§3 Dodd Premice

Def Let $N = \langle J_\nu^E, F\rangle$ be an active ppm.
$D(N) = D_N$ = the set of limit cardinals in N s.t. $\bigwedge \mu \in (\kappa_N, \delta)$ $o(\mu) \leq \delta$ in N.
(Here $o(\mu)$ = the least upper bound of the ν s.t. $E_\nu \neq \emptyset$ & $crit(E_\nu) = \mu$.)
It is easily seen that $D_N \subset \lambda_N + 1$ is closed in $\lambda_N + 1$. Moreover, $D_N \neq \emptyset$ since e.g. $\kappa^{+\omega N} \in D_N$.

Following §2 we then set:

Def $d_N = \begin{cases} \min\{\delta \in D_N \mid \delta \geq s_N\} & \text{if such } \delta \text{ exists;} \\ s_N & \text{if not.} \end{cases}$

The class of D-premice is defined exactly as in §2. As usual we set:

$D(\nu)_M = D_\nu^M = D_{M\|\nu}$ for $E_\nu^M \neq \emptyset$

$d(\nu)_M = d_\nu^M = d_{M\|\nu}$ "

At present, however, we are interested in a subclass of the D-premice which we shall call d-premice or Dodd premice. We call them that because they are, in fact, based on ideas of Tony Dodd.

They are also related to Ralf Schindler's "mice below 0-handgrenade" and we shall make use of Schindler's methods.

Def Let M be a D-premouse. M is a d-premouse iff

(a) $d_\nu^M \in D_\nu^M$ whenever $E_\nu^M \neq \emptyset$

(b) Let $E_\nu^M \neq \emptyset$, $N = M \| \nu$. Let α be a limit cardinal in N s.t. $\alpha^{+N} \leq s_N$. There is $\mu \in (\alpha, \alpha^{+N})$ s.t. $E_\mu^N \neq \emptyset$ and $\mathrm{crit}(E_\mu^N) = \kappa_N$.

Note (a) is a strong restriction on the D-premice. (b) restricts the class of premice but should not restrict the resulting class of mice, since it follows from the condensation lemmas (e.g. Lemma 4' in § IV of [CR]).

Following § 2 we define:

Def Let N be an active ppm.

$\tilde{C}_N = \{\xi < d_N \mid \xi = d_{N_\xi}\}$

$C_N = \{\xi \in \tilde{C}_N \mid N_\xi$ satisfies D-MIS$\}$.

Since d_N is always a limit cardinal in N for d-premice N we have by § 2 Cor 2.4:

<u>Lemma 1</u> Let N be an active d-premouse. Then $\tilde{C}_N = C_N = \{\xi \in \tilde{C}_N \mid N_\xi \in N\}$

We of course set:
$\hat{C}^M(\nu) = \tilde{C}^M_\nu = \tilde{C}_{M\|\nu}$ for $E^M_\nu = \emptyset$
(similarly for $C(\nu)$).

The most important theorem on D-iteration of Dodd premice is:

<u>Lemma 2</u> Let $\mathcal{I}$ be a normal D-iteration of M, where M is a d-premouse. Then if $i < j$, $i, j \in D^{\mathcal{I}}$, then $\kappa_j \notin (\kappa_i, d_i)$.

pf. Suppose not
Let $\mathcal{I} = \langle\langle M_i\rangle, \langle \nu_i \mid i \in D\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be a counterexample of minimal length θ.
We suppose w.l.o.g. that $\mathcal{I}$ is direct (i.e. $D = \theta$). Clearly $\theta = j+1$ where $j > i$ and $\kappa_j \in (\kappa_i, d_i)$

[Recall that by the convention of §2, $d_i =_{\mathrm{nf}} d^{M_i}_{\nu_i}$, $d^+_i =_{\mathrm{nf}} d^{+M_i}_{\nu_i}$.]

Then:

(1) $\neg\, \mathrm{Lim}(j)$

If not there is $h+1 <_T j$ s.t. j is simple above $h+1$ and $T(h+1) > i$. Then $\mathrm{crit}(\pi_{h+1,j}) = \kappa_h$ where $\kappa_h \geq d_i > \kappa_j$ and $\nu_j = \pi_{h+1,j}(\bar\nu)$ for some $\bar\nu \leq ht(M_{h+1})$. Then $\kappa_j = \mathrm{crit}(E^{M_{h+1}}_{\bar\nu})$. Define a new iteration $\mathcal{J}'$ of length $h+2$ by setting: $\mathcal{J}'|h+1 = \mathcal{J}|h+1$, $\nu'_{h+1} = \bar\nu$. Then $\mathcal{J}'$ is a counterexample of smaller length. Contr! QED (1)

Thus:

(2) $j = h+1$ where $h \geq i$.

