§2 T-premice

In the following let T be a function on active ppms N s.t. $T_N = T(N) \subset [\tau_N, \lambda_N]$ is closed in $\lambda_N + 1$ and is uniformly $J^{E^N}_{\kappa_N}$-definable in κ_N.

Def Let N be an active ppm.

$$t_N = t(N) = \begin{cases} \min\{\xi \in T_N \mid s_N \leq \xi\} \\ \quad \text{if such a } \xi \text{ exists;} \\ s_N \text{ if not} \end{cases}$$

If M is a ppm, $E^M_\nu \neq \emptyset$, we set:

$t^M(\nu) = t(\nu)^M = t^M_\nu =_{df} t_{M\|\nu}$.

Def Let N be an active ppm.

$\tilde{C}_N = \tilde{C}^T_N = \{\xi < t_N \mid \xi = t_{N_\xi}\}$

If M is a ppm, $E^M_\nu \neq \emptyset$, set:

$\tilde{C}(\nu) = \tilde{C}^t_M(\nu) =_{df} \tilde{C}_{M\|\nu}$.

Def A ppm M satisfies the *minimal T-initial segment condition* (T-MIS) iff whenever $E^M_\nu \neq \emptyset$, $N = M\|\nu$, and $\xi \in \tilde{C}_N$, then

$(J^E_{\xi^+})^N \neq (J^E_{\xi^+})^{N_\xi}$.

Def $C_N = \{\xi \in \tilde{C}_N \mid N_\xi \text{ satisfies T-MIS}\}$

Similarly for $C(\nu) = C^T_M(\nu) =_{df} C_{M\|\nu}$.

<u>Def</u> A ppm M satisfies the T-<u>initial segment condition</u> (T-IS) iff whenever $E^M_\gamma \neq \emptyset$, $N = M \| \gamma$ and $\xi \in C_N$, then

(a) $N_\xi \in N$

(b) If $\xi' \in \xi \cap C_N$, then $N_{\xi'} \in N_\xi$.

In this case we also call M a T-<u>premouse</u>

If we take $T_N = \{\lambda_N\}$, we get general premice. (For this reason we also call them λ-<u>premice</u>.) If we take $T_N = \emptyset$, we get the s-premice. However, there are many intermediate possibilities

Trivially:

<u>Lemma 1.1</u> Let N satisfy T-IS. Let $\tau_N \leq \xi < t_N$ s.t. $\xi = t_{N_\xi}$ and N_ξ satisfies T-MIS, Then N_ξ satisfies T-IS.

<u>Lemma 1.2</u> Let N satisfy T-IS. Then N satisfies T-MIS.

proof like §1 Lemma 2.2

<u>Lemma 1.3</u> (T-MIS) $\rightarrow$ MIS

prf.

Suppose not. Let $\lambda' < \lambda$ s.t. $\lambda' = \lambda_{N_{\lambda'}}$ and $(\bigcup^E_{\lambda'+})^{N_{\lambda'}} = (\bigcup^E_{\lambda'+})^N$

Clearly $\sigma_{\lambda'}(\lambda') = \lambda$, $\lambda' = \mathrm{crit}(\sigma_{\lambda'})$, so λ' is a limit cardinal in N and $\lambda' \in \mathrm{gen}_N \subset t_N$. Thus $\lambda' < t_N$.

But $t_{N_{\lambda'}} < \lambda'$, since N satisfies T-MIS. Let $\xi = t_{N_{\lambda'}}$. Then $s_{N_{\lambda'}} = s_N \cap \lambda' \subset \xi$ and hence $N_\xi = N_{\lambda'}$, $\sigma_\xi = \sigma_{\lambda'}$. But then $(J^E_{\xi+})^{N_\xi} = (J^E_{\xi+})^{N_{\lambda'}} = (J^E_{\xi+})^N$. Contr!

QED(1.3)

<u>Lemma 1.4</u> Let $\tau_N \leq \eta < t_N$ s.t. η is a cardinal in $N = \langle J^E_\nu, F\rangle$ and N is an active T-premouse. Then N_η is a T-premouse.

pf.
Let $\xi < t_{N_\eta}$ s.t. $\xi = t_{N_\xi}$ + N_ξ satisfies T-MIS. Then $\xi < s_{N_\eta} \leq \eta$, since otherwise $N_\eta = N_\xi$, $\xi = t_{N_\xi} = t_{N_\eta}$. Hence $N_\xi \in J^{E^N}_\eta \subset N_\eta$. Similarly, if $\xi' < \xi$ s.t. $\xi' = t_{N_{\xi'}}$ and $N_{\xi'}$ satisfies T-MIS, then $N_{\xi'} \in N_\xi$, since N satisfies T-IS. QED

Lemma 1.5 Every T-premouse is a premouse.

prf.

Let $N = \langle J_\gamma^E, F \rangle$ be an active T-premouse.

Let $\tau_N \leq \lambda' < \lambda_N$ s.t. $\lambda' = \lambda_{N_{\lambda'}}$.

Claim $F|\lambda' \in N$.

Clearly $\sigma_{\lambda'}(\lambda') = \lambda$, $\lambda' = \text{crit}(\sigma_{\lambda'})$.

Hence λ' is a limit cardinal in N. But, letting $\xi = t_{N_{\lambda'}}$, we have $s_{N_{\lambda'}} \leq \xi \leq \lambda'$.

Hence $N_{\lambda'} = N_\xi$, satisfies T-MIS by Lemma 2.2 and Lemma 2.4. Hence $N_{\lambda'} \in N$.

QED (Lemma 1.5)

Lemma 2.1 Let N be a ppm.

Then $\tilde{C}_N$ is closed in t_N.

proof.

Let η be a limit pt. of $\tilde{C}_N$.

Case 1 There is no $\xi \geq \eta$ s.t. $\xi \in T_{N_\eta}$

Case 1.1 $\eta = s_{N_\eta}$

Then $\eta = t_{N_\eta} = s_{N_\eta}$

Case 1.2 $s_{N_\eta} < \eta$.

Then $s_{N_\eta} = \sup(\text{gen}_N \cap \eta)$. Set $s = s_{N_\eta}$

Let $\xi \in \tilde{C}_N \cap \eta$ s.t. $s < \xi$.

Then $s_{N_\xi} = s$ and $\xi = t_{N_\xi} > s$. Hence $\xi =$ the least $\xi \in T_{N_\xi}$ s.t. $\xi > s$.

Now let $\xi < \xi' \in \tilde{C}_N \cap \eta$. Then $\xi' =$ the least $\xi \in T_{N_{\xi'}}$ s.t. $\xi > s$.

Since $\sigma_{\xi\xi'}(s) = s$ and $\cancel{\sigma}$

$\sigma_{\xi\xi'} : J_{\nu_{N_\xi}}^{E^{N_\xi}} \xrightarrow[\Sigma_\omega]{} J_{\nu_{N_{\xi'}}}^{E^{N_{\xi'}}}$, we have:

$\sigma_{\xi\xi'}(\xi) = \xi'$, where $\sigma_{\xi\xi'} \upharpoonright \xi = \mathrm{id}$.

It follows that:

$$\sigma_{\xi\eta}(\xi) \geq \sigma_{\xi'\eta}\,\sigma_{\xi\xi'}(\xi) \geq \xi'$$

for $\xi' \in \tilde{C}_N \cap \eta$, $\xi' > \xi$. Hence $\sigma_{\xi\eta}(\xi) \geq \eta$ and $\sigma_{\xi\eta}(\xi) =$ the least $\xi' > s$ s.t. $\xi' \in T_{N_\eta}$.

Contr!

<u>Case 2</u> Case 1 fails.

<u>Case 2.1</u> There are arbitrarily large $\xi \in \tilde{C}_N \cap \eta$ s.t. $\xi = \mathrm{crit}(\sigma_{\xi\eta})$.

Let $X =$ the set of such ξ.

Hence $\sigma_{\xi}(\xi) > \xi$ for $\xi \in X$, Then $X \subset gen_N$. Hence $s_{N_\eta} = \sup(gen_N \cap \eta) = \eta$.

But if $\xi \in X$ and $\eta \in rng(\sigma_\xi)$, we have: $\sigma^{-1}_{\xi\eta}(\eta) \geq \xi$ (since $\xi = crit(\sigma_\xi)$). Hence $\sigma_{\xi\eta}(\xi) \leq \eta$ and $\sigma_{\xi\eta}(\xi) \in T_{N_\eta}$.

Either $\eta = \sigma_{\xi\eta}(\xi) \in T_{N_\eta}$ for some such ξ or else η is a limit pt. of T_{N_η}. In either case $\eta \in T_{N_\eta}$ + hence $\eta = t_{N_\eta}$.

<u>Case 2.2</u> Case 2.1 fails.

For sufficiently large $\xi \in \tilde{C}_N \cap \eta$ we have $\sigma_{\xi\eta}(\xi) = \xi$. Hence $\xi \in T_{N_\eta}$. (Hence $\eta \in C_N$ by closure) But if $\xi < \xi' \in \tilde{C}_N \cap \eta$, we have: $\sigma_{\xi\xi'}(\xi) = \xi \in T_{N_{\xi'}}$, where $\xi < \xi = t_{N_{\xi'}}$. Hence $s = lub(\xi' \cap gen_N) > \xi$.

Hence $s_{N_\eta} = \text{lub}(\eta \cap \text{gen}_N) = \eta$.

Hence $\eta = t_{N_\eta}$. QED (Lemma 2.1)

Corresponding to §1 Lemma 3.2 we get:

<u>Lemma 2.2</u> Let N be an active T-premouse. Then C_N is closed in t_N.

prf.

Let $\eta < t_N$ be a limit pt. of C_N. Then $\eta \in \tilde{C}_N$ by Lemma 2.1. Repeating the proof of § Lemma 3.1 we then get: N_η is a T-premouse. QED (2.2)

<u>Lemma 2.3</u> Let $N = \langle J_\nu^E, F \rangle$ be an active T-premouse. Let $\xi \in \tilde{C}_N \setminus C_N$. Then ξ is the $\tilde{C}_N$-successor of $\eta \in C_N$, where $\eta \in \text{gen}_N$ is a limit cardinal in N.

proof. Suppose not.

Let ξ = the least counterexample.

Let μ = the unique cardinal in N s.t. $\tau_N \leq \mu \leq \xi < \mu^{+N}$. Let $\sigma = \sigma_\xi$.

(1) $\sigma(\mu) = \mu$,

since otherwise $\mu \in \text{gen}_N$ is a limit cardinal in N and $\mu = \xi$. Hence N_ξ is a T-premouse by Lemma 1.4. QED(1)

(2) $(J^E_{\xi+})^{N_\xi} \neq (J^E_{\xi+})^{N_\zeta}$ for $\xi \in \tilde{C}_N \cap \mu$.

prf.
$(J^E_{\xi+})^{N_\xi} \neq (J^E_{\xi+})^{N} = (J^E_{\xi+})^{J^{E^N}_\mu} = (J^E_{\xi+})^{N_\zeta}$
since N satisfies T-MIS and μ is a cardinal in N.

(3) Let $\mu < \gamma < \zeta$ with $\gamma \in \tilde{C}_N$. Then $\mu^{+N_\gamma} < \zeta$.

prf. Set $\tilde{\mu} = \mu^{+N_\gamma}$.
If $\tilde{\mu} \le \gamma$ there is nothing to prove, so suppose not. Then $\sigma_\gamma(\mu) = \mu$ & hence $\sigma_\gamma \upharpoonright \tilde{\mu} = id$. Hence $\sigma_\gamma(\gamma) = \gamma$. It suffices to show:

<u>Claim</u> $[\gamma, \zeta) \cap gen_N \neq \emptyset$

If $\zeta = lub(\zeta \cap gen_N)$ this is immediate. Otherwise ζ = the least $\zeta \in T_{N_\zeta}$ s.t. $\zeta \supset gen_{N_\zeta} = gen_N \cap \zeta$. But $\gamma = \sigma_{\gamma\zeta}(\gamma) \in T_\zeta$, where $\sigma_{\gamma\zeta} = \sigma_\zeta^{-1}\sigma_\gamma$, and $\gamma \supset gen_{N_\gamma} = \gamma \cap gen_N$. Hence $\zeta = \gamma < \zeta$ if $\zeta \cap gen_N \subset \gamma$. QED(3)

(4) Let $\mu < \gamma < \zeta$ s.t. $\gamma \in \tilde{C}_N$.
Then $(J^E_{\gamma+})^{N_\gamma} \neq (J^E_{\gamma+})^{N_\zeta}$.

proof.
$\mu^{+N_\gamma} = \mu^{+(J^E_{\gamma+})^{N_\gamma}} < \zeta \le \mu^{+N_\zeta}$.

QED(4)

Hence by (2), (4):

(5) $\mu \in \tilde{C}_N$ and $(J^E_{\mu+})^{N_\mu} = (J^E_{\mu+})^{N_\xi}$.

(6) ξ = the immediate successor of μ in $\tilde{C}_N$

proof. Suppose not.

Let η be least s.t. $\mu < \eta < \xi$ and $\eta \in \tilde{C}_N$.

Then $(J^E_{\mu+})^{N_\mu} = (J^E_{\mu+})^{N_\xi} \neq (J_{\mu+})^{E N_\eta}$,

since $\mu + N_\eta \leq \mathrm{crit}(\sigma_{\eta\xi}) < \xi \leq \mu + N_\xi$.

It follows as in (2) that

$(J^E_{\zeta+})^{N_\zeta} \neq (J^E_{\zeta+})^{N_\eta}$ for $\zeta \in \tilde{C}_N \cap \mu$.

Hence η satisfies T-MIS. Hence

$N_\eta \in N$ and N_μ. Moreover,

$N_\mu \in N_\eta$, since N_μ is a T-premouse

by Lemma 1.4. Hence $\mu + N_\mu < \mu + N_\eta$

since $\bar{\bar{N}}_\mu \leq \mu$ in N_η. Hence

$(J^E_{\mu+})^{N_\mu} \neq (J^E_{\mu+})^{N_\xi}$, since $\mu + N_\xi \geq \xi > \eta$

Contr! QED(6)

(7) $\mu = \mathrm{crit}(\sigma_{\mu\xi})$

proof. Suppose not.

Then $\mathrm{crit}(\sigma_{\mu\xi}) \geq \mu + N_\mu$. But then

$\mu + N_\xi \geq \xi > \mathrm{crit}(\sigma_{\mu\xi}) \geq \mu + N_\mu$. Contr!,

by (5).

But then $\mu = \mathrm{crit}(\sigma_\mu)$. It follows that $\mu \in \mathrm{gen}_N$ and μ is a limit cardinal in N. Hence $\mu \in C_N$. QED (Lemma 2.3)

As a corollary of the proof:

Cor 2.4 Let ξ, η be as in Lemma 2.3. Then $\eta^{+N_\eta} = \eta^{+N_\xi}$ and $\eta < \xi \leq \eta^{+N_\xi}$.

(Hence if $\xi \in T_{N_\xi}$ and T is s.t. T_N is always a class of limit cardinals in N, then the situation of Lemma 2.3 cannot occur.)

Def Let $N = \langle J_\gamma^E, F \rangle$ be an active T-premouse.

N is of type 1 iff $C_N = \emptyset$

N is of type 2 iff C_N has a maximum

N is of type 3 iff $\sup C_N = t_N$

(Note By Lemma 2.3, every limit pt of $\tilde{C}_N$ is a limit pt. of C_N.)

Lemma 2.5 Let N be of type 3. Then $\omega\rho_N^1 = t_N = s_N$. (prf. like §1 Lemma 3.4)

Lemma 2.6 Let N, N' be active ppms. Let $\sigma: N \xrightarrow{\Sigma_0} N$ s.t. $\sup \sigma'' s_N \leq s_{N'} \leq \sigma(s_N)$. Then $\sup \sigma'' t_N \leq t_{N'} \leq \sigma(t_N)$. Moreover $\sigma(t_N) = t_{N'}$ if $s_N < t_N$.

We first prove two auxiliary lemmas.

<u>Lemma 2.6.1</u> Let N, N' be active ppms. Let $\sigma : N \xrightarrow{}_{\Sigma_0} N'$. Then $\sigma : J^{E^N}_{\nu_N} \xrightarrow{}_{\Sigma_\omega} J^{E^{N'}}_{\nu_{N'}}$.

prf.

Let $N = \langle J^E_\nu, F \rangle$, $N' = \langle J^{E'}_{\nu'}, F' \rangle$.

Let $\pi : N \xrightarrow{}_F \tilde{N}$, $\pi' : N' \xrightarrow{}_F \tilde{N}'$.

There is $\tilde{\sigma} : \tilde{N} \xrightarrow{}_{\Sigma_0} \tilde{N}'$ defined by:

$\tilde{\sigma}(\pi(f)(\alpha)) = \pi'(\sigma(f))(\sigma(\alpha))$. The usual methods then show: $\tilde{\sigma}(\nu) = \nu'$, $\tilde{\sigma} \upharpoonright J^E_\nu = \sigma$.

QED (2.6.1)

(<u>Note</u> The proof of 2.6.1 does not require the well foundedness of $\tilde{N}, \tilde{N}'$.)

<u>Lemma 2.6.2</u> Let N, N', σ' be as above. Let $\xi \leq \lambda_N$, $\eta \in N$ s.t. $\xi < \eta$ and $T_N \cap [\xi, \eta) = \emptyset$.

Let $\xi' = \text{lub}\, \sigma''\xi$, $\eta' = \sigma(\eta)$. Then $T_{N'} \cap [\xi', \eta') = \emptyset$.

proof. Suppose not.

For each $\varsigma < \xi$ we have:

$N' \models \bigvee \delta \in T\, (\sigma(\varsigma) < \delta < \eta')$.

Hence the same statement holds of ς, η in N. Hence there is $\delta \in T_N$ s.t. $\varsigma < \delta < \eta$. Hence $\varsigma < \xi$. Hence ξ is a limit point of T_N. Hence $\xi \in T_N \cap [\xi, \eta)$. Contr! QED

The proof of Lemma 2.6. is immediate. If $T_N \cap [s_N, \lambda_N + 1) = \emptyset$, then $T_{N'} \cap [\sup \sigma'' s_N, \lambda_{N'} + 1) = \emptyset$ + hence $t_{N'} = s_{N'}$. Otherwise $\sigma(t_N) \in \in T_{N'}$. If $s_N < t_N$, then $T_N \cap [s_N, t_N) = \emptyset$ + hence $T_{N'} \cap [\sup \sigma'' s_N, \sigma(t_N)) = \emptyset$. Thus $T_{N'} \cap [s_{N'}, \sigma(t_N)) = \emptyset$. Hence $\sigma(t_N) = t_{N'}$. Now let $s_N = t_N$. Then $s_{N'} \leq \sigma(t_N) \in T_{N'}$. Hence $t_{N'} \leq \sigma(t_N)$. But $\sup \sigma'' t_N = \sup \sigma'' s_N \leq s_{N'} \leq t_{N'}$.

