfrak{U}}_{q} \circ \pi^{\bar{G}}_{\bar{s}} = \pi^* \circ \pi^{\bar{G}}_{\bar{s}}$. Then $\pi : s_0 \lhd \langle Q, M, B \rangle$. It remains only to show:

(1) $\pi : \langle M_s, \bar{a} \rangle \prec \langle M, a \rangle$, whenever $\langle a, \bar{a} \rangle \in F$.

Let $a = \pi^*(a')$. Then $\langle a', \bar{a} \rangle \in F^{\bar{s}}$ and $\pi^{\bar{G}}_{\bar{s}} : \langle M_s, \bar{a} \rangle \prec \langle \bar{M}, a' \rangle$. But $\pi^*(\langle \bar{M}, a' \rangle) = \langle M, a \rangle$. Hence $\pi = \pi^* \circ \pi^{\bar{G}}_{\bar{s}} : \langle M_{\bar{s}}, \bar{a} \rangle \prec \langle M, a \rangle$. QED(a)

We now note that the last part of the proof shows that for any $\bar{s} \in \bar{G}$, $s = \pi^*(\bar{s})$ is compatible with q', and hence with q. But $\bar{r} \in \bar{G}$ and $r = \pi^*(\bar{r})$ since $\pi^* : \bar{N}^* \prec N^*$.

QED (3.2)

By Lemmas 3.2 and 3.1 we then easily get:

<u>Lemma 3.3</u> Let G be $\mathbb{P}$-generic. Then

- $H_{\omega_1} = H_{\omega_1}^{V[G]}$
- $cf(\beta) > \omega$ in $V[G]$.

Hence:

Lemma 3.4 §4.1 Lemma 4.4 holds (i.e. if $G \ni p$ is $\mathbb{P}$-generic and $B = B^G$, then $\langle H_\theta^{V[G]}, B \rangle$ models $\mathcal{L}(p)$.

proof.

The only problematical axioms were $H_{\omega_1} = \underline{H_{\omega_1}}$ and $cf(\beta) > \omega$, which are now seen to hold.

By a literal repetition of the proofs:

Lemma 3.5 §4.1 Lemmas 4.5 – 4.9 hold.

We note that $\beta = \beta_\lambda$ and $cf(\beta) > \omega$.

We can then repeat the proof of §4.1 Sublemma 4.10.1, to get

Lemma 3.6 $BA(\mathbb{P})$ has a closure subset of size β. (Hence $\overline{BA(\mathbb{P})} \leq \beta^+$ where $\beta = \beta_\lambda$); Moreover β^+ remains a cardinal in $V[G]$ whenever G is $\mathbb{P}$-generic.)

Repeating the relevant part of the proof (the case $\beta = \beta_\lambda$) we get;

Lemma 3.7 §4.1 Lemma 4.12 holds. (I.e. if G is $\mathbb{P}$-generic, then $\bar{\beta} = \omega_1 = cf(\lambda)$ in $V[G]$ and $\beta^+ = \omega_2^{V[G]}$.)

We then choose σ, $\mathbb{B}_\lambda$ s.t.

$$\begin{tikzcd} \sigma : BA(\mathbb{P}) \arrow[r, "\sim", leftrightarrow] & \mathbb{B}_\lambda \\ \bigcup_i \mathbb{B}_i \arrow[u, "k"] \arrow[ur, hook] & \end{tikzcd}$$

and $\mathbb{B}_\lambda \subset H_{\beta^+}$.

By an almost literal repetition of the proofs we then get:

<u>Lemma 4</u> Lemmas 5 – 10 of §4.1 hold.

(In particular $\langle \mathbb{B}_i \mid i \leq \lambda \rangle$ satisfies (a) – (h) of § 2.3.)

This completes the second limit case.

§4.3 The third limit case

We now deal with the case that λ is not an ω-point. Let $\gamma = \gamma_\lambda$. Then either λ is an ω_1-point (i.e. $cf(\lambda) < \gamma$ and either $cf(\lambda) = \omega_1$ or $cf(\lambda) \in A_0$), or else $\lambda = \gamma$ is strongly inaccessible. In the first case we have $\beta_\lambda = \gamma$. In the second β_λ is undefined. In both cases we let $\mathbb{B}_\lambda$ be a minimal completion of $\bigcup_{i<\lambda} \mathbb{B}_i$, ensuring that $\mathbb{B}_\lambda \subset H_{\gamma^+}$.