(3) $\kappa_h > \kappa_j$, since otherwise $\kappa_j \geq \kappa_h > d_h \geq d_j$. Hence:

(4) $\kappa_h \geq d_i$, since otherwise $\kappa_h \leq \kappa_i$ by the minimality of j. Hence $\kappa_h < \kappa_j$. Contr!

Now let $\xi = T(j)$. Then $\xi \leq h$ and $d_i \leq \kappa_h < d_\xi$. Hence $i < \xi$. Note that

(5) $\nu_j = ht(M_j)$,

since otherwise there is $\mu \in (d_i, d_i^+)$ with $E_\mu \neq \emptyset \wedge \mathrm{crit}(E_\mu) = \kappa_j$ in M_j, hence in M_i as well, since $J^{E^{M_i}}_{d_i^+} = J^{E^{M_j}}_{d^+}$.

Hence $o(\kappa_j) > \alpha_i$ in $M_i \| \nu_i$, where $\kappa_i < \kappa_j < \alpha_i$. Contr! QED (5)

But then, $\pi_{\xi i}(\nu') = \nu_j$ where $\nu' = ht(M_h^*)$. Hence $\kappa_j = crit(E_{\nu'}^{M_h})$. Define an iteration $\mathcal{J}'$ of length $\xi + 2$ as follows: $\mathcal{J}' | (\xi+1) = \mathcal{J} | (\xi+1)$, $\nu_\xi' = \nu'$. Then $\mathcal{J}'$ is shorter than $\mathcal{J}'$ but $\kappa_\xi' = \kappa_j \in (\kappa_i, \alpha_i)$, where $i < \xi$. Contr! QED (Lemma 2)

Def $\Gamma(\mathcal{J})$ = the set of $i < lh(\mathcal{J})$ s.t. for all $j < lh(\mathcal{J})$, $i \leq j \rightarrow i \leq_T j$ in $\mathcal{J}$.
An iteration $\mathcal{J}$ is almost linear iff for all limit $\lambda \leq lh(\mathcal{J})$, $\Gamma(\mathcal{J}|\lambda)$ is cofinal.

Fact Let $\mathcal{J}$ be a d-iteration (i.e. a D-iteration of a d-premouse). $\mathcal{J}$ is almost linear iff for all $\eta < lh(\mathcal{J})$ the set $\{i \mid T(i+1) \leq \eta \leq i\}$ is finite.

proof. (w.l.o.g. take $\mathcal{J}$ as normal)
$(\rightarrow)$ is trivial.
$(\leftarrow)$ Suppose not. Let $\eta < \lambda$. Let $\eta \leq i$ s.t. κ_i is minimal for $\eta \leq i < \lambda$. Then $T(i+1) \leq \eta$ and there at most finitely many j with $\kappa_j = \kappa_i$, $j < \lambda$. Then let j be a maximal such. Then $\kappa_h \geq \alpha_j$ for $h > j$. Hence $j \leq_T h$ for $h > j$. QED

Thus, if $\mathcal{T}$ is almost linear and of limit length, then $\mathcal{T}$ composes into a sequence of successive iterations whose initial points are the models N_i $(i \in \Gamma(\mathcal{T}))$. $\mathcal{T}$ then has the unique cofinal branch $b = \{j \mid \bigvee i \in \Gamma(\mathcal{T})\ j \leq_T i\}$.

As an immediate consequence of Lemma 2 we get:

<u>Lemma 3</u> Let $\mathcal{T}$ be a putative normal d-iteration of N of length $k+1$. Then $\mathcal{T}$ is almost linear.

[<u>Note</u> Following Steel, we call $\mathcal{T}$ a putative iteration if it is like an iteration except that the final model, if there is one, may not be well founded, although it is formed according to the iteration rules. We still take its well founded core as being transitive.]

[<u>Note</u> We shall write "d-iteration" to mean a D-iteration of a d-premouse.]

proof of Lemma 3.