QED (Lemma 2.6)

As a pendant to Lemma 2.6.2 we get:

<u>Lemma 2.6.3</u> Let N, N', σ be as in Lemma 2.6. Let $\xi' \leq \lambda_{N'}$, $\xi' < \eta' \in N'$ s.t. $T_{N'} \cap [\xi', \eta') = \emptyset$. Set: $\xi = \text{lub}\, \sigma^{-1}{}'' \xi'$. Let $\eta' = \sigma(\eta)$. Then $T_N \cap [\xi, \eta) = \emptyset$.

The proof is left to the reader.

Def $C^*_N = \{\xi \mid N_\xi \neq N \wedge \xi \in T_{N_\xi}\}$ for active ppms N.

Note: It is clear that:

- $C^*_N \subset s_N$, since $N_\xi = N$ for $s_N \leq \xi$
- If $\xi \in \tilde{C}_N$ and $\forall \eta \geq \xi \;\; \eta \in T_N$, then $\xi \in C^*_N$
- If $\xi \in C^*_N$, $s = s_{N_\xi}$, then there is $\eta \leq \xi$ s.t. $s = s_{N_\eta}$ and $\eta \in \tilde{C}_N$.

Lemma 2.6.4 Let $\sigma : N \to_{\Sigma_0} N'$, where N, N' are active ppms. If $C^*_N \cap [\eta, \mu) = \emptyset$, then $C^*_{N'} \cap [\sigma(\eta), \sigma(\mu)) = \emptyset$.

proof. Suppose not

Let $\xi' \in C^*_{N'} \cap [\eta', \mu')$ where $\eta' = \sigma(\eta)$, $\mu' = \sigma(\mu)$. Set $\xi = \text{lub}\, \sigma^{-1}{}''\xi'$. Then $\xi \leq \sigma(\xi')$. Define $\tilde{\sigma} : N_\xi \to_{\Sigma_0} N'_{\xi'}$ by:

$\tilde{\sigma}(\pi(f)(\alpha)) = \pi' \sigma(f)(\sigma(\alpha))$, where $f \in N \cap ({}^n\kappa)$, $\kappa = \kappa_N$, $\alpha < \xi$, and

$\pi : J^E_\tau \xrightarrow[F|\xi]{} J^{E^{N_\xi}}_{\nu_{N_\xi}}$, $\pi' : J^E_\tau \xrightarrow[F|\xi]{} J^{E^{N'_{\xi'}}}_{\nu_{N'_{\xi'}}}$.

Then $\tilde{\sigma} \restriction \xi = \text{id}$. Set:

$\tilde{\xi} = \sup \sigma''\xi$. Then $\tilde{\xi} \leq \xi \leq \sigma(\xi)$, Since $T_{N_\xi} \cap [\xi, \xi+1) = \emptyset$, we have $T_{N'_{\xi'}} \cap [\tilde{\xi}, \sigma(\xi)+1) = \emptyset$. Hence $\xi' \notin T_{N'_{\xi'}}$. Contr! QED (2.6.4)

<u>Lemma 3.1</u> Let N be of type 1. Let $\sigma : N \longrightarrow_G N'$, where $crit(G) < t_N$. Then N' is of type 1 and $\sigma(t_N) = t_{N'}$.

prf.

<u>Case 1</u> $T_N = \emptyset$. Then $T_{N'} = \emptyset$ by Lemma 2.6. Hence $t_N = s_N = \tau_N$ and $t_{N'} = s_{N'} = \tau_{N'}$
$= \sigma(\tau_N)$.

<u>Case 2</u> $T_N \neq \emptyset$. Then $t_N = \min T_N$. Then $C_N = \emptyset$ and by Lemma 2.3 $\tilde{C}_N = \emptyset$. Hence $C^*_N = \emptyset$. Hence $C^*_{N'} = \emptyset$ by Lemma 2.6.4, where $C_{N'} \subset \tilde{C}_{N'} \subset C^*_{N'}$. Hence N' is of type 2 and $t_{N'} = \min T_{N'} = \sigma(t_N)$, since $T_{N'} \cap \sigma(t_N) = \emptyset$ by Lemma 2.6.2.

QED (3.1)

Similarly:

<u>Lemma 3.2</u> Let N be of type 1. Let $\sigma : N \longrightarrow^*_G N'$ where $crit(G) < t_N$. Then N' is of type 1 and $\sigma(t_{N'}) = t_N$.

<u>Lemma 3.3</u> Let N be of type 2. Let $\sigma : N \longrightarrow_G N'$, where $\mathrm{crit}(G) < t_N$. Then N is of type 2 and $\sigma(t_N) = t_{N'}$. Moreover, if $\xi = \max C_N$, $\xi' = \max C_{N'}$, then $\sigma(\xi) = \xi'$ and $\sigma(N_\xi) = N'_{\xi'}$.

proof.

<u>Claim 1</u> $\sigma(t_N) = t_{N'}$

<u>Case 1</u> $T_N \setminus s_N = \emptyset$ (hence $t_N = s_N$)

Then $s_{N'} = \sup \sigma '' s_N$ and $T_{N'} \setminus s_{N'} = \emptyset$ by Lemma 2.6.2. It suffices to show:

<u>Claim</u> $s_N = \delta + 1$ for some δ,

since then $s_{N'} = \sigma(\delta) + 1 = \sigma(s_N)$.

Clearly $s_N > t_N$, since otherwise N has type 1. Suppose $\mathrm{Lim}(s_N)$. Then $T_N \setminus \gamma = \emptyset$ for a $\gamma < s_N$, since otherwise s_N is a limit pt. of T_N; hence $s_N \in T_N$. Contr!

QED (.

<u>Case 2</u> Case 1 fails.

(1) $T_N \cap [\delta, t_N) = \emptyset$ for a $\delta < s_N$.

Suppose not. Then s_N is a limit in T_N + hence $s_N \in T_N$. Hence $s_N = t_N$. But then $\sup \tilde{C}_N = t_N$. To see this let $\rho_0 < s_N$ and set:

ρ_1 = the least $\rho > \rho_0$ s.t. $\rho \in \mathrm{you}_N$

ρ = " " $\rho > \rho_1$ s.t. $\rho \in T_N$

Then $\varsigma < t_N$. But $\sigma_\varsigma(\varsigma) = \varsigma$, since otherwise $T_{N_\varsigma} \cap [\varsigma_1, \varsigma) = \emptyset$ + hence $T_N \cap [\varsigma_1, \sigma_\varsigma(\varsigma)) = \emptyset$, by Lemma 2.6.2, where $\varsigma < \sigma_\varsigma(\varsigma)$. Hence $\varsigma \notin T_N$. Contr!

Since only successors of limit pts in $\tilde{C}_N$ can fail to be in C_N, we have $\sup C_N = t_N$. Hence N is of type 3, Contr! QED (1)

But then $T_{N'} \cap [\sigma(\delta), \sigma(t_N)) = \emptyset$, where $\sigma(\delta) < s_{N'}$ and $\sigma(t_N) \in T_{N'}$. Hence $\sigma(t_N) = t_{N'}$. QED (Claim 1)

<u>Claim 2</u> Let $\xi = \max C_N$. Then $\sigma(\xi) \in C_{N'}$ and $\sigma(N_\xi) = N'_{\sigma(\xi)}$.

prf.

Since $N_\xi = (N_\xi)_\xi$ we have: $\sigma(N_\xi) = \sigma(N_\xi)|$.

In N we have:

$$\bigwedge \alpha < \xi \bigwedge x \in \mathcal{P}(n) \, (\alpha \in F(x) \longleftrightarrow \alpha \in F_{N_\xi}(x)),$$

where $N = \langle J^E_\gamma, F \rangle$, $n = \kappa_N$. Hence the same Π_1 statement holds of $\sigma(N_\xi)$ in N'. Hence $\sigma(N_\xi) = N'_{\sigma(\xi)}$. But $\sigma(\xi) = t_{\sigma(N_\xi)}$, since $\xi = t_{N_\xi}$. Hence $\sigma(\xi) \in C_{N'}$. QED (Claim 2

<u>Claim 3</u> Let $\xi = \max C_N$. Then $\sigma(\xi) = \max C_{N'}$ and N' is a T-premouse.

proof. Set $\xi' = \sigma(\xi)$.

<u>Case 1</u> $\xi = \max \tilde{C}_N$.

<u>Case 1.1</u> $T_N \setminus (\xi + 1) = \emptyset$. Then $T_{N'} \setminus (\xi' + 1) = \emptyset$. $s_N = \delta + 1$, where $\delta = \mathrm{crit}(\sigma_\xi^N)$, since otherwise $\delta + 1 \in \tilde{C}_N$. (<u>Note</u> It is easily seen that $\delta = \xi$ or ξ^{+N_ξ}.) Clearly $\xi' = t_{N_{\xi'}'}$, $\delta' = \mathrm{crit}(\sigma_{\xi'}^{N'})$, where $\delta' = \sigma(\delta)$. $t_{N'} = s_{N'} = \sup \sigma'' s_N = \delta' + 1$. If $\delta = \xi$, then $\delta' = \xi'$ and hence $\xi' = \max \tilde{C}_{N'}$. Hence for all $\mu \in \tilde{C}_{N'}$ $(N'_\mu) = (N'_{\xi'})_\mu \in N'$ and N' is a T-premouse. Now let $\delta = \xi^{+N_\xi}$. Then $\delta' = \xi'^{+N'_{\xi'}}$. But then
$$\delta' \leq \xi'^{+N'_{\xi'+1}} \leq \xi'^{+N'_{\delta'}} = \delta',$$
since $\delta' \in \mathrm{gen}_{N'}$. Hence $\sigma_{\xi'+1}^{N'} \upharpoonright \delta' = \mathrm{id}$, and it follows that $(\xi', \delta') \cap \mathrm{gen}_{N'} = \emptyset$. Hence $\tilde{C}_{N'} \cap (\xi', \delta'] \cap \mathrm{gen}_{N'} = \emptyset$, since if $\mu \in \tilde{C}_{N'} \cap (\xi', \delta']$, then

$\mu = \text{lub}(\mu \cap \text{gen}_{N'})$, Contr! Hence $\xi' = \max \tilde{C}_{N'}$ and we conclude as before that N' is a T-premouse.
QED (Case 1.1)

<u>Case 1.2</u> Case 1.1 fails.

Then $t_N \in T_N$ and $C^*_N \cap (\xi, t_N) = \emptyset$. Hence $t_{N'} \in T_{N'}$ and $C^*_{N'} \cap (\xi', t_{N'}) = \emptyset$, where $\tilde{C}_{N'} \subset C^*_{N'}$. Hence $\xi' = \max \tilde{C}_{N'}$ and it follows as before that N' is a T-premouse. QED (Case 1.2)

<u>Case 2</u> Case 1 fails.

Then $\xi \in \text{gen}_N$ is a limit cardinal in N and $\varrho = \max \hat{C}_N$ is the next largest element of $\hat{C}_N$. Moreover $\varrho \leq \delta$ where $\delta = \xi^{+N_\xi}$. But $(J^E_{\xi^+})^{N_\varrho} = (J^E_{\xi^+})^{N_\xi}$. Since $\sigma^N_\varrho(\xi) = \xi$ we then have $\delta = \text{crit}(\sigma^N_\varrho)$. But $\delta = \xi^{+N_\xi} \leq \xi^{+N_\mu} \leq \xi^{+N_\delta} = \delta$ for all $\mu \in [\xi, \delta]$. Hence $(\xi, \delta) \cap \text{gen}_N = \emptyset$. We know that, letting $\varrho', \delta' = \sigma(\varrho, \delta)$, we have $(J^E_{\xi^+})^{N'_{\xi'}} = (J^E_{\delta'})$ and $\delta' \in \text{gen}_{N'}$, since $\delta \in \text{gen}_N$.

~~Since~~ Hence $\delta' = \mathrm{crit}(\sigma_{\delta'}^{N'})$, since $\sigma_{\delta'}^{N'}(\xi') = \xi'$. Hence

$$\delta' = \xi'^{+N'_\xi} \leq \xi^{+N'_\mu} \leq \xi^{+N'_{\delta'}} = \delta'$$

for all $\mu \in [\xi', \delta']$. Hence $\mathrm{gen}_{N'} \cap (\xi', \delta') = \emptyset$ and $\delta' = \mathrm{crit}(\sigma_\mu^{N'})$ for $\xi' < \mu \leq \delta'$; hence

$$(J^E_{\xi'^+})^{N'_\mu} = J^{E^{N'_{\xi'}}}_{\delta'} = (J^E_{\xi^+})^{N_{\xi'}}$$

for $\mu' \in (\xi', \delta']$.

<u>Case 2.1</u> $T_N \setminus (\xi+1) = \emptyset$

Then $t_N = s_N = \delta+1$, since otherwise $\delta+1 \in \tilde{C}_1$

Clearly $T_{N'} \setminus (\xi'+1) = \emptyset$. Hence $t_{N'} = s_{N'} = \sup \sigma'' s_N = \delta'+1$. Moreover $\varsigma = \xi+1$ and $\varsigma' = \xi'+1$. Hence $\varsigma' \in \tilde{C}_N$ and $\varsigma' \notin C_{N'}$ by the above. Since $(\xi', \delta') \cap \mathrm{gen}_{N'} = \emptyset$, we have $\varsigma' = \max \tilde{C}_{N'}$. Hence for each $\mu \in \tilde{C}_{N'}$, either $\mu = \varsigma'$ and $N'_{\varsigma'}$ does not satisfy T-MIS, or else $N'_{\mu'} = (N'_{\xi'})_{\mu'} \in N'$.

Hence N' is a T-premouse. QED (2.1)

<u>Case 2.2</u> Case 1 fails

Then $\varrho' = \sigma(\varrho) = t_{N'_{\xi'+1}}$ = the least $\varrho > \xi'$ s.t. $\varrho' \in T_{N'_{\xi'+1}}$.

(If $\varrho < \delta$, then ϱ = the least $\varrho \in T_N$ s.t. $\varrho > \xi$ and ϱ' = the least $\varrho' \in T_{N'}$ s.t. $\varrho' > \xi'$. Hence $\varrho' = t_{N'_{\xi'+1}}$, since $\varrho' < \operatorname{crit}(\sigma^{N'}_{\xi'+1})$. If $\varrho = \delta$, then $\sigma_\varrho(\delta) = \xi + N$ = the least $\varrho \in T_N$ s.t. $\varrho > \xi$. Hence $\xi' + N' = \sigma(\xi + N) =$ = the least $\varrho \in T_N$ s.t. $\varrho > \xi'$. Hence $\varrho' = \delta' = \xi' + N'_{\xi'+1} = (\sigma^{N'}_{\xi'+1})^{-1}(\xi' + N') =$ = the least $\varrho' \in T_{N'_{\xi'+1}}$ s.t. $\varrho' > \xi' =$ $= t_{N'_{\xi'+1}}$.)

But then $\varrho' \in \tilde{C}_{N'}$, $\varrho' \notin C_{N'}$.

<u>Case 2.2.1</u> $T_N \setminus (\varrho + 1) = \emptyset$.

Then $T_{N'} \setminus (\varrho' + 1) = \emptyset$. As before, $t_N = s_N = \delta + 1$, $t_{N'} = s_{N'} = \delta' + 1$, hence $\sigma(t_N) = t_{N'}$. But $\mathrm{gen}_{N'} \cap (\xi', \delta') = \emptyset$. Hence $\tilde{C}_{N'} \setminus (\varrho' + 1) = \emptyset$, since if $\mu \in \tilde{C}_{N'} \setminus (\varrho' + 1)$, then $\mu = \mathrm{lub}(\mu \cap \mathrm{gen}_{N'})$. But $\tilde{C}_{N'} \cap (\xi', \varrho') = \emptyset$, since if $\mu \in \tilde{C}_{N'} \cap (\xi', \varrho')$, then $N'_\mu = N'_{\varrho'}$

and $t_{N'_\mu} = t_{N'_{\xi'}} = \xi'$. Hence $\zeta' = \max C_{N'}$ and for $\mu \in \tilde{C}_{N'}$ we have either $\mu = \xi'$ and N'_μ does not satisfy T-MIS, or else $N'_\mu = (N'_{\zeta'})_\mu \in N'$. Hence N' is a T-premouse.

QED (2.2.1)

<u>Case 2.2.2</u> Case 2.2.1 fails.

Then $t_N \in T_N$ and $T_N \cap (\xi, t_N) = \emptyset$, since if $\mu = \min(T_N \cap (\xi, t_N))$, then $\mu \in \tilde{C}_N$ by the usual argument. ($\sigma_\mu(\mu) = \mu$ since otherwise $\mu < \sigma_\mu(\mu)$ and $T_N \cap (\xi, \sigma_\mu(\mu)) = \emptyset$.) Hence $T_{N'} \cap (\xi', t_{N'}) = \emptyset$. But $\xi' \leq \delta' \in \mathrm{gen}_{N'} \subset \alpha_{N'}$. Hence $t_{N'} =$ (the least $t \in T_{N'}$ s.t. $t > \delta'$) $=$ $= \sigma(t_N)$. $\tilde{C}_{N'} \cap (\zeta', \xi') = \emptyset$ follows as in Case 2.2.1. Similarly for $\tilde{C}_{N'} \cap (\xi', \delta'] = \emptyset$. But $C^*_N \cap (\delta, t_N) = \emptyset$, since if $\mu =$ $= \min(C^*_N \cap (\delta, t_N))$, then $\mu \in \tilde{C}_N$. Hence $C^*_{N'} \cap (\delta', t_{N'}) = \emptyset$. Hence $\zeta' = \max C_{N'}$ and, as above, N' is a T-premouse.

QED (Lemma 3.3)

Similarly:

<u>Lemma 3.4</u> Let $\sigma : N \xrightarrow[G]{*} N'$, where N is of type 2 and $\mathrm{crit}(\sigma) < t_N$. Then N' is of type 2, $\sigma(t_N) = t_{N'}$ and $\sigma(\zeta) = \zeta'$, $\sigma(N'_\zeta) = N'_{\zeta'}$, where $\zeta = \max C_N$, $\zeta' = \max C_{N'}$.