If $\lambda = \gamma$ is not an ω_1-point, then $\bigcup_{i<\lambda} \mathbb{B}_i$ satisfies the γ-cc. Hence $\mathbb{B}_\lambda = \bigcup_{i<\lambda} \mathbb{B}_i \subset V_\gamma$. By the iteration theorem in §1 we have:

<u>Lemma 1.1</u> $\mathbb{B}_\lambda$ is subcomplete

But then:

<u>Lemma 1.2</u> Let G be $\mathbb{P}$-generic.

(a) If γ is an ω_1-point, then $\bar{\bar{\gamma}} = cf(\gamma) = \omega_1$ in $V[G]$ and $\gamma^+ = \omega_2^{V[G]}$

(b) If γ is not an ω_1-point, then $\bar{\bar{\gamma}} = \omega_2$ in $V[G]$.

proof.
Clearly each $\xi < \gamma$ is collapsed to ω_1, since $\mathbb{B}_i \subseteq \mathbb{B}_\lambda$ for $i < \lambda$. Hence $\bar{\bar{\gamma}} = \omega_1 = cf(\gamma)$ if γ is an ω_1-point. But γ^+ remains a cardinal, since $\mathbb{B}_\lambda$ has a dense subset $\bigcup_{i<\lambda} \mathbb{B}_i$ of size γ. Now let $\lambda = \gamma$ be strongly inaccessible. Then $\mathbb{B}_\lambda$ satisfies γ-cc. Hence γ remains a cardinal. QED (1.2)

Since $\bigcup_{i<\lambda} \mathbb{B}_i$ is dense in $\mathbb{B}_\lambda$, any automorphism of $\bigcup_{i<\lambda} \mathbb{B}_i$ extends uniquely to an automorphism of $\mathbb{B}_\lambda$. Hence

__Lemma 1.3__ $\mathbb{B}_\lambda$ is symmetrical over $\langle \mathbb{B}_i \mid i < \lambda \rangle$.

__Cor 1.4__ $\mathbb{B}_\lambda$ is symmetrical over $\mathbb{B}_i$ for $i < \lambda$.

prf.
Let $\sigma : \mathbb{B}_i \overset{\sim}{\longleftrightarrow} \mathbb{B}_i$ define $\sigma_j : \mathbb{B}_j \overset{\sim}{\longleftrightarrow} \mathbb{B}_j$ for $i \leq j \leq \lambda$ by: $\sigma_0 = \sigma$; $\sigma_{j+1} : \mathbb{B}_{j+1} \overset{\sim}{\longleftrightarrow} \mathbb{B}_j$ s.t. $\sigma_j \subset \sigma_{j+1}$; $\sigma_\eta : \mathbb{B}_\eta \overset{\sim}{\longleftrightarrow} \mathbb{B}_\eta$ s.t. $\bigcup_{j<\eta} \sigma_j \subset \sigma_\eta$ for limit η. QED (1.4)

It remains to prove:

<u>Lemma 2</u> Let $\xi < \lambda$ s.t. $\xi \in Ac$. If $\lambda \in Ac$, then $\mathbb{B}_\lambda$ is symmetrically proud over $\mathbb{B}_\xi$. Otherwise $\mathbb{B}_\lambda$ is symmetrically semiproud over $\mathbb{B}_\xi$.

proof.

Let $\theta > 2^\delta$ be big enough to verify the proudness of $\mathbb{B}_j$ over $\mathbb{B}_i$ for all $i < j < \lambda$ s.t. $i, j \in Ac$.