Suppose not. Let $X = \{i \mid T(i+1) \leq \eta < i\}$ be infinite. Let $\langle i_n \mid n < \omega \rangle$ enumerate the first ω elements of X. Then if $i_n < j < i_{n+1}$ we must have $T(j+1) \geq i_n + 1$, since otherwise $\kappa_j < t_{i_n}$ and $\kappa_j \notin (\kappa_{i_n}, t_{i_n})$. Hence $\kappa_j \leq \kappa_{i_n} < t_\eta$ and $T(j+1) \leq \eta$. Contr! Thus $i_n + 1 \leq_T j$ for $i_n < j \leq i_{n+1}$. In particular, if $\eta < h \leq i_n$, then $h \not<_T j$ for $i_n < j \leq i_{n+1}$, since $T(i_n + 1) \leq \eta$. Now let b be a cofinal branch in $\lambda = \sup_n i_n$. For sufficiently large $j \in b$ there is n_j s.t. $i_{n_j} < j \leq i_{n_j + 1}$. Pick j, j' s.t. $n_j < n_{j'}$. Then $j \not<_T j'$. Contr! QED (Lemma 3)

Thus, if a normal d-iteration of limit length has any cofinal branch at all, it is almost linear and, therefore, has a unique cofinal branch.

Since the property of not having any cofinal branch is absolute in ZF-model it follows by a Löwenheim-Skolem argument that:

<u>Corollary 3.1</u> Let N be a d-premouse. Assume that whenever $\sigma : Q \xrightarrow{\Sigma_\omega} N$ and Q is a countable d-premouse, then Q is countably normally d-iterable. Then N is normally d-iterable.

(Clearly the same holds with "d-iterable" in place of "normally d-iterable".)

. . . .

A d-<u>weasel</u> is of course a structure $W = J^E_\infty$ s.t. each $W \| \xi = \langle J^E_\xi, E_{\omega\xi} \rangle$ is a d-mouse (i.e. a d-iterable premouse). We shall construct a d-weasel K^d which is universal wrt. d-mice and in which many large cardinal properties of V are preserved (i.e. if κ is strong in V, then it is strong in K^d). We construct K^d as the limit of a Steel-type array. The construction will be very much like Ralf Schindler's construction of a universal weasel for his "mice below 0-handgrenade". As in his case we get by with a very weak "background condition" for adding an extender to the array. Schindler requires only that the extender be ω-complete. This enables him to prove that each N_i in the array is normally iterable, but not that it is fully iterable. Normal iterability suffices for many purposes. Schindler's construction would yield full iterability if we modified it to require that each new extender in the array be strongly ω-complete in the following sense:

Def Let F be an extender at κ, λ. F is strongly ω-complete iff whenever U is countable, then there is $g: U \cap \lambda \to \kappa$ s.t. whenever $h \in U$ s.t. $h: \xi \to \mathcal{P}(\kappa) \cap \mathrm{dom}(F)$ for a $\xi < \kappa$, $X \in \mathrm{rng}(h)$, and $\alpha_1, \ldots, \alpha_n < \lambda$, then

$$\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \in X \longleftrightarrow \langle \alpha_1, \ldots, \alpha_n \rangle \in F(X).$$

In the present paper we shall follow Schindler in requiring only ω-completeness and proving only normal iterability. For the reader this has the advantage of avoiding some technical detail. We are convinced - but have not yet fully checked - that a construction using strong ω-completeness will yield an array of fully iterable mice.

. . . .

We define an array as follows:
N_0 and N_λ for $\mathrm{Lim}(\lambda)$ are defined as usual.
Now let N_i be defined. If N_i is not weakly normally iterable, then N_{i+1} is undefined.
Now suppose that N_i is weakly normally iterable. (This is enough to prove solidarity and the condensation lemmas.) Let $M_i = \mathrm{core}(N_i) = \langle J^E_\alpha, E_{\omega\alpha} \rangle$. If $E_{\omega\alpha} = \emptyset$ and there exists a d-premouse $N = \langle J^{E'}_\beta, F \rangle$ s.t. $F \neq \emptyset$ is ω-complete and $\alpha = d_N$, $J^{E'}_\alpha = J^E_\alpha$, set:

Choosing N, if possible, to be of type B.

$N_{i+1} = N$ for such an N. Otherwise set:
$N_{i+1} = \langle J^E_{\alpha+1}, \emptyset \rangle$. This completes the definition.

We claim that N_∞ is defined. Suppose not. Then there is i s.t. N_i is defined and N_{i+1} not. Hence N_i is not weakly normally iterable. We derive a contradiction by showing that if $\sigma : Q \xrightarrow[\Sigma_\omega]{} N_i$, then Q is countably normally iterable. It, in fact, suffices to assume that σ is Σ^*-preserving and <u>fine</u> in the following sense:

<u>Def</u> Let $\sigma : Q \xrightarrow[\Sigma_0]{} N$, where Q, N are d-premice. σ is <u>fine</u> iff $\sigma(d^+_Q) \le d^+_N$ in the case that Q, N are active.
(<u>Note</u> Fineness guarantees that if ν is applicable in Q, then $\sigma(\nu)$ is applicable in N.)