Lemma 3.5 Let $N = \langle J_\nu^E, F\rangle$ be of type 3. Let $t = t_N$ (hence $t = s_N = \omega\rho_N^1$). Let $\delta : J_t^E \longrightarrow_G J_{t'}^{E'}$, where G is an extender on J_ρ^E. Let $\delta' : N \longrightarrow_{\Sigma_0^{(n)}} N'$ be the canonical completion of δ. Then N' is a T-premouse of type 3 and $t' = t_{N'}$. Moreover, $\delta' : N \longrightarrow_G N'$

proof.

As in §1 Lemma 4.5 we get:

(1) $\quad t' = s_{N'} = \omega\rho^1_{N'}$

(2) $\xi < t' \longrightarrow N'_\xi \in N'$

(3) $\xi < t \longrightarrow \delta(N_\xi) = N'_{\delta(\xi)}$.

But there are arb. large $\xi < t$ s.t. N_ξ is a T-promouse and $\xi \in C_N$ - i.e. N_ξ is a T-premouse $\xi = t_{N_\xi}$. Hence $N'_{\delta(\xi)}$ is a T-premouse and $\delta(\xi) = t_{N'_{\delta(\xi)}}$. Hence $\delta(\xi) \in C_N$ and $C_{N'}$ is cofinal in t'. But $C_{N'}$ is closed in $t_{N'}$ and $C_{N'} \setminus t' = \emptyset$, since $t' = s_{N'}$. Hence $t' = t_{N'}$ QED (3.5)

(Note Using the fact that $\delta'' \tau_N$ is cofinal in $\tau_{N'}$, we can, in fact, show that $\delta' : N \longrightarrow_G N'$.)

<u>Lemma 3.6</u> Let $N = \langle J_\nu^E, F\rangle$ be of type 3. Let $\sigma : N \xrightarrow[\sigma]{*} N'$ where $\mathrm{crit}(\sigma) < t_N$. Then N' is of type 3. Moreover,

$$t_{N'} = \begin{cases} \sup \sigma'' t_N & \text{if } \mathrm{crit}(\sigma) \geq \omega\rho_N^2 \\ \sigma(t_N) & \text{if not} \end{cases}$$

proof.

If $\mathrm{crit}(\sigma) \geq \omega\rho_N^2$, this follows by Lemma 3.

Now let $\mathrm{crit}(\sigma) < \omega\rho_N^2$. It follows as in Lemma 4.6 of §1 that $N_\xi \in N'$ for $\xi < t' = \sigma(t_N)$. Hence $t' = s_{N'} = \omega\rho_{N'}^1$, as before. The statement:

$Q = N'_\xi$ is uniformly $\Pi_1(N')$ in Q, ξ, since

$Q = N'_\xi$ iff Q is a ppm $\wedge$ $\xi \geq s_Q$ $\wedge$ $\underbrace{F_N|\xi = F_Q|\xi}_{\Pi_1(N)}$

Since $N'_\xi \in J^{E^{N'}}_{\rho^1_{N'}}$ for $\xi < t' = \omega\rho^1_{N'}$,

$\{\langle Q, \xi\rangle \mid Q = N'_\xi\}$ is rudimentary in $N'^{(1)} = \langle J^{E^{N'}}_{\rho^1_{N'}}, A^1_{N'}\rangle$. (Note: Since $s_{N'} = \omega\rho^1_{N'}$, $A^1_{N'}$ involves no parameter.) $\mathrm{lub}\, C_{N'} = t'$ is then expressible over $N'^{(1)}$ by the Π_2 statement:

$\bigwedge \eta \bigvee \xi > \eta\, (N'_\xi$ is a J-premouse $\wedge$ $\xi = t_{N'_\xi})$.

But the corresponding statement holds in $N^{(1)}$, where $\sigma' : N^{(1)} \xrightarrow[\Sigma_2]{} N'^{(1)}$.

QED (3.6)

It is apparent from the above proofs that $\{\langle Q, \zeta\rangle \mid Q = N_\zeta\}$ is uniformly $\Pi_1(N)$ for active ppms N. Hence the set: $\hat{C}_N = \{\zeta \mid N_\zeta \in N\}$ is uniformly $\Sigma_1(N)$. Hence $\sigma''\hat{C}_{\bar{N}} \subset \hat{C}_N$ if $\sigma : \bar{N} \xrightarrow[\Sigma_1]{} N$.

<u>Lemma 4.1</u> Let $\sigma : \bar{N} \xrightarrow[\Sigma_1]{} N$ where N is of type 1. Then $\bar{N}$ is of type 1 and $\sigma(t_{\bar{N}}) = t_N$.

proof.

<u>Case 1</u> $T_N = \emptyset$

Then $T_{\bar{N}} = \sigma^{-1}{}'' T_N = \emptyset$. Moreover $gen_N = \emptyset$ and hence $gen_{\bar{N}} = \sigma^{-1}{}'' gen_N = \emptyset$. Hence $t_N = s_N = \tau_N$, $t_{\bar{N}} = s_{\bar{N}} = \tau_{\bar{N}}$, where $\sigma(\tau_{\bar{N}}) = \tau_N$. Moreover, $\tilde{C}_{\bar{N}} = \emptyset$, since if $\zeta \in \tilde{C}_{\bar{N}}$, then $T_{\bar{N}_\zeta} = \emptyset$ & hence $\zeta = \sup(gen_{\bar{N}} \cap \zeta) = 0$, Contr! Hence $\bar{N}$ is a T-premouse vacuously.

<u>Case 2</u> Case 1 fails.

Then $t_N = \min T_N$, since otherwise $\zeta \in C_N$, where $\zeta = \min T_N$. Hence $\sigma(\bar{t}) = t$, where $\bar{t} = \min T_{\bar{N}}$. Moreover $gen_{\bar{N}} = \sigma^{-1}{}'' gen_N \subset \bar{t}$. Hence $\bar{t} = t_{\bar{N}}$. It remains only to show:

<u>Claim</u> $\tilde{C}_{\bar{N}} = \emptyset$.

Suppose not. Let $\zeta \in \tilde{C}_{\bar{N}}$. Then

$\bar{\zeta} = t_{\bar{N}_{\bar{\zeta}}} = \min T_{\bar{N}_{\bar{\zeta}}}$ and hence $\sigma_{\bar{\zeta}}^{\bar{N}}(\bar{\zeta}) = t_{\bar{N}} = \min T_{\bar{N}}$.

Set $\zeta = \sigma(\bar{\zeta})$. Then $\zeta \in \mathrm{gen}_N$, since $\bar{\zeta} \in \mathrm{gen}_{\bar{N}}$

But $\sigma_{\zeta}^{N}(\zeta) < t_N$, since otherwise $\sigma_{\zeta}^{N}(\zeta) = t_N$ and hence $\zeta = \min T_{N_\zeta} = t_{N_\zeta}$. But $\tilde{C}_{N_\zeta} = \zeta \cap \tilde{C}_N = \emptyset$. Hence N_ζ satisfies T-MIS vacuously. Hence $\zeta \in C_N$, $\zeta < t_N$. Contr!

Since $\sigma_{\zeta}^{N}(\zeta) < t_N$, there exist $\eta < t_N$, $f \in (\kappa_n) \cap N$, $\alpha < \zeta$ s.t.

$$\{\langle \beta, \gamma \rangle < n \mid f(\beta) = \gamma\} \in F_{\langle \alpha, \eta \rangle}.$$

This is a $\Sigma_1(N)$ statement about ζ. Hence the same $\Sigma_1(\bar{N})$ statement holds of $\bar{\zeta}$. Hence $\sigma_{\bar{\zeta}}^{\bar{N}}(\bar{\zeta}) < t_{\bar{N}}$. Contr!

QED (Lemma 4.1)

<u>Lemma 4.2</u> Let N be of type 2. Let $\sigma : \bar N \longrightarrow_{\Sigma_1} N$ s.t. $N_\xi \in \mathrm{rng}(\sigma)$, where $\xi = \max C_N$. Then $\bar N$ is of type 2 and $\sigma(t_{\bar N}) = t_N$. Moreover $\sigma(\bar\xi) = \xi$, $\sigma(\bar N_{\bar\xi}) = N_\xi$ where $\bar\xi = \max C_{\bar N}$.

prf.

<u>Case 1</u> $\xi = \max \tilde C_N$.

<u>Case 1.1</u> $T_N \setminus (\xi+1) = \emptyset$. Then $T_{\bar N} \setminus (\bar\xi + 1) = \emptyset$. $s_N = \delta + 1$, where $\delta = \mathrm{crit}(\sigma_\xi)$; since otherwise $\delta + 1 \in \tilde C_N$, $\delta+1 > \xi$. (Note: It is easily seen that $\delta = \xi$ or $\xi + N_\xi$.)

$\sigma(\bar\xi) = \xi$ where $\bar\xi = t_{\sigma^{-1}(N_\xi)}$. Moreover, $\sigma(\bar\delta) = \delta$, where $\bar\delta = \bar\xi$ if $\delta = \xi$; $\bar\delta = \bar\xi + \sigma^{-1}(N_\xi)$ if $\delta = \xi + N_\xi$. Clearly $\sigma^{-1}(N_\xi) = \bar N_{\bar\xi}$, since $Q = N_\xi$ is a $\Pi_1(N)$ condition, $s_{\bar N} = \sigma^{-1}$"s_N $= \bar\delta + 1$.

Hence $t_{\bar N} = s_{\bar N} = \sigma^{-1}(t_N)$

$\bar\xi \in C_{\bar N}$, since $\bar\xi = t_{\bar N_{\bar\xi}}$. If $\xi = \delta$, then $\bar\xi \in \mathrm{gen}_{\bar N}$, since $\xi \in \mathrm{gen}_N$. Hence $\bar\xi = \mathrm{crit}(\sigma_{\bar\xi}^{\bar N})$. If $\xi < \delta$, then $\bar\xi \notin \mathrm{gen}_{\bar N}$; hence $\bar\xi < \bar\delta \in \mathrm{gen}_{\bar N}$.

Hence $\sigma_{\bar{\xi}}^{\bar{N}}(\bar{\xi}) = \bar{\xi}$. Hence $\sigma_{\bar{\xi}}^{\bar{N}} \upharpoonright \bar{\xi}^{+\bar{N}_\xi} = \mathrm{id}$. Hence $\bar{\delta} = \mathrm{crit}(\sigma_{\bar{\xi}}^{\bar{N}}) =$ the least $\bar{\delta} \in \mathrm{gen}_{\bar{N}}$ s.t. $\bar{\delta} \geq \bar{\xi}$. Hence $\bar{\xi} = \max \tilde{C}_{\bar{N}}$, since if $\bar{\xi} < \xi \in \tilde{C}_{\bar{N}}$ we would have: $\xi = \mathrm{lub}(\xi \cap \mathrm{gen}_{\bar{N}})$ (since $T_{\bar{N}} \setminus \xi = \emptyset$). Hence for all $\xi \in \tilde{C}_{\bar{N}}$ we have: $\bar{N}_\xi = (\bar{N}_{\bar{\xi}})_\xi \in \bar{N}$. Hence $\bar{N}$ is a T-premouse.

<u>Case 1.2</u> Case 1.1 fails.

It follows as before that: $\xi \in C_{\bar{N}}$, $\sigma(\bar{N}_{\bar{\xi}}) = N_\xi$, $\sigma(\bar{\delta}) = \delta$, where $\bar{\delta} = \mathrm{crit}(\sigma_{\bar{N}_{\bar{\xi}}})$ and $\delta = \mathrm{crit}(\sigma_{N_\xi})$. Clearly $T_N \cap (\delta, T_N) = \emptyset$, since otherwise, letting $\xi =$ the least $\xi \in T_N \setminus \delta + 1$, we have $\sigma_\xi^N(\xi) = \xi$ (otherwise $\xi < \sigma_\xi^N(\xi) = \min T_N \setminus \delta + 1$), hence $\xi = t_{N_\xi}$ and $\xi \in \tilde{C}_N$. Contr!

But then $T_{\bar{N}} \cap (\bar{\delta}, \bar{t}) = \emptyset$, where $\sigma(\bar{t}) = t_{\bar{N}}$ and $\bar{\delta} < s_{\bar{N}}$. Hence $\bar{t} = t_{\bar{N}}$. But $\tilde{C}_{\bar{N}} \setminus \bar{\xi} + 1 = \emptyset$ by the argument of Lemma 4.1, since $\tilde{C}_N \setminus \xi + 1 = \emptyset$. Hence $\bar{\xi} = \max \tilde{C}_{\bar{N}} = \max C_{\bar{N}}$ and $\bar{N}_\xi = (\bar{N}_{\bar{\xi}})_\xi \in N$ for all $\xi \in \tilde{C}_{\bar{N}}$.

QED (Case 1)

Case 2 Case 1 fails.

Then $\xi \in \mathrm{gen}_N$ is a limit cardinal in N and $\varrho = \max \tilde{C}_N$ is the next largest element of $\tilde{C}_N$. Moreover $\varrho \leq \delta$, where $\delta = \xi^{+N_\xi}$. But $(J^E_{\xi^+})^{N_\varrho} = (J^E_{\xi^+})^{N_\xi}$.

Hence $\delta = \mathrm{crit}(\sigma_\varrho^N)$, since $\sigma_\varrho^N(\bar{\xi}) = \xi$.
Hence $\delta = \mathrm{crit}(\sigma_\delta^N)$. But:

$$\delta = \xi^{+N_\xi} \leq \xi^{+N_\mu} \leq \xi^{+N_\delta} = \delta$$

for all $\mu \in [\xi, \delta]$. Hence $(\xi, \delta) \cap \mathrm{gen}_N = \emptyset$.
As before: $\bar{\xi} \in C_{\bar{N}}$ and $\sigma(\bar{N}_{\bar{\xi}}) = N_\xi$, where $\sigma(\bar{\xi}) = \xi$. Moreover $\sigma(\bar{\delta}) = \delta$, where $\bar{\delta} = \bar{\xi}^{+\bar{N}_{\bar{\xi}}}$. $\bar{\xi}, \bar{\delta} \in \mathrm{gen}_{\bar{N}}$ and $(\bar{\xi}, \bar{\delta}) \cap \mathrm{gen}_{\bar{N}} = \emptyset$. Hence $\bar{\delta} = \mathrm{crit}(\sigma_{\bar{\mu}}^{\bar{N}})$ for $\bar{\xi} < \bar{\mu} \leq \bar{\delta}$. Clearly:
$J^{E^{\bar{N}_{\bar{\mu}}}}_{\bar{\delta}} = J^{E^{\bar{N}}}_{\bar{\delta}}$ for $\bar{\xi} < \bar{\mu} \leq \bar{\delta}$, since $\sigma_{\bar{\mu}}^{\bar{N}} \restriction \bar{\delta} = \mathrm{id}$. Hence $(J^E_{\bar{\xi}^+})^{\bar{N}_{\bar{\mu}}} = J^{E^{\bar{N}}}_{\bar{\delta}} = J^{E^{\bar{N}_{\bar{\xi}}}}_{\bar{\delta}} = (J^E_{\bar{\xi}^+})^{N_{\bar{\xi}}}$ for $\bar{\mu} \in (\bar{\xi}, \bar{\delta}]$.

Case 2.1 $T_N \setminus (\xi+1) = \emptyset$.

Then $t_N = s_N = \delta+1$, since otherwise $\delta+1 \in \tilde{C}_N$. $T_{\bar{N}} \setminus (\bar{\xi}+1) = \emptyset$. Hence $t_{\bar{N}} = s_{\bar{N}} = \bar{\delta}+1$, since $\mathrm{gen}'_{\bar{N}} = \sigma^{-1}{}''\mathrm{gen}_N$. Hence $\sigma(t_{\bar{N}}) = t_N$. Clearly $\varsigma = \xi+1$. Let $\bar{\varsigma} = \bar{\xi}+1$. Then $\bar{\varsigma} \in \tilde{C}_{\bar{N}}$ by the above. But $\bar{\varsigma} \notin C_{\bar{N}}$, since $\bar{N}_{\bar{\varsigma}}$ does not satisfy T-MIS by the above. Since $(\bar{\xi}, \bar{\delta}) \cap \mathrm{gen}_{\bar{N}} = \emptyset$, we have: $\bar{\varsigma} = \max \tilde{C}_{\bar{N}}$. Hence for each $\mu \in \tilde{C}_{\bar{N}}$, either $\mu = \bar{\varsigma}$ or $\bar{N}_\mu = (\bar{N}_{\bar{\xi}})_{\bar{\mu}} \in \bar{N}$. Hence $\bar{N}$ is a T-premouse. QED (Case 2.1)

Case 2.2 Case 1 fails.

Then $\varsigma =$ the least $\varsigma > \xi$ s.t. $\varsigma \in T_{N_{\xi+1}}$. But then $\varsigma = \sigma(\bar{\varsigma})$, where $\bar{\varsigma} = t_{\bar{N}_{\bar{\xi}+1}}$, (If $\varsigma < \delta$, then $\varsigma =$ the least $\varsigma > \xi$ s.t. $\xi \in T_N$; hence $\sigma(\bar{\varsigma}) = \varsigma$ where $\bar{\varsigma} =$ the least $\bar{\varsigma} > \bar{\xi}$ s.t. $\bar{\varsigma} \in T_{\bar{N}}$. Hence $\bar{\varsigma} =$ the least $\bar{\varsigma} > \bar{\xi}$ s.t. $\bar{\varsigma} \in T_{\bar{N}_{\bar{\xi}+1}}$, since $\sigma^{\bar{N}}_{\bar{\xi}+1}(\bar{\varsigma}) = \bar{\varsigma}$, If $\varsigma = \delta$, then $\sigma^N_{\bar{\xi}+1}(\varsigma) = \xi + N \in \mathrm{rng}(\sigma)$

Hence $\sigma_{\bar{\xi}+1}(\bar{\delta}) = \bar{\xi} + \bar{N} = \sigma^{-1}(\xi + N) =$

the least $\bar{\varsigma} > \bar{\xi}$ s.t. $\bar{\varsigma} \in T_{\bar{N}}$.)

Hence $\bar{\varsigma} \in \tilde{C}_{\bar{N}}$ and $\bar{\varsigma} \notin C_{\bar{N}}$.