Let B be $\mathbb{B}_\lambda$ - generic and σ - conforming, where $\sigma \in V[B]$ s.t. $\sigma : \bar{W} \prec W = H_\theta$, and $\bar{W}$ is countable, $\sigma(\bar{\xi}) = \xi$, and $\sigma(\langle \bar{\mathbb{B}}_i \mid i \leq \bar{\lambda} \rangle) = \langle \mathbb{B}_i \mid i \leq \lambda \rangle$. Let $\bar{B}'_{\bar{\xi}}$ be $\mathbb{B}_{\bar{\xi}}$ - generic over $\bar{W}$, $\bar{B}'_{\bar{\xi}} \in V$, Let B'_ξ be $\mathbb{B}_\xi$ - generic s.t. $\sigma'' \bar{B}'_{\bar{\xi}} \subset B'_\xi$ and $V[B'_\xi] = V[B_\xi]$ (where $B_i = B \cap \mathbb{B}_i$ for $i \leq \lambda$). Let $\bar{B}' \supset \bar{B}'_{\bar{\xi}}$ be $\bar{\mathbb{B}}_\lambda$ - generic over $\bar{W}$.

<u>Claim</u> There exist B', π s.t.

- $B' \supset B_\xi$ is $\mathbb{B}_\lambda$ - generic.
- π is an automorphism of $\mathbb{B}_\lambda$ s.t. $B' = \pi'' B$.
- $\pi \in V$.

Set: $\tilde{\lambda} = \sup \sigma''\bar{\lambda}$. Then $cf(\tilde{\lambda}) = \omega$ in $V[B]$. Hence $\tilde{\lambda}$ is not an ω_1-point of the iteration. Hence $\tilde{\lambda}$ is either an ω-point or $\tilde{\lambda} = \gamma_{\tilde{\lambda}}$ is strongly inaccessible in V. We handle these cases separately.

<u>Case 1</u> $\tilde{\lambda}$ is an ω-point

We work in $V[B]$. Pick $\langle \bar{\xi}_i \mid i < \omega \rangle$ monotone and cofinal in $\bar{\lambda}$ s.t. $\bar{\xi}_0 = \bar{\xi}$ and $\xi_i' = \sigma(\bar{\xi}_i) \in A$ for all i.

Set $\bar{B}_i' = \bar{B} \cap \bar{\mathbb{B}}_i$ for $i \leq \bar{\lambda}$ and $B_i = B \cap \mathbb{B}_i$ for $i \leq \lambda$. Since $\mathbb{B}_{\xi_{i+1}}$ is proper over $\mathbb{B}_{\xi_i}$ we can successively pick B'_{ξ_i} s.t.

- $B'_{\xi_{i+1}} \supset B'_{\xi_i}$ with B_{ξ_0} as given
- B'_{ξ_i} is $\mathbb{B}_{\xi_i}$ - generic
- $V[B'_{\xi_i}] = V[B_{\xi_i}]$
- $\sigma''\bar{B}'_{\bar{\xi}_i} \subset B'_{\xi_i}$

Set $\tilde{B}' = \bigcup_i B'_{\xi_i}$; $\tilde{B} = \bigcup_i B_{\xi_i}$.

Since $V[B'_{\xi_i}] = V[B_{\xi_i}]$ for $i < \omega$, we have: $H_{\gamma_{\tilde{\lambda}}}[\tilde{B}'] = H_{\gamma_{\tilde{\lambda}}}[\tilde{B}]$.

By §4.1 Lemma 7.3 we conclude that there are $B'_{\tilde\lambda} \supset \tilde{B}'$ and $\tilde\pi : \mathbb{B}_{\tilde\lambda} \overset{\sim}{\longleftrightarrow} \mathbb{B}_{\tilde\lambda}$ s.t. $B'_{\tilde\lambda}$ is $\mathbb{B}_{\tilde\lambda}$ - generic, $\tilde\pi \in V$, and $\tilde\pi '' B_{\tilde\lambda} = B'_{\tilde\lambda}$. But then $\tilde\pi$ extends to $\pi : \mathbb{B}_\lambda \overset{\sim}{\longleftrightarrow} \mathbb{B}_\lambda$, since $\mathbb{B}_\lambda$ is symmetrical over $\mathbb{B}_{\tilde\lambda}$. Set: $B' = \pi '' B$. Then B' is $\mathbb{B}_\lambda$ - generic. Moreover, if $B^* = \bigcup_i \bar{B}'_{\bar\xi_i}$, then $\sigma '' B^* \subset \tilde{B}' \subset B'_{\tilde\lambda} \subset B'$.
But $\bar{B}' = \{ b \in \bar{\mathbb{B}}_{\bar\lambda} \mid \bigvee a \in B^* \; a \subset b \}$, since $\bigcup_i \bar{\mathbb{B}}_{\bar\xi_i}$ is dense in $\bar{\mathbb{B}}_{\bar\lambda}$. Hence $\sigma '' \bar{B}' \subset B'$. QED (Case 1)

<u>Case 2</u> Case 1 fails.