Def Let $\delta : Q \xrightarrow[\Sigma^*]{} N_\gamma^v$ be fine. Let $\mathcal{I}$ be a putative normal d-iteration of Q. $\mathcal{I}$ is <u>realizable wrt. δ</u> iff

(a) If $\mathcal{I}$ is of length $k+1$, then there is a fine $\delta' : Q_k \longrightarrow N_{\gamma'}$, where

- $\gamma' = \gamma$ and $\delta' \pi_{0k} = \delta$ if i is simple in $\mathcal{I}$
- $\gamma' < \gamma$ if i is not simple in $\mathcal{I}$
- δ' is $\Sigma_0^{(n)}$-preserving whenever

$$\operatorname{lub}_{i<k} \kappa_i \leq \omega\rho^n_{Q_k}.$$

(b) If $\mathcal{I}$ is of limit length λ, then $\mathcal{I}$ has a putative extension of length $\lambda+1$ which is realizable in the sense of (a).

Def If (a) holds, we say that $\langle \delta', \gamma' \rangle$ is a <u>realization</u> of $\mathcal{I}$ wrt δ.

<u>Lemma 4</u> N_∞ is defined.

proof.
It suffices to show that N_γ is weakly normally iterable whenever $\gamma < \infty$ is defined. Hence it suffices to show:

<u>Lemma 4.1</u> Let N_γ be defined, $\gamma < \infty$. Let Q be a d-premouse and let $\delta : Q \xrightarrow[\Sigma^*]{} N_\gamma$, where Q is countable and $\mathcal{I}$ is a countable normal putative d-iteration of Q. Then $\mathcal{I}$ is realizable wrt. δ.

proof of Lemma 4.1 (w.l.o.g $\mathcal{T}$ is direct)

We first construct a sequence $\langle \delta_i, \gamma_i \rangle$ $(i < lh(\mathcal{T}))$ s.t.

(a) $\langle \delta_i, \gamma_i \rangle$ is a realization of $\mathcal{T}|i+1$ wrt. δ

(b) If $j \leq_T i$ and i is simple above j, then $\gamma_i = \gamma_j$ and $\delta_i \pi_{ji} = \delta_j$.

We note that if $h = T(i+1)$ and $\eta < ht(Q_h)$, (hence $Q_i^* \neq Q_h$), then η_i is the least $\beta \geq \nu_h$ s.t. $\omega\rho^{\omega}_{Q_h \| \beta} < \tau_i$. It follows easily, using the notation of the "resurrection sequence" in §2, that $\eta_i = \bar{\beta}_n(Q_h, \nu_h)$ for some $n > 0$. Setting $\gamma' = \eta_n[\gamma_h, \delta_h(\nu_h)]$, we then have; $\sigma^{(n)}\delta_h : Q_i^* \xrightarrow[\Sigma^*]{} N_{\gamma'}$, where $\sigma^{(n)} = \sigma^{(n)}[\gamma_h, \delta(\nu_h)]$. This suggests the following clause:

(c) If $i = j+1$, $\eta_j = \bar{\beta}_n(Q_h, \nu_h)$, then $\gamma_i = \eta_n[\gamma_h, \delta_h(\nu_h)]$ and $\delta_i \pi_{hi} = \sigma^{(n)}\delta_h$ where $\sigma^{(n)} = \sigma^{(n)}[\gamma_h, \delta_h(\nu_h)]$.

(<u>Note</u> If $n = 0$, this gives us no more information than was contained in (b)

We shall construct $\langle \delta_i, \gamma_i \rangle$ inductively verifying (a)-(c) at each stage.