<u>Case 2.2.1</u> $T_N \setminus (\varsigma + 1) = \emptyset$,

Then $T_{\bar{N}} \setminus (\bar{\varsigma} + 1) = \emptyset$. As before,

$t_N = s_N = \delta + 1$, $t_{\bar{N}} = s_{\bar{N}} = \bar{\delta} + 1$;

hence $\sigma(t_{\bar{N}}) = t_{\bar{N}}$. But $\mathrm{gen}_{\bar{N}} \cap (\bar{\xi}, \bar{\delta}) = \emptyset$

Hence $\tilde{C}_{\bar{N}} \setminus (\bar{\varsigma} + 1) = \emptyset$, since for

$\mu \in \tilde{C}_{\bar{N}} \setminus (\bar{\varsigma} + 1)$ we would have $\mu < s_{\bar{N}}$

and $\mu = \mathrm{lub}(\mathrm{gen}_{\bar{N}} \cap \mu)$. Obviously

$\tilde{C}_{\bar{N}} \cap (\bar{\xi}, \bar{\varsigma}) = \emptyset$, since if $\mu \in \tilde{C}_{\bar{N}} \cap (\bar{\xi}, \bar{\varsigma})$,

then $\bar{N}_\mu = \bar{N}_{\bar{\varsigma}}$ and $t_{\bar{N}_\mu} = t_{\bar{N}_{\bar{\varsigma}}} = \bar{\varsigma}$.

Hence $\bar{\xi} = \max C_{\bar{N}}$ and for

$\mu \in \tilde{C}_{\bar{N}}$ we have either $\mu = \bar{\varsigma}$

and $\bar{N}_\mu$ does not satisfy T-MIS

or else $\bar{N}_\mu = (\bar{N}_{\bar{\xi}})_\mu \in \bar{N}$. Hence

$\bar{N}$ is a T-premouse. QED (Case 2.2.1)

Case 2.2.2 Case 2.2.1 fails.

Then $t_N \in T_N$ but $T_N \cap (\varrho, t_N) = \emptyset$, since if $\mu \in T_N \cap (\varrho, t_N)$ is minimal, then $\mu \in \tilde{C}_N$ by the usual argument. Hence $T_{\bar{N}} \cap (\bar{\varrho}, t_{\bar{N}}) = \emptyset$. But $\bar{\varrho} \leq \bar{\delta} \in \mathrm{gen}_{\bar{N}} \subset s_{\bar{N}} \subset \sigma^{-1}{}'' t_{\bar{N}}$. Hence $t_{\bar{N}} =$ (the least $t \in T_{\bar{N}}$ s.t. $t > \bar{\delta}$) $= \sigma^{-1}(t_N)$. $\tilde{C}_{\bar{N}} \cap (\bar{\xi}, \bar{\varrho}) = \emptyset$ follows as in Case 2.2.1. But then $\tilde{C}_N \cap (\bar{\varrho}, t_{\bar{N}}) = \emptyset$ by the argument of Lemma 4.1. Hence $\bar{\xi} = \max C_N$, and for all $\mu \in \tilde{C}_N$, either $\mu = \bar{\varrho}$, where $\bar{N}_{\bar{\varrho}}$ does not satisfy T-MIS, or else $\bar{N}_\mu = (\bar{N}_{\bar{\xi}})_{\bar{\mu}} \in \bar{N}$. Hence $\bar{N}$ is a T-premouse.

QED (Lemma 4.2)

The best we can do for type 3 mice is:

<u>Lemma 4.3</u> Let N be of type 3. Let $\sigma : \bar{N} \xrightarrow[\Sigma_1^{(1)}]{} N$. Then $\bar{N}$ is of type 3 and $t_{\bar{N}} = \sigma^{-1}{}''t_N$.

proof.
Let $N = \langle J_\nu^E, F\rangle$, $\bar{N} = \langle J_{\bar{\nu}}^{\bar{E}}, \bar{F}\rangle$.
Let $s = \omega\rho_N^1 = s_N = t_N$; $\bar{s} = \sigma^{-1}{}''s$.

<u>Claim 1</u> $\bar{s} = s_{\bar{N}}$.

$(\leq)$ $\xi \in \mathrm{gen}_{\bar{N}} \to \sigma(\xi) \in \mathrm{gen}_N$
$\to \sigma(\xi) < s \leq \sigma(\bar{s})$
$\to \xi < \bar{s}$

$(\geq)$ Sei $\eta < \bar{s}$. In N gilt:

$$\bigvee \xi' > \sigma(\eta)\ \xi \in \mathrm{gen}_N.$$

Hence the corresponding $\Sigma_1^{(1)}$ condition holds of η in $\bar{N}$. Hence there is $\xi > \eta$ s.t. $\xi \in \mathrm{gen}_{\bar{N}}$.

QED (Claim 1)

<u>Claim 2</u> $\bar{s} = t_{\bar{N}}$

<u>Case 1</u> $T_N \setminus s = \emptyset$.

Then $T_N \setminus \sigma(\bar{s}) = \emptyset$; hence $T_{\bar{N}} \setminus \bar{s} = \emptyset$

Hence $t_{\bar{N}} = s_{\bar{N}} = \bar{s}$

<u>Case 2</u> Case 1 fails

We must prove: $\bar{\tau} \in T_{\bar{N}}$.

If $\text{lub}\, T_{\bar{N}} \cap \bar{\tau} = \bar{\tau}$, this is immediate.

Otherwise there is $\eta < \bar{\tau}$ s.t. $T_{\bar{N}} \cap (\eta, \bar{\tau}) = \emptyset$

Hence $T_N \cap (\sigma(\eta), \sigma(\bar{\tau})) = \emptyset$, where $\sigma(\eta) < \tau \leq \sigma(\bar{\tau})$ and $\tau \in T_{\bar{N}}$.

Hence $\sigma(\bar{\tau}) = \tau \in T_N$. Hence $\bar{\tau} \in T_{\bar{N}}$

QED (Claim 2)

<u>Claim 3</u> $\bar{\tau} = \sup C_{\bar{N}}$ and $\bar{N}$ is a T-premouse.

Let $\eta < \bar{\tau}$. Then in N we have:

$$V \xi^{1} > \sigma(\eta) \, V Q^{1} (Q = N_{\xi} \text{ is a T-premouse} \wedge \xi = t_{Q})$$

Hence the same $\Sigma_1^{(1)}$ statement holds of η in $\bar{N}$. Hence there is $\xi > \eta$ s.t. $\bar{N}_\xi \in \bar{N}$ (hence $\xi < \bar{\tau} = \tau_{\bar{N}}$), $\xi = t_{\bar{N}_\xi}$ and $\bar{N}_\xi$ is a T-premouse. Hence $\xi \in C_{\bar{N}}$. Hence ~~lub~~ $C_{\bar{N}} = \bar{\tau}$. Moreover $\bar{N}_\eta = (\bar{N}_\xi)_\eta \in N$. Hence $\bar{N}$ is a T-premouse. QED (4.3)

We now consider the iterations appropriate to T-premice. In these iterations, if $\nu = \nu_i$ indexes an extender used at stage i, then $t_i = t(\nu)$ will play the role of λ_i and $t_i^+ = t^+(\nu)$ will play the role of ν_i:

As with s-premice we set:

Def $t^+(N) = t(N)^{+N}$

$t^+(\nu)^N = t^+(N\|\nu)$ for $E_\nu^N \neq \emptyset$.

<u>Lemma ~~4~~.4</u> Let N be a T-premouse s.t. $E_\nu, E_{\nu'} \neq \emptyset$ in N. Then $\nu \neq \nu' \to t^+(\nu) \neq t^+(\nu')$.

prf. Let $\nu < \nu'$. Then $t^+(\nu)$ is not a cardinal in $M\|\nu'$, since $s(\nu) \leq t(\nu) < t^+(\nu)$, whereas $t^+(\nu')$ is.

QED (4.4)

Def Let $\mathcal{I} = \langle\langle M_i\rangle, \langle \nu_i \mid i \in D\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, D\rangle$ be an iteration of M of length θ, where M is a T-premouse. We call $\mathcal{I}$ a <u>normal</u> <u>t-iteration</u> iff, letting $t_i = t(\nu_i)^{M_i}$, $t_i^+ = t^+(\nu_i)^{M_i}$

(i) $\mathcal{I}$ is standard

(ii) $\nu_i > t_h^+$ for $i \in D$, $h \in D \cap i$

(iii) $T(i+1) =$ the largest $\xi \in D$ s.t.
$$\sup_{h<\xi} t_h \leq \kappa_i \quad \text{for } i \in D \quad (\text{hence } \kappa_i < t_\xi)$$

(iv) At $i \in D$ there is no $\nu > \nu_i$ s.t.
$$E_\nu^{M_i} \neq \emptyset \text{ and } t^+(\nu) < t_i^+ \text{ in } M_i$$

<u>Note</u> (iv) is called the <u>applicability condition</u>. $\bar\nu$ with $E_{\bar\nu}^{M_i} \neq \emptyset$ is called <u>applicable</u> or <u>free in</u> M_i iff there is no ν s.t. $E_\nu^{M_i} \neq \emptyset$ and $t^+(\nu) < t^+(\bar\nu) \leq \bar\nu < \nu$.

In the following we develop the propert of normal T-iterations. The initial lemmas do not even assume that M is a T-premouse, but merely a ppm,

<u>Lemma 5</u> Let $j \in D \cap i$. Then $J^{E^{M_j}}_{t_j^+} = J^{E^{M_i}}_{t_j^+}$ and t_j^+ is a cardinal in M_i.

pf. (w.l.o.g. assume that $\mathcal{T}$ is direct)
Suppose not. Let i be a minimal counterexample. Then $i > 0$.

<u>Case 1</u> $i = \lambda$, $\mathrm{Lim}(\lambda)$. It suffices to show: <u>Claim</u> There is $h \leq_T \lambda$ s.t. $\mathrm{crit}(\pi_{h\lambda}) \geq t$

Suppose not. Pick $h \leq_T \lambda$ s.t. $h > j$ and i is simple above h for $h \leq_T i \leq_T \lambda$.
Let $h = T(i+1)$, $i+1 \leq_T \lambda$.
Then $\kappa_i = \mathrm{crit}(\pi_{h\lambda}) < t_j^+$, but $t_j \leq \kappa_i$. Since τ_i is a cardinal in M_h, $J^{E^{M_h}}_{\tau_i} = J^{E^{M_i}}_{\tau_i}$, and t_j^+ is a cardinal in M_h, we have $\tau_i \leq t_j^+$. But $J^{E^{M_h}}_{t_j^+} = J^{E^{M_i}}_{t_j^+}$; hence no element of (t_j, t_j^+) is a cardinal in M_h. Hence $\tau_i = t_j^+$ and $\kappa_i = t_j$. Now let $i+1 = T(\ell+1)$, $\ell + 1 \leq_T \lambda$.
The same argument shows: $\kappa_\ell = t_j$.
Hence $T(\ell+1) = h < i+1$. Contr!
QED (Case 1)

<u>Case 2</u> $i = h+1$. Then ν_h is a cardinal in M_i & $J^{E^{M_i}}_{\nu_h} = J^{E^{M_h}}_{\nu_h}$, where $t_h^+ \leq \nu_h$. Hence it holds for $j = h$. Now let $j < h$. Then t_j^+ is a cardinal in M_h and $J^{E^{M_i}}_{t_j^+} =$
$= J^{E^{M_h}}_{t_j^+} = J^{E^{M_j}}_{t_j^+}$, since $t_j^+ < \nu_h$.

QED (Lemma 5)

As a corollary of the proof of Case 1 we have:

<u>Cor 5.1</u> Let $j < h \leq_T i \leq_T l$; $h, i, l \in D$, where l is simple above h. Then $\mathrm{crit}(\pi_{il}) \geq t_j^+$

prf.
If $\kappa_i \geq t_j^+$, we are done. Otherwise $\kappa_i = t_j$ and hence $\kappa_l \geq t_j^+$ by the above argument.

QED (Cor 5.1)

<u>Cor 5.2</u> Let $\lambda < \theta$, $\mathrm{Lim}(\lambda)$, $\sup D \cap \lambda = \lambda$. Set: $\tilde{\kappa} = \sup\{\kappa_i \mid i \in D,\ i \leq_T \lambda\}$. Then
$$\tilde{\kappa} = \sup_{i \in D \cap \lambda} \kappa_i = \sup_{i \in D \cap \lambda} t_i = \sup_{i \in D \cap \lambda} t_i^+$$

<u>Cor 5.3</u> Let $i \in D$, $\xi = T(i+1)$. Then $\tau_i \leq t^+_\xi$.

prf. of Cor 5.3. t_ξ^+ is a cardinal in $J_{\nu_i}^{E^{M_i}}$ or else $t_\xi^+ = \nu_i$. For $i=\xi$ this is immediate. Otherwise use Lemma 5. QED (Cor 5.3)

<u>Lemma 6</u> Let $j, i \in D$, $j \le i$. Then $t_j^+ \le t_i^+$

proof. Suppose not.

Assume w.l.o.g. that I is direct. Let i be the least counterexample. Then $i > 0$ and $j < i$. By Cor 5.1 we have: $\neg \mathrm{Lim}(i)$. Let $i = h+1$. Then $t_i^+ < t_h^+$ by the minimality of i,

(1) $\nu_i = ht(M_i)$, since otherwise $t_i \ge s(\nu_i)^{M_i} \ge t$ since $t_h^+ < \nu_i$ is a cardinal in M_i.

Let $\xi = T(i)$, $\bar\nu = ht(M_h^*) = \eta_h$. Then $E_{\bar\nu}^{M_h^*} \neq \emptyset$. Let $\pi_{\xi, i}(\bar\kappa, \bar\tau) = \kappa_i, \tau_i$.

(2) $\kappa_h > \bar\kappa$, since otherwise $t_i^+ > t_i \ge \tau_i =$
$= \pi_{\xi i}(\bar\tau) \ge \pi_{\xi i}(\tau_h) = \nu_h \ge t_h^+$.

Clearly: $\tau_h \le t_\xi^+ \le t^+(\bar\nu)$ in M_h^*, since $\kappa_h < t_\xi$ and $t_\xi^+ < t^+(\bar\nu)$ if $\bar\nu \ne \nu_\xi$ by the applicability condition. Hence:

(3) $\tau_h = t^+(\bar\nu)$ in M_h^*

prf. Otherwise $\tau_h \le t(\bar\nu)$, since τ_h is a cardinal in M_h^* and the interval $(t(\bar\nu), t^+(\nu))$ has no cardinals. But

then $t_h^+ \leq \nu_h = \text{lub}\, \pi_{\mathfrak{z}i} {}'' \tau_h \leq \text{lub}\, \pi_{\mathfrak{z}i} {}'' t(\bar{\nu}) \leq$
$\leq t_i < t_i^+$. QED(3)

Hence:

(4) $\nu_{\mathfrak{z}} = \bar{\nu}$ (hence $t_{\mathfrak{z}} = t(\bar{\nu})$ in M_h^*), since otherwise $\tau_h \leq t_{\mathfrak{z}}^+ < t^+(\bar{\nu})$ by the applicability condition.

(5) $\lambda_h < t_i$ (since $\lambda_h = \pi_{\mathfrak{z}i}(\kappa_h) <$
$< \text{lub}\, \pi_{\mathfrak{z}i} {}'' t_{\mathfrak{z}} \leq t_i$.).

(6) $t_h = \lambda_h$, since otherwise
$t_h^+ < \lambda_h < t_i < t_i^+$. But then:

(7) $t_h < t_i$ by (5)

Hence $t_h^+ \leq t_i^+$, since t_h^+, t_i^+ are the successor cardinals of t_h, t_i in M_i. QED (Lemma 6)

(Note: The proofs of (1) - (7) use only: $i = h+1$ and $t_i^+ \ngtr t_h^+$.)

As a corollary of the proof of Lemma we get:

<u>Cor 6.1</u> Let $t_i^+ = t_h^+$ for an $h < i$. Then $h = \max(D \cap i)$ and $\lambda_h = t_h < t_i$.
(Hence $\lambda_i > t_i$, since $\nu_h = t_h^+ = t_i^+$ + hence $\lambda_h < t_i < \nu_h$; hence t_i is not a cardinal in $M \| \nu_i$.)

prf.
Assume w.l.o.g. that $\mathcal{I}$ is direct. Our claim then reads:

<u>Claim</u> $h+1 = i$ and $\lambda_h = t_h < t_i$.

Clearly $i > 0$. $\neg \lim(i)$ by Cor 5.1. Let $i = h+1$. Then $t_i^+ = t_h^+$ by Lemma 6. Using only the fact that $t_i^+ \ngtr t_h^+$ we can repeat the proofs of (1) – (7). Hence $t_h = \lambda_h < t_i$. As above, we conclude that $\lambda_i > t_i$. Now suppose there were a $j < h$ s.t. $t_j^+ = t_h^+$. As before, we can take $h = j+1$. We can then repeat the proof of (1) – (7) with h, j in place of i, h. But then as above we can conclude: $\lambda_h > t_h$. Contr!

QED (Cor 6.1)

<u>Cor 6.2</u> $t_h < t_i$ for $h < i$, $h, i \in D$

proof of Cor. 6.2

By Cor 6.1 for $t_h^+ = t_i^+$. Otherwise $t_h^+ < t_i^+$ where t_h^+ is a cardinal and t_i^+ the cardinal successor of t_i in $J_{\nu_i}^{E^{M_i}}$ (or $t_i^+ = \nu_i$). Hence $t_h \leq t_h^+ \leq t_i$.

QED (Cor 6.2)

<u>Note</u> Lemmas 5 – 6.2 hold for arbitrary ppms.

Note that in a T-iteration we don't necessarily have: $\lambda_i \leq \lambda_j$ for $i < j$. We can squeeze out a bit more information about these iterations by defining:

<u>Def</u> For $h < i < lh(\mathcal{J})$, $h \in D$ set:
$$\tilde{\lambda}_{hi} = \inf\{\lambda_l \mid h \leq l < i \wedge l \in D\}.$$

<u>Lemma 7</u>

(a) Let $h \leq l < i$, $h \in D$. Then $J_{\tilde{\lambda}_{hi}}^{E^{M_l}} = J_{\tilde{\lambda}_{hi}}^{E^{M_i}}$ and $\tilde{\lambda}_{hi}$ is a limit cardinal in M_i.