Then $\tilde\lambda = \gamma_{\tilde\lambda}$ is strongly inaccessible in V, The second successor case applies to $\mathbb{B}_{\tilde\lambda + 1}$, since otherwise $\tilde\lambda$ would have cofinality ω_1 in $V[B]$. Define $\langle \bar\xi_i \mid i < \omega \rangle, \langle B'_{\bar\xi_i} \mid i < \omega \rangle$ exactly as before and set $B'_{\tilde\lambda} = \bigcup_i B'_{\bar\xi_i}$. Then by §3.2 Lemma 6.12, $B'_{\tilde\lambda}$ is $\mathbb{B}_{\tilde\lambda}$ - generic and since $V[B'_{\bar\xi_i}] = V[B_{\bar\xi_i}]$ for $i < \omega$ and $B_{\tilde\lambda} = \bigcup_i B_{\bar\xi_i}$, we have: $H_{\tilde\lambda}[B'_{\tilde\lambda}] = H_{\tilde\lambda}[B_{\tilde\lambda}]$ where

$$\tilde\lambda = \gamma_{\tilde\lambda} = \gamma_{\tilde\lambda + 1}.$$

By § 3.2 Lemma 7.2 we conclude:
There is $\tilde{\pi} \in V$ s.t. $\tilde{\pi} : \mathbb{B}_{\tilde{\lambda}+1} \overset{\sim}{\longleftrightarrow} \mathbb{B}_{\tilde{\lambda}+1}$
and $\tilde{\pi}'' B_{\tilde{\lambda}} = B'_{\tilde{\lambda}}$. Hence $B'_{\tilde{\lambda}+1} =$
$=_{\mathrm{Df}} \tilde{\pi}'' B_{\tilde{\lambda}+1}$ is $\mathbb{B}_{\tilde{\lambda}+1}$ - generic and
$B'_{\tilde{\lambda}} \subset B_{\tilde{\lambda}+1}$. But $\tilde{\pi}$ extends to
$\pi : \mathbb{B}_\lambda \overset{\sim}{\longleftrightarrow} \mathbb{B}_\lambda$. Set: $B' = \pi'' B$.
Then B' is $\mathbb{B}_\lambda$ - generic and
$\sigma'' \bar{B}' \subset B'_{\tilde{\lambda}} \subset B'$, since $\bar{B}' = \bigcup_i \bar{B}'_{\bar{\xi}_i}$.

QED (Lemma 2).

Thus we get:

<u>Lemma 3</u> $\langle \mathbb{B}_i \mid i \leq \lambda \rangle$ satisfies (a) - (h) of § 2.3.

proof:
(h) is immediate, (a) is by Lemma 1.2, (b) is by Lemma 1.1, (c) is immediate, (d) is vacuous; (e) is by Lemma 2, (f) is by Lemma 1.3. (g) is immediate since $\mathbb{B}_\lambda / B$ is the direct limit of $\langle \mathbb{B}_{h+i} / B \mid i < \lambda - h \rangle$ whenever B is $\mathbb{B}_h$ - generic. Thus the same proof can be carried out in $V[B]$.

QED (Lemma 3)

Thus, if G is $\mathbb{B}_\kappa$-generic, then $\kappa = \omega_2$ in $V[G]$ and every regular $\tau \in (\omega_1, \kappa)$ (in V) becomes ω-cofinal in $V[G]$. Since $\mathbb{B}_\kappa = \bigcup_{i<\kappa} \mathbb{B}_i$ satisfies κ-cc, every club subset $B \subset \kappa$ in $V[G]$ will contain a club A lying in V. Hence all stationary subsets of κ will remain stationary in $V[G]$.

This completes the proof.