We first note a consequence of (a)-(c):

(1) Let $h \leq_T i$. Let $h = T(j+1)$, $j+1 \leq_T i$. Let $\sigma^{(m)}$ be as in (c) (i.e. $\delta_{j+1} = \sigma^{(m)} \delta_h$.) Let $\kappa = \kappa_j = \mathrm{crit}(\pi_{hi})$. Then $\delta_i(\kappa) < \sigma^{(m)} \delta_h(\kappa)$, $\delta_i \restriction \kappa = \sigma^{(m)} \delta_h \restriction \kappa$, and $\delta_i(X) = \delta_i(\kappa) \cap \sigma^{(m)} \delta_h(X)$ for all $X \in \mathcal{P}(\kappa) \cap Q_i$.

proof. Suppose not. Let i be the least counterexample. By (b) we have: $i = l+1$ where $\eta_l < Q_{T(i)}$. Let $k = T(i)$. If $k > h$ then d_h^+ is a cardinal in Q_k and $\tau = \tau_j$ is a cardinal in $J_{d_h^+}^{E^{Q_h}} = J_{d_h^+}^{E^{Q_k}}$. Hence τ is a cardinal in Q_k, where $\delta_i = \tilde{\sigma} \delta_k$, $\tilde{\sigma} = \sigma^{(m)}[\gamma_k, \delta_k(\nu_k)]$ for some m. By Fact 6 about arrays in §2, $\tilde{\sigma} \restriction \tau = \mathrm{id}$. Hence $\delta_i \restriction \tau = \delta_k \restriction \tau$ and $k < i$ must be a counterexample. Contr! Thus $k = h$. But then $\delta_i(\kappa) < \delta_i(\lambda_j) = \delta_i \pi_{hi}(\kappa) = \sigma^{(m)} \delta_h(\kappa)$. Moreover, $\delta_i \restriction \kappa = \delta_i \pi_{hi} \restriction \kappa = \sigma^{(m)} \delta_h \restriction \kappa$.

If $X \subset u$, then $\delta_i(X) = \delta_i(u \cap \pi_{h_i}(X)) =$
$= \delta_i(u) \cap \sigma^{(n)} \delta_h(X)$. QED (1)

We also note the following fact:

<u>Lemma 4.1.1</u> Let $E_\nu^{N_\gamma} \neq \emptyset$ s.t. τ_ν is a cardinal in N_γ and ν is applicable in N_γ. Then $E_\nu^{N_\gamma}$ is ω-complete.

proof.

Let U be countable. We claim that there is $g : U \cap \lambda_\nu \longrightarrow \kappa_\nu$ which verifies ω-completeness wrt. U in the following sense: If $\alpha_1, \ldots, \alpha_n < \lambda_\nu$, $X \in \mathbb{P}(\kappa) \cap U \cap N_\gamma$ then $\langle g(\vec{\alpha})\rangle \in X \longleftrightarrow \langle \vec{\alpha} \rangle \in E_\nu^{N_\gamma}(X)$.

Set: $\gamma^* = \gamma^*[\gamma, \nu]$, $\sigma^* = \sigma^*[\gamma, \nu]$.

Then $\sigma^* : N_\gamma \| \nu \underset{\Sigma^*}{\longrightarrow} N_{\gamma^*}$ and the top extender F^* of N_{γ^*} is ω-complete. Let $g^* : \sigma^{*\,\prime\prime}(U \cap \lambda_\nu) \longrightarrow \kappa_\nu$ verify the ω-completeness of F^* wrt. $U^* = (U \cap \mathbb{P}(\kappa_\nu)) \cup \sigma^{*\,\prime\prime}(U \cap \lambda_\nu)$. Note that $\sigma^* \upharpoonright (\tau_\nu + 1) = \mathrm{id}$ by Fact 8 of §2, since τ_ν is a cardinal in N_γ. Hence $g(\alpha) = g^* \sigma^*(\alpha)$ verifies the ω-completeness of $E_\nu^{N_\gamma}$ wrt. U.

QED (4.4.1)

We now define δ_i by induction on i.

Case 1 $i=0$. Set: $\delta_0 = \delta$

Case 2 $i = j+1$.

Let $h = T(i)$. For $h \le l < j$ we have $\kappa_l \ge \kappa_j$, since otherwise $\kappa_l < \kappa_j < d_l$. Hence $T(l+1) \ge h$. Hence $h \le_T l$ for $h \le l \le j$.