(b) Let $h = T(i+1) < i$, $i \in D$. Then $\tau_i < \tilde{\lambda}_{hi}$

(c) $J_{\tilde{\lambda}_{h,i+1}}^{E^{M_i}} = J_{\tilde{\lambda}_{h,i+1}}^{E^{M_i^*}}$ if $h = T(i+1)$, $i \in D$

(d) $h \leq j < i \longrightarrow \tilde{\lambda}_{hi} \leq \tilde{\lambda}_{ji}$

(e) $h < j \leq i \longrightarrow \tilde{\lambda}_{hi} \leq \tilde{\lambda}_{hj}$

proof of Lemma 7. (w.l.o.g. $\mathcal{I}$ is direct)

(a) ~~follows~~ by a straightforward modification of the proof of Lemma 5.

(b) $\kappa_i < t_i \leq t_\ell \leq \lambda_\ell$ and $\tau_i \leq t_i^+ \leq t_\ell^+ \leq \nu_\ell$ for $h \leq \ell < i$.

Hence $\tau_i = \kappa_i^{+J_{\nu_\ell}^{EM_\ell}} = \kappa^{+J_{\lambda_\ell}^{EM_\ell}} < \lambda_\ell$.

(c) $\tau_i \in J_{\tilde{\lambda}_{h,i+1}}^{EM_i} = J_{\tilde{\lambda}_{h,i+1}}^{EM_h}$, where $\tilde{\lambda}_{h,i+1} \leq \lambda_i$

and τ_i is a cardinal in $J_{\lambda_i}^{EM_i}$.

Hence $\eta_i \geq \tilde{\lambda}_{h,i+1}$.

(d) and (e) are trivial. QED (Lemma 7)

Note This, too, holds for T-iterations of an arbitrary ppm. The same is true of:

Lemma 8 Let $\mathcal{I} = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\eta_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal T-iteration of M. If $i \in D$, then E_{ν_i} is close to M_i^*.

(Here $M_i^* =_{\text{Df}} M_{T(i+1)} \| \eta_i$)

proof. (Assume w.l.o.g. that $\mathcal{I}$ is direct.)

The case $\nu_i \in M_i$ is trivial, since then $E_{\nu_i} \in M_i$. Hence $E_{\nu_i\alpha} \in \mathcal{P}(\tau_i) \cap M_i =$
$= \mathcal{P}(\tau_i) \cap M_i^*$ for $\alpha < \lambda_i$.

Hence it suffices to prove:

- <u>Lemma 8.1</u> Let $\nu_i = ht(M_i)$. Let $A \subset \tau_i$ be $\underline{\Sigma}_1(M_i)$. Then A is $\underline{\Sigma}_1(M_i^*)$.

prf. Suppose not.
Let $\mathcal{J}$ be a counterexample of minimal length θ. We derive a contradiction.
Clearly $\theta = i+2$, where Lemma 8.1 fails at i and holds at all $j < i$. Let $\delta = T(i+1)$. Then $\nu_i = ht(M_i)$ and $\delta < i$, since otherwise Lemma 8.1 would hold at i. It is easily seen that $i = h+1$ for some h. (Otherwise pick a $j < i$ s.t. $j > \delta$, $j+1 \, T \, i$, $\mathcal{J}$ is simple above $j+1$ in $\mathcal{J}$, and A is $\Sigma_1(M_i)$ in $p \in rng(\pi_{j+1,i})$.
Then $\kappa_j \geq t_\delta > \kappa_i$. Hence $\tau_j > \tau_i$, since κ_j, κ_i are cardinals in $J_{\nu_j}^{EM_j}$.
Hence A is $\Sigma_1(M_{j+1})$ in $p' = \pi_{j+1}^{-1}(p)$.
Define an iteration $\mathcal{J}'$ of length $j+3$ by setting $\mathcal{J}' | j+2 = \mathcal{J} | j+2$ and $\nu'_{j+1} = ht(M_{j+1})$. This is well founded, since there is a

canonical map $\sigma : M'_{j+1} \underset{\Sigma_0}{\longrightarrow} M_i$.
But clearly $T'(j+2) = \delta$ and A is $\Sigma_1(M_i)$, where $lh(\mathcal{Y}') < \theta$. Hence A is $\Sigma_1(M_i^*)$ by minimality, since $M_i^* = M'^{*}_{j+1}$. Contr!)

Let $\xi = T(h+1)$ where $h+1 = i$. Lemma 8. holds at all $j \leq h$ + hence so does Lemma 8. Hence π_{kj} is Σ^*-preserving whenever $k \leq_T j \leq i$ and j is simple above k. Then:

(1) $\kappa_i < \kappa_h$ (hence $\pi_{\xi i} \upharpoonright \tau_i^+ = \mathrm{id}$ where $\tau_i^+ = \tau_i^+$)

proof
Let $\kappa' = \pi_{\xi i}^{-1}(\kappa_i) = \mathrm{crit}(E_{h+}^{M_h^*})$. It suffices to show $\kappa' < \kappa_h$. If not,

$\kappa_i = \pi_{\xi i}(\kappa') \geq \pi_{\xi i}(\kappa_h) = \lambda_h \geq t_h$,

Hence $\delta = i$. Contr!

(2) $\delta \leq \xi$, since $\kappa_i < \kappa_h < t_h$.

(3) $\omega\rho^1_{M_i} \leq \tau_i$

Suppose not. Let $A \subset \tau_i$ be $\Sigma_1(M_i)$. Then $A \in \mathcal{P}(\tau_i) \cap M_i \subset J^{E^{M_i}}_{\tilde\lambda_{\delta i}} = J^{E^{M_i^*}}_{\tilde\lambda_{\delta i}}$.
Hence $A \in M_i$. Contr!

(4) $\omega\rho^1_{M_h^*} \leq \tau_i$

proof. $\pi_{\xi i}$ is Σ^*-preserving and $\pi_{\xi,i} \upharpoonright (\tau_i + 1) = \mathrm{id}$

(5) $\mathbb{P}(\kappa_h) \cap \Sigma_1(M_i) \subset \Sigma_1(M_h^*)$

prf.

$\pi_{\xi i} : M_h^* \xrightarrow[E_{\nu_h}]{} M_i$ is a Σ_0 ultrapower by (4) and (1). The conclusion follows by [NFS] ("A New Fine Structure Theory for Higher Core Models") §1 Lemma 8 and the closeness of E_{ν_h} to M_h^*.

(6) $\xi > \delta$

prf. Suppose not. Then $\xi = \delta$. But $\eta_h \leq \eta_i$ since $\tau_i < \kappa_h < \tau_h$. But then $\mathbb{P}(\tau_i) \cap \Sigma_1(M_i) \subset \Sigma_1(M_h^*) \subset \Sigma_1(M_i^*)$, since $M_h^* = M_i^* \| \eta_h$. Contr!

(7) $M_h^* = M_\xi$ (i.e. $\eta_h = ht(M_\xi)$)

prf. Suppose not. Then $\tau_i < \tilde{\lambda}_{\delta,\xi}$, where $\tilde{\lambda}_{\delta,\xi}$ is a limit cardinal in M_ξ. Let $A \subset \tau_i$ be $\Sigma_1(M_i)$. Then $A \in \mathbb{P}(\tau_i) \cap \Sigma_1(M_h^*) \subset \mathbb{P}(\tau_i) \cap M_\xi = \mathbb{P}(\tau) \cap J^{E^{M_\xi}}_{\tilde{\lambda}_{\delta,\xi}} = \mathbb{P}(\tau_i) \cap J^{E^{M_\delta}}_{\tilde{\lambda}_{\delta,\xi}} \subset M_i^*$. Contr!

QED(7)

We now define a new iteration $\bar{\mathcal{I}}$ of length $\xi + 2$. Set: $\bar{\mathcal{I}} | \xi+1 = \mathcal{I} | \xi+1$

and $\bar{\nu}_\xi = \mathrm{ht}(M_\xi)$. Then $\bar{\kappa}_\xi = \kappa_i$ and $\bar{M}^*_\xi = M^*_i$. $\bar{M}_{\xi+1}$ is well founded, since there is a canonical $\sigma : M_{\xi+1} \longrightarrow_{\Sigma^{(n)}_i} M_{i+1}$ defined by:

$$\sigma(\bar{\pi}_{\delta,\xi+1}(f)(\alpha)) = \pi_{\delta,i+1}(f)(\pi_{\xi i}(\alpha)),$$

since $\langle \mathrm{id} \restriction M^*_i, \pi_{\xi i} \restriction \bar{\lambda}_\xi \rangle : \langle \bar{M}^*_\xi, \bar{E}_{\bar{\nu}_\xi} \rangle \longrightarrow \langle M^*_i, E_{\nu_i} \rangle$ (where $\omega\rho^{n+1}_{M^*_i} \leq \kappa_i < \omega\rho^n_{M^*_i}$). Now let $A \subset \tau_i$ be $\Sigma_1(M_i)$. Then A is $\Sigma_1(M_\xi)$ by (5). But $\mathcal{J}$ is of length $\xi+2 < i+2$. Hence Lemma 8.1 holds at $\bar{\mathcal{J}}$ and A is $\Sigma_1(\bar{M}^*_\xi)$, where $\bar{M}^*_\xi = M^*_i$. Contr!

QED (Lemma 8)

<u>Def</u> M is <u>normally</u> T-<u>iterable</u> (up to ξ) iff there is a successful strategy for normal T-iterations of M (of length $< \xi$)

The concept of a <u>good</u> T-iteration is defined in the usual way. (I.e. these can be decomposed into a linear sequen. of normal T-iterations.)

Def M is T-*iterable* (up to ξ) iff there is a successful strategy for good T-iterations (of length $< \xi$).

Def Let $\mathcal{I} = \langle\langle M_i\rangle, \langle\nu_i\rangle, \ldots, T\rangle$ be a normal T-iteration. i is *singular* in $\mathcal{I}$ iff $E_{\nu_i} \neq \emptyset$ and $t_i^+ = t^+_{M_{i+1}}$. (This means that the situation in Cor 6.1 occurs at i: $\nu_i = t_i^+$, $\lambda_i = t_i$, $\tau_i = t^+_{p_i^*}$, $\kappa_i < t_{p_i^*}$. If i is singular, $i < j$ and $t_j^+ = t_i^+$, then j is not singular and $h \notin D$ for $h \in (i, j]$.

Def Let M^0, M^1 be premice which are normally T-iterable up to $\gamma \leq \infty$ with successful strategies S_0, S_1 resp. The T-*coiteration* of M^0, M^1 *up to* γ *given by* $\langle S_0, S_1\rangle$ *with coiteration indices* $\langle t_i^+ \mid i+1 < \theta\rangle$ in the pair $\langle \mathcal{I}^0, \mathcal{I}^1\rangle$ defined by:

(a) $\mathcal{I}^h = \langle\langle M_i^h\rangle, \langle \nu_i^h \mid i \in D^h\rangle, \langle \eta_i^h\rangle, \langle \pi_{ij}^h\rangle, T^h\rangle$ is a normal T-iteration of M of length $\theta \leq \gamma$ $(h = 0, 1)$

Def Let $M = \langle J_\nu^E, F\rangle$ be a T-ppm, $\eta \le \nu$.

$$\hat{E}_\eta = \begin{cases} E_\nu & \text{if } E_\nu \neq \emptyset \text{ and } \eta = t_\nu^+ ; \\ \emptyset & \text{if no such } \nu \text{ exists.} \end{cases}$$

(b) Let M_i^h be defined, $i < \delta$. Set:

$t_i^+ \simeq$ the least $\eta \le \max(\mathrm{ht}(M_i^0), \mathrm{ht}(M_i^1))$ s.t. $\hat{E}_\eta^{M_i^0} \neq \hat{E}_\eta^{M_i^1}$.

(c) We set: $i \in D^h \longleftrightarrow \hat{E}_{t_i^+}^{M_i^h} \neq \emptyset$

and $\nu_i =$ that ν s.t. $E_\nu^{M_i^h} = \hat{E}_{t_i^+}^{M_i^h}$,

<u>except</u> in the case that this convention would make i singular in $\mathcal{J}^h$ but not in $\mathcal{J}^{1-h}$. If that happens, we set $i \in D^h$, $E_{\nu_i}^{M_i^h} = \hat{E}_{t_i^+}^{M_i^h}$, but $i \notin D^{1-h}$.

(d) $\mathcal{J}^h$ conforms to S_h.

<u>Note</u> The normality of $\mathcal{J}^0, \mathcal{J}^1$ follows easily.

<u>Lemma 9</u> Let $\bar{\bar{M}}^0, \bar{\bar{M}}^1 < \kappa$, where κ is regular and M^0, M^1 are normally T-iterable up to $\kappa+1$. Then the T-coiteration terminates below κ.

proof. Suppose not.

Let $\mathcal{J} = \langle \mathcal{J}^0, \mathcal{J}^1\rangle$ be the coiteration.

Let $\tau = \kappa^+$. Let $X \prec H_\tau$ s.t. $\overline{\overline{X}} < \kappa$, $X \cap \kappa$ is transitive and $\mathcal{T}^0, \mathcal{T}^1 \in X$.
Let $\sigma : \bar{H} \stackrel{\sim}{\longleftrightarrow} X$ where $\bar{H}$ is transitive,
Let $\bar{\mathcal{T}}^h = \sigma^{-1}(\mathcal{T}^h) = \langle \langle \bar{M}_i^h \rangle, \langle \bar{\nu}_i^h \rangle .., \langle \bar{\pi}_{ij}^h \rangle, \bar{T}^h \rangle$.
Then $\bar{M}_i^h = M_i^h$, $\bar{\pi}_{ij}^h = \pi_{ij}^h$ for $i, j < \bar{\kappa} = \kappa \cap X = $
$= \mathrm{crit}(X)$. Thus:

$$\bar{M}_{\bar{\kappa}}^h, \langle \bar{\pi}_{i\bar{\kappa}}^h \mid i \leq_{T^h} \kappa \rangle = \lim_{i \leq_T j < \bar{\kappa}} (M_i^h, \pi_{ij}^h).$$

Hence:

(1) $\bar{M}_{\bar{\kappa}}^h = M_{\bar{\kappa}}^h$, $\bar{\pi}_{i\bar{\kappa}}^h = \pi_{i\bar{\kappa}}^h$,

(2) $\sigma \upharpoonright M_{\bar{\kappa}}^h = \pi_{\bar{\kappa}\kappa}^h$

prf. Let $x \in M_{\bar{\kappa}}^h$, $x = \pi_{i\kappa}(\bar{x})$. Then
$\sigma(x) = \sigma(\pi_{i\bar{\kappa}}^h(\bar{x})) = \pi_{i\kappa}^h(\bar{x}) = \pi_{\bar{\kappa}\kappa}^h \pi_{i\bar{\kappa}}^h(\bar{x}) =$

$= \pi_{\bar{\kappa}\kappa}^h(x)$. QED(2)

Any truncation in the branch $b^h = \{i \mid i \leq_{T^h} \kappa\}$ must occur below $\bar{\kappa}$. It follows that $\mathbb{P}(\bar{\kappa}) \cap M_{\bar{\kappa}}^0 = \mathbb{P}(\bar{\kappa}) \cap M_{\bar{\kappa}}^1$. (Otherwise $\mathbb{P}(\bar{\kappa}) \cap M_\kappa^0 \neq \mathbb{P}(\bar{\kappa}) \cap M_\kappa^1$ and $t_\kappa^+ < \bar{\kappa}^{+M_\kappa^h} =$
$= \bar{\kappa}^{+M_{\bar{\kappa}}^h} \leq t_{\bar{\kappa}}^+$.)

Now let $i_h + 1 \leq_{T^h} \kappa$ s.t. $\bar{\kappa} = T^h(i_h + 1)$.
Since $\pi_{\bar{\kappa}\kappa} = \pi_{i_h+1,\kappa} \pi_{\bar{\kappa}, i_h+1}$ and $\mathrm{crit}(\sigma) = \bar{\kappa}$, we have:

(3) $\mathrm{crit}(\hat{E}) = \bar{\kappa}$, $\hat{E}(X) \cap t_{i_h}^h = \sigma(X) \cap t_{i_h}^h$ for:
$\hat{E} = \hat{E}_{t_i^h}^{M_{i_h}^h}$, $X \in \mathbb{P}(\bar{\kappa}) \cap M_{i_h}^h$

From this we derive a contradiction:

<u>Case 1</u> $i_0 = i_1$. Let $i = i_0 = i_1$.

If $t_i^0 = t_i^1$, then $\hat{E}^0_{t_i^+} = \hat{E}^1_{t_i^+}$. Contr!

Let e.g. $t_i^0 < t_i^1$. Set: $\tilde{M}^h = M^h \| \nu_i^h$.

Then $\tilde{M}^0 = (\tilde{M}^1)_{t_i^0}$, $t_i^0 = t_{\tilde{M}^0}$. Hence

$\tilde{M}^0 \in \tilde{M}^1$. Hence $t_i^+ = t^+_{\tilde{M}^0} < t^+_{\tilde{M}^1} = t_i^+$.

Contr!

<u>Case 2</u> $i_0 \neq i_1$. Let e.g. $i_0 < i_1$.

Then $t^+_{i_0} \leq t^+_{i_1}$.

<u>Case 2.1</u> $t^0_{i_0} < t^1_{i_1}$.

We obtain a contradiction exactly as in Case 1, using the fact that, setting $\tilde{M}^h = M^h_{i_h} \| \nu^h_{i_h}$ we would have $\tilde{M}^0 \in \tilde{M}^1$ although $t^+_{i_0} = (t^0_{i_0})^{+\tilde{M}^0} = (t^0_{i_0})^{+\tilde{M}^1}$.

<u>Case 2.2</u> Case 2.1 fails.

Then $t^+_{i_1} = t^+_{i_0}$, since otherwise $t^0_{i_0} < t^+_{i_1} \leq t^1_{i_1}$. Hence $t^+_{i_0+1} = t^+_{i_0}$, since $i_1 \geq i_0 + 1$. Hence $\hat{E}^{M^0_{i+1}}_{t^+_{i_0}} \neq \hat{E}^{M^1_{i+1}}_{t^+_{i_0}}$, so i_0 must be singular in either $\mathcal{J}^0$ or $\mathcal{J}^1$. But if $i_0 \in D^1$, it must be singular in both. It follows easily that i_0+1 is not singular in $\mathcal{Y}^0$ or $\mathcal{Y}^1$, hence that $\hat{E}^{M^0_{i_0+1}}_{t^+_{i_0}} = \hat{E}^{M^1_{i_0+1}}_{t^+_{i_0}} = \emptyset$,

Hence $t^+_{i_0+1} > t^+_{i_0}$. Hence $i_1 = i_0 + 1$.

If $t^1_{i_1} < t^0_{i_0}$ we would obtain a contradiction exactly as in Case 1. Hence $t^1_{i_1} = t^0_{i_0} = t$ and $\tilde{M}^0 = \tilde{M}^1 = \tilde{M}$. Thus $\hat{E}^{M^1_{i_0}}_{t^+} \neq \emptyset$, since otherwise $\hat{E}^{M^1_{i_0+1}}_{t^+} = \emptyset$.

Hence $i_0 \in D^1$, since otherwise $\tilde{M} = M^1_{i_0} \| \nu^1_{i_0}$ and $\hat{E}^{M^0_{i_0}}_{t^+} = \hat{E}^{M^1_{i_0}}_{t^+}$. Contr!

But then $t = t^1_{i_0+1} > t^1_{i_0} = t$. Contr!

QED (Lemma 9)

Similar lemmas can be worked out for "double rooted iterations", though we have not checked this in detail. It should then be possible to prove solidity and condensation lemmas for iterable T-premice. The proofs are likely, however, to be somewhat more complicated.

For now we take this for granted and consider criteria of iterability.

<u>Def</u> By a <u>weakly T-iterable T-premouse</u> (or <u>weak T-mouse</u>) we mean a T-premouse M with the property: Let $\sigma : \bar{M} \xrightarrow[\Sigma_\omega]{} M$, where $\bar{M}$ is a countable T-premouse. Then $\bar{M}$ is T-iterable up to $\omega_1 + 1$.

By a Löwenheim-Skolem argument it will follow that weak T-mice satisfy as much solidity and condensation as fully iterable T-mice. We assume that this will be enough to carry out the iterability arguments developed later in this paper.

We shall obtain weak T-mice by Steel's technique of constructing what we call "arrays".

<u>Def</u> A T-*array* is a sequence $\langle N_i \mid i < \theta \rangle$ of T-premice s.t.

(a) N_i is a weak T-mouse for $i+1 < \theta$.

(b) $N_0 = \langle \emptyset, \emptyset \rangle$

(c) Let $i+1 < \theta$, $\mathrm{core}(N_i) = \langle J^E_\beta, E_{\omega\beta} \rangle$. Either $N_{i+1} = \langle J^E_{\beta+1}, \emptyset \rangle$ or $E_{\omega\beta} = \emptyset$ and $N_{i+1} = \langle J^{E'}_\alpha, F \rangle$, where $F \neq \emptyset$, $\beta = t^+_{N_{i+1}}$, and $J^E_\beta = J^{E'}_\beta$.

<u>Def</u> For $\xi < i \leq \theta$ set:

$\kappa_\xi = \kappa_{\xi,i} = \inf \{ \omega\rho^\omega_{N_h} \mid \xi \leq h < i \}$;

$\mu_\xi = \mu_{\xi,i} = \kappa_\xi^{+N_\xi}$ (adopting the convention that $\mu_\xi = \mathrm{ht}(N_\xi)$ if $\kappa_\xi = \mathrm{ht}(N_\xi)$ or κ_ξ is the largest cardinal in M_ξ).

(d) Let $\lambda < \theta$, $\mathrm{Lim}(\lambda)$. Then $J^{E^{N_\xi}}_{\mu_{\xi,\lambda}} = J^{E^{N_i}}_{\mu_{\xi,\lambda}}$ for $\xi \leq i \leq \lambda$ and

$N_\lambda = \langle \bigcup_{\xi<\lambda} J^{E^{N_\xi}}_{\mu_{\xi,\lambda}}, \emptyset \rangle$.

We also set: $M_i = \mathrm{core}(N_i)$ if $i < \theta$ and N_i is a weak T-mouse.

Note $\zeta \leq \xi < i \leq \theta \to \kappa_{\zeta i} \leq \kappa_{\xi i}$.

Note Suppose $\theta = i+1$ and that N_i is a weak T-mouse. We can extend our array to one of length $\theta+1$ simply by setting: $N_\theta = \langle J^E_{\alpha+1}, \emptyset \rangle$ where $M_i = \langle J^E_\alpha, E_{\omega\alpha} \rangle$. If $E_{\omega\alpha} = \emptyset$ there may be another alternative: Suppose that F is an extender on M_i with critical point κ. Let $\tau = \kappa^{+M}$ and let $\pi : J^E_\tau \to_F J^{E'}_\nu$. Set: $F' = \pi \upharpoonright \mathbb{P}(\kappa)$. If $N = \langle J^{E'}_\nu, F' \rangle$ is a T-premouse with $t^+_N = \alpha$ we may set: $N_\theta = N$.

. . . .

We summarize the main facts about arrays:

Fact 1 Set: $\mu_h = \mu_{hi}$. For $h \leq j < i \leq \theta$:

(a) $\mu_h \leq \mu_j$

(b) $J^{E^{M_h}}_{\mu_h} = J^{E^{N_h}}_{\mu_h} = J^{E^{N_j}}_{\mu_h}$.

(c) $(\kappa_h = \kappa_j = \omega\rho^\omega_j$ and $j < h < i) \to \mu_h < \mu_k$

(d) $N_h \neq N_j$ if $h < j$.

Note If $\mathrm{Lim}(\theta)$ it follows that we can extend the array by setting:

$$N_\theta = \langle \bigcup_{\xi<\theta} J^{E^{N_\xi}}_{\mu_{\xi,\theta}}, \emptyset \rangle.$$

If $\theta = \infty$, we get a weasel by setting:

$$N_\infty = \bigcup_{\xi<\infty} J^{E^{N_\xi}}_{\mu_{\xi,\infty}}.$$

Def Let $\gamma \le \theta \le \infty$, $\mathrm{Lim}(\gamma)$. Let $ht(N_\gamma) = \mu$. Let $\omega < \lambda < \mu$ s.t. λ is a limit ordinal and is cardinally absolute in N_γ (i.e. if $\tau < \lambda$ is a cardinal in $J^{E^{N_\gamma}}_\lambda$, then it is a cardinal in N_γ) and $\lambda \le t^+_{N_\gamma}$ if N_γ is active, . Set:

$$\delta = \delta(\lambda) = \delta(\lambda,\gamma) = \sup\{\xi<\gamma \mid \mu_{\xi,\gamma} < \lambda\}.$$

Fact 2 δ is a limit ordinal

Fact 3 $\mu_{i\gamma} = \mu_{i\delta}$ for $i<\delta$.

Hence:

Fact 4 $N_\delta = \langle \bigcup_{i<\delta} J^{E^{N_i}}_{\mu_{i,\delta}}, \emptyset \rangle = \langle J^{E^{N_\gamma}}_\lambda, \emptyset \rangle.$

Fact 5 $M_\delta = N_\delta$ and $\mu_\delta = \lambda = ht(M_\delta)$

Fact 6 If λ is a limit cardinal in N_γ, then $\mu_\delta = \kappa_\delta = \lambda$.

We gave rather garbled proofs of these facts in §10 of [NFS]. A better proof is contained in §1 of [MOI].

. . . . .

In Steel's arrays, every extender $E_\nu^{N_i}$ can be traced back to its point of origin as the top extender of some $N_{\xi+1}$ by means of the "resurrection sequence". In T arrays, $N_{i+1} = \langle J_\beta^E, F \rangle$ may be much longer than $M_i = \langle J_\alpha^E, \emptyset \rangle$, so extenders $E_\nu^{N_{i+1}}$ with $\alpha \leq \nu < \omega\beta$ cannot be traced. But these extenders are not applicable, since $\alpha = t^+$ is then a cardinal in J_β^E and hence $t_\beta^+ < t_\nu^+ \leq \nu < \beta$.

In fact every applicable extender does have a resurrection sequence. Using this we can imitate the known constructions to get T-iterable premice (e.g. for 1-small T-premice).

Def Let N be a T-premouse, $\nu \leq ht(N)$. ν is free in N iff there is no $\nu' \leq ht(N)$ s.t. $E^N_{\nu'} \neq \emptyset$ and $t^+_{\nu'} < \nu < \nu'$ in N.

[Thus, if $E^N_\nu \neq \emptyset$, ν is free iff ν is applicable]

Def Let N be a T-premouse. Let $\nu < ht(N)$ be free in N. Set:

$\beta = \beta(N,\nu) =$ the maximal $\beta \in [\nu, ht(N))$ s.t. $\omega\rho^\omega_{N\|\beta} < \omega\rho^\omega_{N\|\xi}$ for all $\xi \in [\nu, \beta)$.

[β is then the place where $\omega\rho^\omega_{N_i\|\beta}$ becomes minimal for $\beta \in [\nu, ht(N_i))$.]

Fact 7 There is exactly one $\eta < i$ s.t. $N_i\|\beta = M_\eta$ ($\beta = \beta(N_i, \nu)$)

Def Let M be a T-premouse. Let $\nu \leq ht(M)$ be free in M.

$\beta^+ = \beta^+(M,\nu) =$ the maximal $\beta \in [\nu, ht(M)]$ s.t. $\omega\rho^\omega_{M\|\beta} < \omega\rho^\omega_{M\|\xi}$ for all $\xi \in [\nu, \beta)$.

Fact 8 If N_i is a weak mouse, then there is exactly one $\eta \leq i$ s.t. $M_i\|\beta^+ = M_\eta$ ($\beta^+ = \beta^+(M_i, \nu)$).

Fact 7 and Fact 8 are proven by simultaneous induction on i. The proof is virtually the same as in §10 of [NFS]

Def Let $\xi < \Theta$, $E_\nu^{N_\xi} \neq \emptyset$, where ν is free in N_ξ. The resurrection sequence:

$S(\nu, \xi) = \langle\langle \eta_1, \beta_1, \sigma_1 \rangle, \ldots, \langle \eta_{\tilde{p}}, \beta_{\tilde{p}}, \sigma_{\tilde{p}} \rangle\rangle$

(where $\tilde{p} = \tilde{p}[\xi, \nu] < \omega$) is defined by:

Case 1 $\nu = ht(N_\xi)$. $S(\nu, \xi) = \emptyset$ (hence $\tilde{p} = 0$).

Case 2 $\nu < ht(N_\xi)$. Set:

$\beta_1 = \beta(N_\xi, \nu)$, $\eta_1 =$ that $\eta < \xi$ s.t. $N_\xi \| \beta_1 = M_\eta$

$\sigma_1 =$ the core map $\sigma: M_{\eta_1} \longrightarrow N_{\eta_1}$.

$S(\nu, \xi) =_{\text{Df}} \langle \eta_1, \beta_1, \sigma_1 \rangle {}^\frown S(\sigma_1(\nu), \eta_1)$.

(here $\sigma_1(\nu) =_{\text{Df}} ht(N_{\eta_1})$ if $\nu = ht(M_{\eta_1})$.)

We also write:
$\eta_h[\nu, \xi] = \eta_h$, $\beta_h[\nu, \xi] = \beta_h$, $\sigma_h[\nu, \xi] = \sigma_h$. In addition we set:
for $1 \leq h \leq \tilde{p} = \tilde{p}[\nu, \xi]$. In addition we set:

$\eta_0 = \eta_0[\nu, \xi] =_{\text{Df}} \xi$; $\beta_0 =_{\text{Df}} ht(N_\xi)$, $\sigma_0 =_{\text{Df}} id \restriction N_\xi$,

Def $\sigma^{(m)} = \sigma^{(m)}[\nu, \xi] =_{\text{Df}} \sigma_m \circ \cdots \circ \sigma_0$ for $m \leq \tilde{p}$

$\sigma^* = \sigma^{(\tilde{p})}$. Then

Fact 9 $S(\nu, \xi) = \langle\langle \eta_1, \beta_1, \sigma_1 \rangle, \ldots, \langle \eta_m, \beta_m, \sigma_m \rangle\rangle {}^\frown$
$S(\sigma^{(m)}(\nu), \eta_m)$

for $0 \leq m \leq \tilde{p}$. Hence:

<u>Fact 10</u> $\langle \eta_{m+n}, \beta_{m+n}, \sigma_{m+n} \rangle =$

$= \langle \eta_n, \beta_n, \sigma_n \rangle [\sigma^m(\nu), \eta_m]$

<u>Def</u> Let N be a T-premouse. Let $E_\nu^N \neq \emptyset$ and $\nu \in N$ where ν is free in N. $\bar{\beta}_i[\nu, N]$ $(i \leq p = p[\nu, N]$ is defined by:

$\bar{\beta}_0 = ht(N)$; $\bar{\beta}_{i+1} \simeq \beta(\nu, N \| \bar{\beta}_i)$.

Then:

<u>Fact 11</u> $\tilde{p} = p[\nu, N_\xi]$, $\beta_h = \sigma^{(h)}(\bar{\beta}_h)$ $(1 \leq h \leq$

<u>Fact 12</u> $\bar{\beta}_p = \nu$, $\beta_p = \sigma^{(p)}(\nu)$

<u>Def</u> $\sigma^* = \sigma^*[\nu, \xi] = \sigma^{(p)}$

$\eta^* = \eta^*[\nu, \xi] = \eta_p$

<u>Fact 13</u> $\sigma^* : N_\xi \| \nu \xrightarrow[\Sigma^*]{} N_{\eta^*}$

<u>Fact 14</u> Let $\lambda < \nu$ be a cardinal in N_ξ, $m \leq p$. Then $\sigma^{(m)} \upharpoonright \lambda = id$.

<u>Fact 15</u> Let $\lambda < \nu$ be a successor cardinal in N_ξ. Then $\sigma^{(m)} \upharpoonright \lambda + 1 = id$.

(Facts 9–13 are straightforward. Facts 14 and 15 are proven in §10 of [NFS].)

In forming arrays we shall employ formation rules which are intended to guarantee that each N_i is iterable (or at least a weak mouse). We will also want to know that each N_i is unique: If $N_i = \langle J^E_\beta, \emptyset \rangle$ we might find two different possible extensions $N^h_{i+1} = \langle J^{E^h}_{\alpha_h}, F^h \rangle$ $(h=0,1)$. To show that this cannot happen we use the theory of *bicephali*. The proof that N^h_{i+1} is iterable should also give us the iterability of the pair $\langle N^0_{i+1}, N^1_{i+1} \rangle$. (Roughly speaking, this means that we can do normal iterations of the pair, choosing an extender from one structure or the other and applying it to both.) Such a pair is called a bicephalus. By coiterating it against itself we then discover that, in fact, $N^0_{i+1} = N^1_{i+1}$. The appropriate iterations for this purpose are Σ_0-iterations – i.e. we take only Σ_0 ultraproducts until a truncation occurs. In the following pages we first develop the Σ_0-iterations of T-premice. Then we introduce bicephali and develop their iteration theory.

Σ_0 - Iteration

Lemma 10. Let N be a T-premouse, $\sigma : N \to_G N'$ where $crit(G) < t_N$. Then N' is a T-premouse of the same type.

proof

For N of type 1 this follows by Lemma 3.1.
" " " type 2 " " " Lemma 3.3
For N of type 3 we can repeat the proof of Lemma 3.5 (or use Lemma 3.5' and the remark following it.) QED

. . . . . .

Def A Σ_0 iteration $\mathcal{Y} = \langle\langle M_i \rangle, \dots, T\rangle$ of N is defined as before except that we take $\pi_{\zeta, i+1} : N_\zeta \to_{E_{\nu_i}} N_{i+1}$ whenever $i \in D$ and $i+1$ is simple in $\mathcal{Y}$, and otherwise $\pi_{\zeta, i+1} : N_\zeta \to^*_{E_{\nu_i}} N_{i+1}$. (This notion is developed in [NFS].) Normal and good Σ_0 T-iterations are defined correspondingly.

Lemmas 5 - 7 then go through for Σ_0 T-iterations of T-premice. Lemma 8 holds only in the form: If $i \in D$ and $i+1$ is not simple in $\mathcal{Y}$, then E_{ν_i} is close to M_i^*.

If M^0, M^1 are T-premice which are normally Σ_0 T-iterable, we define the Σ_0-coiteration exactly as before. The corresponding version of Lemma 9 goes through as before. From this we get:

Lemma 9' Let M^0, M^1 be T-premice which are presolid and normally Σ_0 T-iterable. Let $\tilde{M}^0, \tilde{M}^1$ be the coiterates. Either $\tilde{M}^0$ is a simple iterate of M^0 and a segment of $\tilde{M}^1$, or conversely.

Note Lemmas 9, 9' hold under the assumption that M^0, M^1 are $\beta^+ + 1$-iterable, where $\overline{\overline{M^0}}, \overline{\overline{M^1}} \leq \beta$.

Note Lemmas 9, 9' also hold for mixed coiteration where one side is a Σ_0 iteration and the other a $*$-iteration

Bicephali

In this section we develop a technique which we shall use to show that the extender choices in the arrays developed in §3 and §4 are unique. The premice used there all have the property:

• If $E_\nu^M \neq \emptyset$, then t_ν^M is a cardinal in M.

If we didn't have this property we would presumably have to use the method of [MS] §11 as well as the "bicephalus method".

Def Let N be an active T-premouse. N is of type A iff t_N is a limit cardinal in N and $N_\xi \in N$ for arbitrarily large $\xi < t_N$.
Otherwise N is of type B.

Lemma 11.1 Let N be of type A. Then $s_N = t_N$.

prf. trivial.

<u>Lemma 11.2</u> Let N be of type B.
Let $\sigma : N \longrightarrow_G N'$ where $\mathrm{crit}(G) < t_N$.
Then $t^+_N = \sigma(t^+_N)$ and N' is of type B.

proof.
Let α = the cardinal predecessor of t^+_N.
If $t_N > \alpha$, then:
$\sigma(\alpha) < \sup \sigma'' t_N \leq t_{N'} \leq \sigma(t_N) < \sigma(t^+_N)$;
hence $t^+_{N'} = \sigma(t^+_N)$.
Now let $\alpha = t_N$. If α is a successor cardinal then $\sigma(\alpha) = \sup \sigma'' \alpha \leq t_{N'} \leq \sigma(\alpha)$.
Hence $\sigma(\alpha) = t_{N'}$, $\sigma(t^+_N) = t^+_{N'}$.
Now let α be a limit cardinal.
There is $\gamma < s_N$ s.t. $(\gamma, \alpha) \cap T_N \neq \emptyset$, since otherwise $s_N = \mathrm{lub}\, T_N \cap \alpha = \alpha$, hence $\sup C_N = \alpha$ and N is of type A. But then $\alpha \in T_N$, since otherwise $s_N = \alpha$, $\sup C_N = \alpha$.
Hence:
$\sigma(\gamma) < \sup \sigma'' s_N = s_{N'} \leq \sigma(\alpha) \in T_{N'}$,
where $(\sigma(\gamma), \sigma(\alpha)) \cap T_{N'} = \emptyset$. Hence
$\sigma(\alpha) = t_{N'}$, $\sigma(t^+_N) = \alpha^{+N'} = t^+_{N'}$.