Case 1 $h = j$

Let $\eta_h = \bar\beta_n(Q_h, \nu_h)$. Following (c) we set;
$\gamma_i = \eta_m[\gamma_h, \delta_h(\nu_h)]$, $\sigma^{(n)} = \sigma^{(n)}[\gamma_h, \delta_h(\nu_h)]$

Then $\sigma^{(n)}\delta_h : Q_h^* \to N_{\gamma_i}$. τ_h is a cardinal in Q_h^*. Hence $\sigma^{(n)}\delta_h(\tau_h)$ is a cardinal in N_{γ_i}. Hence $F^* = E^{N_{\gamma_i}}_{\sigma^{(n)}\delta_h(\nu_h)}$ is ω-complete. Hence there is $g : \lambda_h \to \sigma^{(n)}\delta_h(\kappa_h)$ s.t. whenever $\alpha_1, \dots, \alpha_n < \lambda_h$ and $X \in \mathcal{P}(\kappa_h) \cap Q_h^*$, then: $\langle g(\vec\alpha)\rangle \in \sigma^{(n)}\delta_h(X) \longleftrightarrow \langle\vec\alpha\rangle \in F(X)$, where $F = E^{Q_h}_{\nu_h}$. (This is because $\sigma^{(n)}\delta_h(F(X)) = F^*(\sigma^{(n)}\delta_h(X))$.)

Note that $\sigma^{(n)}\delta_h : Q_h^* \to N_{\gamma_i}$ is $\Sigma_0^{(m)}$-preserving whenever $\kappa_h < \omega\rho^m_{Q_h^*}$.

(If $n > 0$, then $\sigma^{(n)}\delta_h$ is Σ^*-preserving. If not, then $\sigma^{(n)}$ is Σ^*-preserving and δ_h is $\Sigma_0^{(m)}$-preserving whenever $\kappa_\ell < \omega\rho^m_{Q_h}$ for all $\ell \leq_T h$.) Let φ be a $\Sigma_0^{(m)}$-fmla and let $\alpha_1, \ldots, \alpha_n < \lambda_h$, $f_1, \ldots, f_n \in \Gamma^*(\kappa_h, Q_h^*)$. Then

$$Q_i \models \varphi[\pi_{hi}(\vec{f})(\vec{\alpha})] \longleftrightarrow \langle\vec{\alpha}\rangle \in F(X)$$

where $X = \{\langle\vec{\xi}\rangle \mid Q_h^* \models \varphi[\vec{f}(\vec{\xi})]\}$

$$\longleftrightarrow \langle g(\vec{\alpha})\rangle \in \sigma^{(n)}\delta_h(X)$$

$$\longleftrightarrow N_{\gamma_i} \models \varphi[\sigma^{(n)}\delta_h(\vec{f})(g(\vec{\alpha}))].$$

Hence by setting:

$$\delta_i(\pi_{hi}(f)(\alpha)) = \sigma^{(n)}\delta_h(f)(g(\alpha))$$

we get a map $\delta_i : Q_i \longrightarrow N_{\gamma_i}$ with the right preservation properties.

Moreover, $\delta_i \pi_{hi} = \sigma^{(n)}\delta_h$.

Case 2 $h \leq_T i$

We again set: $\sigma^{(m)} = \sigma^{(n)}[\gamma_h, \delta_h(\nu_h)]$, $\gamma_i = \eta_m[\gamma_h, \delta_h(\nu_h)]$, where $\eta_i = \bar{\beta}_m(Q_h, \nu_h)$.

Using (1) we prove:

Claim $\delta_i(\kappa_i) \leq \sigma^{(n)}\delta_h(\kappa_i)$, $\delta_i \upharpoonright \kappa_i = \sigma^{(n)}\delta_h \upharpoonright \kappa_i$, and $\delta_i(X) = \delta_i(\kappa_i) \cap \sigma^{(n)}\delta_h(X)$ for $X \in \mathcal{P}(\kappa_i) \cap Q_i^*$.

proof of Claim.

Let $h = T(\ell+1)$, $\ell+1 \leq i$. Then $\kappa_\ell \geq \kappa_i$ and hence $\tau_\ell \geq \tau_i$. Hence $\eta_\ell \leq \eta_i$. Let

$\eta_\ell = \bar{\beta}_{n+m}(Q_h, \nu_h)$. Set:

$\sigma^{(n+m)} = \sigma^{(n+m)}[\gamma_h, \delta_h(\nu_h)]$. Then $\sigma^{(n+m)} = \sigma' \sigma^{(n)}$ where $\sigma' =$

$= \sigma^{(m)}[\gamma_i, \sigma^{(n)}\delta_h(\nu_h)]$. Since τ_i is a cardinal in Q_i^* and $\sigma^{(n)}\delta_h : Q_i^* \to N_{\gamma_i}$, $\sigma^{(n)}\delta_h(\tau_i)$ is a cardinal in N_{γ_i} and hence $\sigma' \upharpoonright \sigma^{(n)}\delta_h(\tau_i) = \mathrm{id}$ by § 2 Fact 8.