QED (11.2)

Def By a T-prebicephalus (T-pb) we mean a triple $P = \langle P^0, P^1, t \rangle$ s.t.

(i) P^0, P^1 are T-premice

(ii) t is a successor cardinal in P^0, P^1 and $J_t^{EP^0} = J_t^{EP^1}$

(iii) Let α be the cardinal predecessor of t in P^h $(h=0,1)$. Then α is a limit cardinal in P^h $(h=0,1)$ and one of the following holds:

(A) $\alpha = s_{P^0} = s_{P^1}$ and $P^h_\xi \in P^h$ for arbitrarily large $\xi < \alpha$.

(B) P^0, P^1 are of type B and $t = t^+_{P^h}$ $(h=0,1)$.

P is a T-pb of type A if (A) holds and otherwise of type B.

We set: $t^+_P = t$, $\alpha_P = \alpha$.

We use obvious abbreviations like:

$\beta^{+P} = \beta^{+P^h}$ (for $\beta < t^+_P$), $J_\beta^{EP} = J_\beta^{EP^h}$ (for $\beta \leq t^+_P$),

$\mathcal{P}(\kappa)_P = \mathcal{P}(\kappa)_{P^h}$ (for $\kappa < t^+_P$), etc.

Def A strict T-pb is a T-pb s.t. $t^+_P = t^+_{P^h}$ for $h = 0,1$. (Thus if P is strict and of type A, then P^0, P^1 are of type A.)

We are primarily interested in strict pb's. However, we shall have to consider the structures which arise from them by iteration.

<u>Lemma 11.3</u> Let P be a pb. Let $\sigma^h : P^h \xrightarrow[G]{} Q^h$ where $\mathrm{crit}(G) < t_p$. Then:

(a) $\sigma^0 \upharpoonright t_p^+ + 1 = \sigma^1 \upharpoonright t_p^+ + 1$

(b) If P is of type A, then $Q = \langle Q^0, Q^1, t \rangle$ is a T-pb of type A, where: $\alpha = \sup \sigma^h{}'' t_p$, $t = \alpha^{+Q^h}$ $(h = 0, 1)$

(c) If P is of type B, then $Q = \langle Q^0, Q^1, t \rangle$ is a T-pb of the same type, where $t = \sigma^h(t_p^+)$ $(h = 0, 1)$.

The proof is left to the reader.

<u>Def</u> $\sigma : P \xrightarrow[G]{} Q$ iff $\sigma = \langle \sigma^0, \sigma^1 \rangle$ and σ^0, σ^1, P, Q are as above.

<u>Note</u> We also use obvious abbreviations like $\sigma \upharpoonright J_\zeta^E = \sigma^h \upharpoonright J_\zeta^E$ $(\zeta \leq t_p^+)$ and $\sigma \upharpoonright \mathbb{P}(\tau) = \sigma^h \upharpoonright \mathbb{P}(\tau)$ $(\tau < t_p^+)$.

Def An extender e is in P iff e is either a top extender of P^0 or P^1 or $e = E_\nu^P$ for a $\nu < t_P^+$.

e is applicable in P iff e is a top extender or $e = E_\nu^P$ and $\nu < t_P^+$ is applicable in P.

(Note that there is no $\nu < ht(P)$ with $t_P^+ = t_{P\|\nu}^+$, since then t_P^+ would not be a cardinal in P^h.)

For $\xi \leq t_P^+$, $h < 2$ set:

$$e_\xi^h = e_\xi^{h,P} = \begin{cases} E_\nu^P & \text{if } \nu < t_P^+,\ E_\nu^P \neq \emptyset,\ \xi = t_{P\|\nu}; \\ E_{ht}^{P^h} & \text{if } \xi = t_P^+; \\ \emptyset & \text{otherwise} \end{cases}$$

(Note $e_\xi^h = E_\nu^{P^h}$ for $E_\nu^{P^h} \neq \emptyset$, $\xi = t_{P\|\nu}$ unless P is of type A and $\xi = t_P^+ < t_{P^h}^+$.)

If Q is a T-premouse set: $Q^0 = Q^1 = Q$.

If Q is active, let t_Q^+ have its usual meaning.

Otherwise set: $t_Q^+ = ht(Q)$.

For $\xi \leq t_Q^+$ define e_ξ^h $(h = 0, 1)$ as above.

(Thus $e_\xi^h = E_\nu^Q$ if $E_\nu^Q \neq \emptyset$ and $\xi = t_{Q\|\nu}^+$.)

If P is a T-ph or T-premouse we then set:

$$U_\xi^P = \{e_\xi^0, e_\xi^1\} \text{ for } \xi \leq t_P^+.$$

Def Let P be a T-pb or premouse. Let $e \in U^P_\xi$, $e \neq \emptyset$.

$\nu_e = \nu(e, P) = \nu$ where $e = P^h_\nu$

$\lambda_e = \lambda(e, P) = lh(e) = \lambda_{P^h \| \nu}$

$\tau_e = \tau(e, P) = \tau_{P^h \| \nu} = \tau^{+P}$

$$t_e = t(e, P) = \begin{cases} \alpha_P & \text{if } P \text{ is a T-pb of type A and } \xi = t^+_P\,; \\ t_{P^h \| \nu} & \text{otherwise.} \end{cases}$$

$t_e^+ = (t_e^+)^P$.

(Note that ν_e, t_e etc. depend only on e, P and not on the $h < 2$ in the definition.)

. . . . . .

Def Let P, Q be T-pb's or T-premice. P is a segment of Q (P seg Q) iff either $P^0 = P^1 = Q^0 = Q^1$ or else iff $P^0 = P^1 = Q \| \xi$ for a $\xi < t^+_Q$.

It is easily checked that;

Lemma 11.3 If $\neg(P \text{ seg } Q)$ and $\neg(Q \text{ seg } P)$, then there is $\xi \leq \min(t^+_P, t^+_Q)$ s.t. there are $e \in U^P_\xi$, $e' \in U^Q_\xi$ with $e \neq e'$.

We are at present uncertain how to deal with iterations of T-pb's if the situation described in Cor 6.1 is permitted, so we shall make an assumption which forbids it:

From now on assume that all of the premice we deal with satisfy one of the following conditions:

(*) If $E_\nu^M \neq \emptyset$, then $t_{M\|\nu}$ is a cardinal in $M\|\nu$,

(**) If $E_\nu^M \neq \emptyset$, then $t_{M\|\nu} < \lambda_{M\|\nu}$.

This limits the applicability of our theory, since e.g. very large s-mice may not satisfy (*) or (**). On the other hand, if we take $T_N =$ the set of $\alpha \geq \tau_N$ which are cardinals in N, we get a very general class of premice satisfying (*). In any case our assumption does not endanger the applications we shall make in §3, §4, since all of the premice used there satisfy (*).

We now define the notion of a <u>normal iteration</u> of a T-pb P. This is a structure $\mathcal{J} = \langle\langle P_i\rangle, \langle e_i \mid i \in D\rangle, \langle \pi_{ij}\rangle, T\rangle$ which is like a normal iteration of a T-premouse with the following differences:

- P_i may be either a premouse or a pb
- e_i is some extender applicable in P_i
- Set $t_i = t_{P_i}$ if e_i is a top extender of P_i. Otherwise set $t_i = t_{P_i \| \nu_i}$, where $e_i = E^{P_i}_{\nu_i}$, $\nu_i < t^+_{P_i}$. The t_i's determine $T(i+1)$.
- P_i is a pb if i is simple in $\mathcal{J}$ and is otherwise a premouse - i.e. when we truncate, we truncate to a premouse.
- If $i+1$ is simple in $\mathcal{J}$ and $\xi = T(i+1)$, then $\pi_{\xi, i+1} : P_\xi \longrightarrow_{e_i} P_{i+1}$. (Hence this is really a Σ_0-iteration.)
- If $i+1$ is not simple in $\mathcal{J}$, then $\pi_{\xi, i+1} : P^*_i \longrightarrow^*_{e_i} P_{i+1}$.

The formal definition follows:

Def Let P be a T-pb. By a <u>normal iteration</u> of P <u>of length</u> θ we mean

$$\mathcal{I} = \langle\langle P_i \mid i < \theta\rangle, \langle e_i \mid i \in D\rangle, \langle \pi_{ij} \mid i \leq_T j < \theta\rangle, T\rangle$$

s.t.

(a) $T \subset \theta^2$ is an iteration tree

(b) P_i is either a T-pb or a T-premouse and $P_0 = P$.

(c) If $i \in D$, then $i+1 < \theta$ and e_i is applicable in P_i.

Def $t_i = t_{e_i} =_{\text{Df}} \begin{cases} t_{P_i} & \text{if } e_i \text{ is a top extender;} \\ t_{P_i \| \nu_i} & \text{if not, where } e_i = E^{P}_{\nu_i} \end{cases}$

$t_i^+ = t_{e_i}^+ = t_i^{+P_i}$

$\nu_i = \nu_{e_i} = \nu$ where $e_i = P_i^h \| \nu$ $\quad (h = 0, 1)$

$\lambda_i = \lambda_{e_i} = \lambda$ " " , $\lambda = \lambda_{P_i^h \| \nu}$

$\kappa_i = \operatorname{crit}(e_i)$; $\tau_i = \kappa_i^{+P_i}$.

(Clearly $t_i, t_i^+, \ldots, \tau_i$ are uniquely determined by the pair: P_i, e_i.)

(d) If P_i is a pb, then for all $h \leq_T i$, P_h is a pb and $\pi_{hi}: P_h \underset{\Sigma_1}{\longrightarrow} P_i$.

(i.e. $\pi^l_{hi}: P^l_h \underset{\Sigma_1}{\longrightarrow} P^l_i$ and $\pi^0_{hi} \upharpoonright t^+_{P_h} = \pi^1_{hi} \upharpoonright t^+_{P_h}$)

(Note $\sigma : P \to Q$ means $\sigma = \langle \sigma^0, \sigma^1 \rangle$ with $\sigma^h : P^h \to Q^h$ and $\sigma^0 \upharpoonright t_P = \sigma^1 \upharpoonright t_P$, $\sigma^h{}'' t_P \subset t_Q$ for pb's P and Q. Similarly for $\xrightarrow[\Sigma_1]{}$, $\xrightarrow[\Sigma^*]{}$ etc.)

Def i is <u>simple</u> in $\mathcal{I}$ iff π_{hi} is total for all $h \leq_T i$. i is <u>simple above</u> k in $\mathcal{I}$ iff π_{hi} is total for $k \leq h \leq_T i$. (Hence i is simple if P_h is a pb.)

(e) The π_{ij} commute – i.e.

$$i \leq_T j \leq_T k \to \pi_{jk}\pi_{ij} = \pi_{ik}$$

(by an obvious abuse of notation.)

(f) If $\lim(\lambda)$, then λ is simple above j for some $j < \lambda$. Moreover:

M_λ, $\langle \pi_{i\lambda} \mid i \leq_T \lambda \rangle$ = the direct limit of

$$\langle M_i \mid i \leq_T \rangle, \langle \pi_{ij} \mid i \leq_T j <_T \lambda \rangle.$$

(g) If $i \notin D$, $i+1 < \theta$, then $i = T(i+1)$, $P_i = P_{i+1}$, $\pi_{i,i+1} = \mathrm{id}$

(h) If $i \in D$, then $T(i+1)$ = the least $\zeta \in D$ s.t. $\kappa_i < t_\zeta$ (hence $\zeta \leq i$).

Def Let $i \in D$, $\xi = T(i+1)$

$$P_i^* = \begin{cases} P_\xi & \text{if } \tau_i \text{ is a cardinal in } P_\xi \\ P_\xi \| \gamma & \text{if not, where } \gamma = \gamma_i \text{ is maximal s.t. } \tau_i \text{ is a cardinal in } P_\xi \| \gamma \end{cases}$$

(Note We must show that this definition makes sense if P_i is a T-pb. Clearly $\tau_i \le t^+_{P_i}$, since $\tau_i \le s_{P_i^h}$ for $h = 0, 1$. If τ_i is not a cardinal in P_i^h, then $\tau_i < t^+_{P_i}$ is not a cardinal in $J^{E^{P_i}}_{t^+_{P_i}}$, since $t^+_{P_i}$ is a cardinal in P_i^h. Hence $\gamma_i < t^+_{P_i}$.)

(i) If $\xi = T(i+1)$ and $P_i^* \neq P_\xi$ or ξ is not simple in $\mathcal{I}$, then $\pi_{\xi, i+1} : P_i^* \longrightarrow^*_{e_i} P_{i+1}$.

(j) If $\xi = T(i+1)$ and $P_i^* = P_\xi$ and ξ is simple in $\mathcal{I}$, then $\pi_{\xi, i+1} : P_\xi \longrightarrow_{e_i} P_{i+1}$.

(k) If $i \in D$, then $\nu_i > t^+_j$ for all $j \in D \cap i$.

(Note If P_i is a pb and $\pi_{ij} : P_h \longrightarrow P_j$, then $\pi^0_{ij} \restriction t^+_{P_i} = \pi^1_{ij} \restriction t^+_{P_i}$, where $\text{crit}(\pi^h_{ij}) < t^+_{P_i}$ for $h = 0, 1$. Hence we may set: $\text{crit}(\pi_{ij}) = \text{crit}(\pi^h_{ij})$.)

An examination of the proofs of Lemma 5 – Lemma 7 shows that with slight modification they go through for normal iterations of T-prebicephali. In particular we get: $i < j \to t_i^+ < t_j^+$. Hence $t_i^0, t_i^1 < t_i^+ \leq t_j^0, t_j^1$, since there is no cardinal between t_j^h and t_j^+ in P_j.

The proof of Lemma 8 then yields:

<u>Lemma 8'</u> Let $\mathcal{I} = \langle\langle P_i\rangle, \langle e_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be a normal iteration of the T-pb P. If $i+1$ is not simple in $\mathcal{I}$ and $i \in D$, then e_i is close to P_i^*.

<u>Cor 8.1'</u> If i is not simple in $\mathcal{I}$ and j is simple above i, $j \underset{T}{\geq} i$, then $\pi_{ij} : P_i \underset{\Sigma^*}{\to} P_j$.

We also have:

<u>Lemma 11.4</u> Let $\mathcal{I}$ be a normal it. of a T-pb. P s.t. $P^0 \neq P^1$. Let i be simple in $\mathcal{I}$. Then $P_i^0 \neq P_i^1$.

prf. Suppose not.

Then $\pi_{0i} \upharpoonright P_{t_0^+} : P_{t_0^+} \underset{\Sigma_0}{\longrightarrow} P_{t_i^+}$. Hence, letting e^0, e^1 be the top extenders of P and e_i^0, e_i^1 the top extenders of P_i, we have: $e_i^0 | t_P^+ = e_i^1 | t_P^+$, since for $\alpha < t_P^+$ we have:

$$\alpha \in e^0(X) \longleftrightarrow \pi_{0i}(\alpha) \in e_i^0(\pi_{0i}(X)) = e_i^1(\pi_{0i}(X))$$
$$\longleftrightarrow \alpha \in e^1(X).$$

Since $s \leq t_P^+$ $(h = 0, 1)$, it follows that $e^0 = e^1$; hence $P^0 = P^1$, since $P^h = \langle J_\nu^E, e^h \rangle$, where $\sigma : J_\tau^{E^P} \to_{e^h} J_\nu^E$.

QED (Lemma 11.4)

* →

<u>Def</u> A <u>normal</u> <u>iteration</u> <u>strategy</u> S for the T-pb P is a partial function on normal iterations $\mathcal{I}$ of P of limit length s.t. $b = S(\mathcal{I})$ is a cofinal branch in $T^{\mathcal{I}}$ with a well founded (hence transitive) limit model P_b, whenever $S(\mathcal{I})$ is defined.

By an S-<u>iteration</u> we mean an $\mathcal{I}$ s.t. whenever $\lambda < lh(\mathcal{I})$, $Lim(\lambda)$, then $\{i \mid i \leq_T \lambda\} = S(\mathcal{I}|\lambda)$.

---

* It is clear that each P_i has the same type (A or B) as P if P_i is a T-pb.

Def We say that S is a successful iteration strategy (up to θ) iff whenever $\mathcal{I}$ is an S-iteration (of length $< \theta$), then $\mathcal{I}$ can be continued in the following sense:

- If $lh(\mathcal{I}) = h+1$, $h+2 < \theta$, and e is applicable in P_h with $\nu_e > t_l^+$ for all $l < h$ s.t. $l \in D$, then $\mathcal{I}$ can be extended to $\mathcal{I}'$ of length $h+2$ by setting: $e_h = e$.
- At $lh(\mathcal{I}) = \lambda < \theta$, $Lim(\lambda)$, then $S(\mathcal{I})$ is defined.

Def P is normally iterable (up to θ) iff P possesses a successful iteration strategy (up to θ).

P is weakly iterable iff whenever $\bar{P}$ is countable and $\sigma: \bar{P} \prec P$ and $\bar{P}$ is transitive, then $\bar{P}$ is normally iterable up to $\omega_1 + 1$.

Def By a T-bicephalus, we mean a weakly iterable T-pb P s.t. P^h is preroticl for $h = 0, 1$.

The main result on bicephali is that they are trivial.

<u>Lemma 12</u> Let P be a T-bicephalus. Then $P^0 = P^1$.

proof. Suppose not.

By a Löwenheim–Skolem argument we may assume P to be countable. Hence P is ω_1+1-iterable. Fix an iteration strategy S which is successful up to ω_1+1. We form an S-<u>coiteration</u> of P – i.e. a pair $\mathcal{I} = \langle \mathcal{I}^0, \mathcal{I}^1 \rangle$ of normal S-iterations of length $\theta \leq \omega_1+1$ s.t.

$$\mathcal{I}^h = \langle \langle P_i^h \rangle, \langle e_i^h \mid i \in D^h \rangle, \langle \pi_{ij}^h \rangle, T^h \rangle$$

with $P_i^h = \langle P_i^{h0}, P_i^{h1} \rangle$ if P_i^h is a pb

(a) $P_0^h = P$ $(h=0,1)$

(b) Let $i < \omega_1$ s.t. P_i^h is defined $(h=0,1)$. Set: $t_i^+ \simeq$ the least $\gamma \leq \min(t^+_{P_i^0}, t^+_{P_i^1})$ s.t. there are $e^0 \in \mathcal{U}_\gamma^{P^0}$, $e^1 \in \mathcal{U}_\gamma^{P^1}$ with $e^0 \neq e^1$. Choose such e^0, e^1 and set: $i \in D^h$ iff $e^h \neq \emptyset$. If $i \in D^h$, set $e_i^h = e^h$.

Claim The coiteration terminates below ω_1.

prf. Suppose not.

Then $lh(\mathcal{J}^h) = \omega_1 + 1$. Let $X \prec H_{\omega_2}$ s.t. X is countable. Let $y^0, y^1 \in X$ and X is countable. Then $\sigma : \bar{H} \overset{\sim}{\longleftrightarrow} X$, $\bar{H}$ transitive. Then $\sigma \upharpoonright H_{\omega_1}^{\bar{H}} = id$. Let $\alpha = \omega_1^{\bar{H}}$. Then $\alpha = crit(\sigma)$, $\sigma(\alpha) = \omega_1$. Let $\sigma(\bar{\mathcal{J}}^h) = \mathcal{J}^h$, where

$\bar{\mathcal{J}}^h = \langle\langle \bar{P}_i^h \rangle, \langle \bar{e}_i^h \mid i \in \bar{D}^h \rangle, \langle \bar{\pi}_{ij}^h \rangle, \bar{T}^h \rangle$,

Then $\bar{P}_i^h = P_i^h$ for $i < \alpha$, $\bar{e}_i^h = e_i^h$ for $i < \alpha$, $\bar{D}^h = \alpha \cap D^h$, $\bar{T}^h = T^h \cap \alpha^2$, and $\bar{\pi}_{ij}^h = \pi_{ij}^h$ for $i \leq_{T^h} j$.

But if $b^h = \{ i \mid i \leq_T \omega_1 \}$, $\bar{b}^h = \sigma^{-1}(b^h)$, then $\bar{b}^h = b^h \cap \alpha$. Clearly $\alpha \in b^h$, since α is a limit pt. of b^h; hence $\bar{b}^h = \{ i \mid i \leq_{\bar{T}^h} \alpha \} = \{ i \mid i <_{T^h} \alpha \}$. This characterizes $\bar{T}^h$. Since:

$\bar{P}_\alpha^h$, $\langle \bar{\pi}_{i\alpha}^h \mid i <_{T^h} \alpha \rangle$ = the direct limit of $\langle P_i^h \mid i <_{T^h} \alpha \rangle$, $\langle \pi_{ij}^h \mid i \leq_{T^h} j \leq_{T^h} \alpha \rangle$,

we conclude: $\bar{P}_\alpha^h = P_\alpha^h$, $\bar{\pi}_{i\alpha}^h = \pi_{i\alpha}^h$, We know that any truncation on the branch b^h occurred below α. Hence $\pi^h_{\alpha,\omega_1} : P^h_\alpha \to P^h_{\omega_1}$. Let $x \in P^h_\alpha$,

$x = \pi^h_{i\alpha}(\bar{x})$, $i < \alpha$. Then $\pi^h_{\alpha\omega_1}(x) = \pi^h_{i\omega_1}(\bar{x}) =$
$= \sigma(\pi^h_{i\alpha}).(\bar{x}) = \sigma(\pi^h_{i\alpha}(\bar{x})) = \sigma(x)$. Hence;

(1) $\sigma \restriction P^h_\alpha = \pi^h_{\alpha,\omega_1}$ (hence $\mathrm{crit}(\pi^h_{\alpha,\omega_1}) = \alpha$).

Now let $\eta_h + 1 \leq_{T^h} \omega_1$ s.t. $\alpha = T^h(\eta_h + 1)$.

Then $\pi^h_{\alpha,\eta_h+1} : P^h_\alpha \to_{e_{\eta_h}} P^h_{\eta_h+1}$ (or $\to^*_{e_{\eta_h}}$ if P^h_α is not a p b). But then $\alpha = \kappa_{e_{\eta_h}} < t_\alpha$ and $\tau = \tau_e \leq t^+_\alpha$, where t^+_α is a cardinal in P^{hl}_α for $h, l < 2$ and $J^{E^{P^0_\alpha}}_{t^+_\alpha} = J^{E^{P^1_\alpha}}_{t^+_\alpha}$. But then

(2) $\tau = \alpha^{+P^h_\alpha} = \tau_{e^h_{\eta_h}}$ $(h = 0, 1)$ and

(3) $J^{E^{P^0_\alpha}}_\tau = J^{E^{P^1_\alpha}}_\tau$ (hence $\mathbb{P}(\alpha) \cap P^0_\alpha = \mathbb{P}(\alpha) \cap P^1_\alpha$).

Let $X \in \mathbb{P}(\alpha) \cap P^h_\alpha$. Let $\xi < \min(t^0_{\eta_0}, t^1_{\eta_1})$ (where $t^h_\xi =_{\mathrm{Df}} t_{P^h_\xi}$). Then

(4) $\xi \in e^h_{\eta_h}(X) \longleftrightarrow \xi \in \pi^h_{\alpha,\eta_h+1}(X)$
$\longleftrightarrow \xi \in \pi_{\alpha,\omega_1}(X)$
$\longleftrightarrow \xi \in \sigma(X)$,

since $crit(\pi^h_{\eta_h+1}, \omega_1) \geq t^h_{\eta_h}$. Thus

(5) $e^0_{\eta_0} | \xi = e^1_{\eta_1} | \xi$ where $\xi = \min(t^0_{\eta_0}, t^1_{\eta_1})$.

From this we derive a contradiction.

<u>Case 1</u> $\eta_0 = \eta_1 = \eta$.

<u>Case 1.1</u> $t^0_\eta = t^1_\eta$. Then $e^0_\eta = e^1_\eta \neq \emptyset$, since $t^h_\eta \geq s\tilde{p}^h_\eta$. Contr!

<u>Case 1.2</u> $t^0_\eta \neq t^1_\eta$ (let e.g. $t^0_\eta < t^1_\eta$)

<u>Case 1.2.1</u> P^1_η is a type A pb.

Then $\tilde{P}^0_\eta = (\tilde{P}^0_\eta)_{t^0_\eta} = (\tilde{P}^1_\eta)_{t^0_\eta} \in \tilde{P}^1_\eta$.

Hence $\tilde{P}^0_\eta \in \tilde{P}^1_\eta \| t^+_\eta$. Hence $t^+_\eta =$
$= (t^0_\eta)^{+\tilde{P}^0_\eta} < t^+_\eta$. Contr!

<u>Case 1.2.2</u> Case 1.2.1 fails and $t^0_\eta = t_{\tilde{p}^0_\eta}$.

Then $P^0_\eta = (P^1_\eta)_{t^0_\eta}$ where $t^0_\eta < t^1_\eta = t_{\tilde{p}^1_\eta}$;

hence $P^0_\eta \in P^1_\eta$ and we get a contradiction as before.

<u>Case 1.2.3</u> The above fail.

Then P^0_η is a type A pb. Hence so is P. If P^1_η were a pb it would also have to be of type A + hence Case 1.2.1 would apply. Hence P^1_η is a premouse. Since $t^0_i = t_{p^0_i} = s_{\tilde{p}^0_i} <$ $< t_{\tilde{p}^0_i}$, we have $t \in T_{\tilde{p}^0_i}$ where $t = t_{\tilde{p}^0_i}$.

But $T_{\tilde{p}^0_i}$ is bounded in t^0_i, since otherwise $t^0_i \in T_{\tilde{p}^0_i}$ and $t^0_i = t_{\tilde{p}^0_i}$.

Hence there is $\xi < t^0_i$ s.t. $(\xi, t) \cap T_{\tilde{p}^0_i} = \emptyset$. Since $\tilde{p}^0_i = (\tilde{p}^1_i)_{t^0_i}$, there is a canonical $\sigma : \tilde{p}^0_i \xrightarrow[\Sigma_1]{} \tilde{p}^1_i$ s.t. $\sigma \restriction t^0_i = id$. Hence $\sigma(t) \in T_{\tilde{p}^1_i}$ and $(\xi, \sigma(t))_{T_{\tilde{p}^1_i}} = \emptyset$

$s_{\tilde{p}^1_i} \leq t^1_i = t_{\tilde{p}^1_i} < t^+_i$. But t is a cardinal in $\tilde{p}^0_i$ since p^0 is of type A. Hence $t^+_i \leq t \leq \sigma(t)$ and $\sigma(t) = t^1_i < t^+_i$. Contr!

QED (Case 1)

<u>Case 2</u> $\eta_0 \neq \eta_1$ (Let e.g. $\eta_0 < \eta_1$)

Then $t^+_{\eta_0} < t^+_{\eta_1}$ and $t^h_{\eta_0} < t^k_{\eta_1}$ $(h, k = 0, 1)$

<u>Case 2.1</u> $P^1_{\eta_1}$ is a type A pb

Exactly like Case 1.2.1

<u>Case 2.2:</u> Case 1.2.1 fails and $t^0_{\eta_0} = t_{\tilde{P}^0_{\eta_0}}$,

Exactly like Case 1.2.2

<u>Case 2.3</u> The above fail.

We again have: $P^0_{\eta_0}$ is a type A pb and $P^1_{\eta_1}$ is a premouse. Set:

$b = \{i \mid i <_{T^1} \omega_1\}$. Then $\alpha, \eta_1 \in b$ and any truncations in b must occur below α.

Let $j+1 \leq_{T^1} \eta_1$ be minimal s.t. $\eta_0 \leq j$. Then $\xi = T^1(j+1) \leq \eta_0$.

<u>Case 2.3.1</u> $\kappa^1_j < t^0_{\eta_0}$.

(1) $\kappa^1_j > \kappa^0_{\eta_0} = \kappa^1_{\eta_1}$.

Suppose not. P^*_j is an active premouse. Set $\bar{\kappa} = \kappa_{P^*_j}$. Then

$\pi_{\xi, j+1}(\bar{\kappa}) = \kappa^1_{j+1} \leq \kappa^1_{\eta_1} = \kappa^0_{\eta_0}$.

Hence $\kappa^1_i > \bar{\kappa}$, since otherwise

$$\kappa^1_{i+1} = \lambda^1_i \geq t^1_i > \kappa^0_{\eta_0} = \kappa^1_{\eta_1} \leq \kappa^1_{i+1}.$$

Contr! Hence $\bar{\kappa} = \pi_{\xi, i+1}(\bar{\kappa}) = \kappa_{i+1}$.

But $\mathrm{crit}(\pi_{i+1, \eta_1}) \geq t^1_{i+1} > t^1_{\eta_0} > \kappa^0_{\eta_0} =$

$= \kappa^1_{\eta_1}$. QED (1)

But then, since $\mathrm{crit}(\pi_{\xi, \eta_1}) = \kappa^1_i$,

we have $e^1_\xi | \kappa^1_i = e^1_{\eta_1} | \kappa^1_i = e^0_{\eta_0} | \kappa^1_i$.

Let $N = (\tilde{P}^0_{\eta_0})_{\kappa^1_i}$. Then $N \in$

$$\in J^{E^{\tilde{P}^0_{\eta_0}}}_{\tau^1_i} = J^{E^{\tilde{P}^1_{\eta_1}}}_{\tau^1_i} = J^{E^{\tilde{P}^1_\xi}}_{\tau^1_i} \in \tilde{P}^1_\xi.$$

and $N = (\tilde{P}^1_{\eta_1})_{\kappa^1_i} = (\tilde{P}^1_\xi)_{\kappa^1_i}$. Let

$\pi_{\xi, \ldots}(N) = N'$. Then $N' =$

$= (\tilde{P}^1_{\eta_1})_{\kappa'}$, where $\kappa' = \pi_{\xi, \eta_1}(\kappa^1_i) \geq$

$\geq \lambda^1_i \geq t^1_i \geq t^1_{\eta_0} \geq t^0_{\eta_0}$. Hence

$$\therefore \tilde{P}^0_{\eta_0} = (\tilde{P}^1_{\eta_1})_{t^0_{\eta_0}} \in J^{E^{\tilde{P}^1_{\eta_1}}}_{t^+_{\eta_0}} = J^{E^{\tilde{P}^0_{\eta_0}}}_{t^+_{\eta_0}} \in \tilde{P}^0_{\eta_0},$$

Contr! QED (Case 2.3.1)

Hence $t^0_{\gamma_0} \le \kappa^1_1$. Hence $\xi = \gamma_0 = i$, since otherwise $\kappa^1_1 < t^1_\xi < t^+_\xi \le t^0_{\gamma_0}$. But $\kappa^1_1 < t^1_\xi$. Hence $t^0_{\gamma_0} < t^1_{\gamma_0}$ and $e^0_{\gamma_0} | t^0_{\gamma_0} = e^1_{\gamma_1} | t^0_{\gamma_0} = e^1_{\gamma_0} | t^0_{\gamma_0}$, since $\mathrm{crit}(\pi_{\gamma_0 \gamma_1}) = \kappa^1_1 \ge t^0_{\gamma_0}$. But this leads to a contradiction just as in Case 1.2.

QED (Claim)

Now let P'^0, P'^1 be the ultimate coiterates. Using presolidity it follows as usual that at least one P'^h is a simple iterate of P. Moreover, P'^h is a segment of P'^{1-h} if P'^{1-h} is not simple. Let $Q = P'^h$ where Q is a simple iterate of P and a segment of P'^{1-h}. Then $Q^0 = Q^1$. Using the iteration map $\pi : P \to Q$ we conclude that $e^0 | t^+_P = e^1 | t^+_P$, where e^0, e^1 are the top extenders of P. Hence $P^0 = P^1$. Contr!

QED (Lemma 12)

Remark Lemma 12 can be proven without assuming (*) or (**).

Def Let P be a T-pb. Let $\xi \leq t_P$ and let $e \in U_\xi^P$. e is superstrong in P iff $\nu_e = \xi$ (hence $\lambda_e = t_e^P$, as is easily seen).

Note If $\mathcal{I} = \langle\langle P_i\rangle, \langle e_i \mid i \in D\rangle, \langle \pi_{ij}\rangle, T\rangle$ is a normal iteration of a T-pb, then $t_i^+ = t_j^+$, $i < j$ can only occur if e_i is superstrong in P_i, since then the situation of Cor 6.1 must be present.

Def Let $\mathcal{I}$ be as above. i is superstrong in $\mathcal{I}$ iff e_i is superstrong in P_i.

It is easily seen that:

Fact Let i be superstrong in $\mathcal{I}$. No $e \in U_{t_i^+}^{P^{i+1}}$ is superstrong in P^{i+1}.

We now modify (b) in the definition of coiteration to read:

(b') Let $i < \omega_1$ s.t. P_i^h is defined $(h = 0, 1)$. Set: $t_i^+ \simeq$ the least $\gamma \leq \min(t^+_{p_i^0}, t^+_{p_i^1})$ s.t. there are $e^0 \in U_\gamma^{p^0}$, $e^1 \in U_\gamma^{p^1}$ with $e^0 \neq e^1$. If possible, choose e^0, e^1 s.t. both are superstrong or both are not superstrong. Set:

$i \in D^h$ iff $e^h \neq \emptyset$, <u>unless</u> e^{1-h} is superstrong in P_i^{1-h} and e^h is not superstrong in P_i^h, in which case we set: $i \notin D^h$.

Set: $e_i^h = e^h$ if $i \in D^h$.

The proof of Case 1 remains unchanged. The proof of Case 2 remains unchanged for $t^+_{\gamma_0} < t^+_{\gamma_1}$.

Now let $t^+_{\gamma_0} = t^+_{\gamma_1}$. If $t_{\gamma_0} \neq t_{\gamma_1}$ we get a contradiction exactly as in case 1, so $t_{\gamma_0} = t_{\gamma_1}$.

Hence $e_{\gamma_0} = e_{\gamma_1}$ and $\tilde{P}_{\gamma_0} = \tilde{P}_{\gamma_1}$.

Since $t^+_{\gamma_1} \geq t^+_{\gamma_0+1}$, we have: $t^+_{\gamma_0+1} = t^+_{\gamma_0}$.

Hence γ_0 is superstrong in $\mathcal{Y}^0$ or $\mathcal{Y}^1$. But if it were superstrong in $\mathcal{Y}^1$ it would have to be so in $\mathcal{Y}^0$ as well, since $\gamma_0 \in D^0$. Hence $e^0_{\gamma_0}$ is superstrong in $P^0_{\gamma_0}$.

<u>Claim</u> $\gamma_0 + 1$ is not superstrong in $\mathcal{Y}^0$ or $\mathcal{Y}^1$.

proof. Suppose not. $\gamma_0 + 1$ is not superstrong in $\mathcal{Y}^0$ by the above Fact. Hence it is superstrong in $\mathcal{Y}^1$. Hence $\gamma_0 \notin D^1$ by the above Fact. Thus $P^1_{\gamma_0} = P^1_{\gamma_0+1}$. Let $e = e^1_{\gamma_0+1}$. Since $e \in \mathcal{U}^{P^1_{\gamma_0}}_{\gamma_0}$ is superstrong and was not chosen as $e^1_{\gamma_0}$, we must have: $e = e^0_{\gamma_0}$. Since the iteration did not terminate at γ_0, there must be $e' \in \mathcal{U}^{P^1_{\gamma_0}}_{\gamma_0}$ s.t. e' is not

superstrong in $P^1_{\eta_0}$. But since e' was not chosen as $e^1_{\eta_0+1}$, we must have: $e' = e^0_{\eta_0+1}$. We then have the following situation: Set $Q = P^1_{\eta_0}$, $F^0 = e$, $F^1 = e'$. Then Q is a T-pb*
$Q = \langle Q^0, Q^1, t \rangle$ with e.g. F^h the top extender of Q^h and

$$\pi : Q^0 \longrightarrow_{F^0} Q^1.$$

But then $\kappa_{Q^1} \geq \lambda_{Q^0}$ + hence ,

$$\tau_{Q^1} \geq \nu_{Q^0} \geq t = t^+_Q.$$

This is impossible in prebicephali. Contr! QED (Claim)

But then $U^{P^h_{\eta_0+2}}_{t^+_{\eta_0}} = \{\emptyset\}$ for $h = 0, 1$.

Hence $t^+_{\eta_0+2} > t^+_{\eta_0} = t^+_{\eta_1}$. Hence

$\eta_1 = \eta_0 + 1$ and $e^1_{\eta_0+1} = e^0_{\eta_0}$,

$P^1_{\eta_0+1} = P^0_{\eta_0}$. This is impossible since $e^1_{\eta_0+1}$ is not superstrong in $P^1_{\eta_0+1}$

Contr! QED