The conclusion follows easily by (1) (applied to κ_ℓ, $\sigma^{(n+m)}\delta_h$ in place κ_i, $\sigma^{(n)}\delta_h$) and the fact that $\kappa_i \leq \kappa_\ell$.

QED (Claim)

τ_i is a cardinal in $J_{d_h^+}^{E^{Q_h}} = J_{d_h^+}^{E^{Q_i}}$, where d_h^+ is a cardinal in Q_i. Hence τ_i is a cardinal in Q_i and $\delta_i(\tau_i)$ is a cardinal in N_{γ_i}. Hence $F^* =$ $= E_{\delta_i(\nu_i)}^{N_{\gamma_i}}$ is ω-complete. Set:

$F = E_{\nu_i}^{Q_i}$. Then $\delta_i(F(X)) = F^*(\delta_i(X))$

for $X \in \mathcal{P}(\kappa_i) \cap Q_i = \mathcal{P}(\kappa_i) \cap Q_i^*$. By the ω-completeness of F^* there is $g : \lambda_i \to \delta_i(\kappa_i)$ s.t. whenever $\alpha_1, \dots, \alpha_n < \lambda_i$ and $X \in \mathcal{P}(\kappa_i) \cap Q_i$, then:

$$\langle g(\vec{\alpha}) \rangle \in \delta_i(X) \longleftrightarrow \langle \vec{\alpha} \rangle \in F(X).$$

We know: $\sigma^{(n)} \delta_h : Q_i^* \to N_{\gamma_i}$ is $\Sigma_0^{(m)}$-preserving for $\kappa_i < \omega\rho^{\omega}_{Q_i^*}$. Let φ be $\Sigma_0^{(n)}$ and $\alpha_1, \dots, \alpha_n < \lambda_i$, $f_1, \dots, f_n \in \Gamma^*(\kappa_i, Q_i^*)$.

Then: $Q_i \models \varphi[\pi_{h_i}(\vec{f})(\vec{\alpha})] \longleftrightarrow$

$\longleftrightarrow \langle \vec{\alpha} \rangle \in F(X) \quad (X = \{\langle \vec{\xi} \rangle \mid Q_i^* \models \varphi[\vec{f}(\vec{\xi})]\})$

$\longleftrightarrow \langle g(\vec{\alpha}) \rangle \in \delta_i(X) = \delta_i(\kappa_i) \cap \sigma^{(n)} \delta_h(X)$

$\longleftrightarrow \langle g(\vec{\alpha}) \rangle \in \sigma^{(n)} \delta_h(X)$

$\longleftrightarrow N_{\gamma_i} \models \varphi[\sigma^{(n)} \delta_h(\vec{f})(\vec{\alpha})]$.

Hence, setting $\delta_i(\pi_{h_i}(f)(\alpha)) = \sigma^{(n)} \delta_h(f)(g(\alpha))$, we get $\delta_i : Q_i \to N_{\gamma_i}$ with the right preservation property. Clearly $\delta_i \pi_{h_i} = \sigma^{(n)} \delta_h$.

. . . .

This completes the construction in Case 2. The verifications are straightforward.

Case 3 $i = \lambda$, Lim (λ).

Pick $j \underset{T}{<} \lambda$ s.t. λ is simple above j.
Then $\gamma_j = \gamma_\ell = \gamma$ for $j \underset{T}{\leq} \ell \underset{T}{<} \lambda$. Moreover
$\delta_\ell \pi_{hj} = \delta_h$ for $j \underset{T}{\leq} h \underset{T}{\leq} \ell \underset{T}{<} \lambda$. Set:
$\gamma_\lambda = \gamma$. Define $\sigma_\lambda : Q_\lambda \to N_\gamma$ by:
$\sigma_\lambda \pi_{\ell\lambda} = \delta_\ell$ for $j \underset{T}{\leq} \ell \underset{T}{<} \lambda$. The verifications are straightforward.

. . . . . .

This completes the construction. If $\mathcal{I}$ is of successor length, there is nothing more to prove. Now let $\mathcal{I}$ be of limit length θ.
By our construction:
(2) Let $h = T(j+1) = T(j'+1)$, $j < j'$, where $\kappa_j = \kappa_{j'}$ and $\kappa_j < \kappa_\ell$ for $j < \ell < j'$. Then
$\delta_{j+1}(\kappa_j) > \delta_{j'+1}(\kappa_j)$.

proof.
Claim $\delta_{j+1}(\kappa_j) = \delta_{j'}(\kappa_j)$
Suppose not. Then $j+1 < j'$. But $d_j \leq \kappa_\ell$ for $j < \ell < j'$, since otherwise $\kappa_j < \kappa_\ell < d_j$.
Hence $j+1 \underset{T}{\leq} j'$ and $\kappa \geq d_j$, where $\kappa = $
$=$ crit $(\pi_{j+1, j'})$. By (1): $\delta_j \upharpoonright \kappa = \delta_{j+1} \upharpoonright \kappa$,
hence $\delta_{j'}(\kappa_j) = \delta_{j+1}(\kappa_j)$. Contr!
But then, since $\kappa_{j'} = \kappa_j$, our construction gives $g : \lambda_{j'} \to \delta_{j'}(\kappa_j)$ s.t. $\delta_{j'+1} \upharpoonright \lambda_j = g$.
Hence $\delta_{j'+1}(\kappa_j) = g(\kappa_j) < \delta_{j'}(\kappa_j) = \delta_{j+1}(\kappa_j)$.

QED (2)

(3) $\mathcal{T}$ is almost linear.

proof. Suppose not.

Let $X = \{i \mid T(i+1) \leq \eta \leq i\}$ be infinite.

Then $(j, j' \in X,\ j < j') \to \kappa_{j'} \leq \kappa_j$, since otherwise $\kappa_j < \kappa_{j'} < \delta_\eta \leq d_j$. Hence there is $j_0 \in X$ s.t. $X \setminus j_0$ is infinite and $\kappa_j = \kappa_{j_0}$ for all $j \in X \setminus j_0$. Let $\langle j_n \mid n < \omega \rangle$ be the first ω elements of $X \setminus j_0$. If $j_n < l < j_{n+1}$, then $\kappa_{j_n} < \kappa_l$. Hence by (2) we have:

$\sigma_{j_n+1}(\kappa_{j_0}) > \sigma_{j_{n+1}+1}(\kappa_{j_0})$ $(n < \omega)$. Contr!

QED(3)

But then $\mathcal{T}$ has a unique cofinal branch b. It is easily checked that our construction gives:

(4) If $i \leq_T j$ and j is not simple above i, then $\gamma_j < \gamma_i$.

Hence there is $i_0 \in b$ s.t. i is simple above i_0 for $i \in b$, $i \geq i_0$. We can extend $\mathcal{T}$ to a putative $\mathcal{T}'$ of length $\theta + 1$ by adding the limit model Q_b and maps $\pi_{i,\theta} = \pi_i^b$ $(i \in b)$. $\mathcal{T}'$ is realizable since, letting $\gamma = \gamma_{i_0}$, we can define $\sigma : Q_b \longrightarrow N_\gamma$ by:

$\sigma \pi_i^b = \sigma_i$ for $i_0 \leq_T i$. QED (4.1)

QED (Lemma 4)

We denote N_∞ by K^d. K^d is a weasel. Since each N_i ($i<\infty$) is normally d-iterable it follows easily that K^d is normally d-iterable (since for any cardinal κ of K^d there is an $i<\infty$ s.t. $N_i = K^d|\kappa$, by the array facts in §2). [Note We could use the method of [MOI] to show that every normally d-iterable premouse is fully d-iterable. Thus K^d is fully d-iterable.]

It is not hard to show:

Lemma 5 K^d is universal for normally d-iterable premice (i.e. the coiteration of K^d with a normally d-iterable premouse terminates below ∞).

However, we skip the proof of this and turn to the question of large cardinal preservation. Certain large cardinal properties in V are retained in the inner model K^d. We first prove some further lemmas about the construction of K^d.

Lemma 6.1 Let $M_i = \langle J_\alpha^E, \emptyset \rangle$. There is at most one N s.t. $N = \langle J_\beta^{E'}, F \rangle$, $F \neq \emptyset$ and N is a candidate for N_{i+1}.

proof.
Let N^0, N^1 be two such. By our condition, N^0, N^1 are both of the same type (A or B) and $N = \langle N^0, N^1 \rangle$ is a prebicephalus. Our iterability proof can easily be modified to show that N is a bicephalus. Hence $N^0 = N^1$. QED (6.1)

Cor 6.1.1 The definition of $\langle N_i \mid i < \gamma \rangle$ is uniform over every V_γ s.t. $\overline{\overline{V_\gamma}} = \gamma$.

Lemma 6.2 Let $i \leq \infty$ and let γ be a cardinal in N_i. There is at most one $\bar{N}$ s.t. $\bar{N} = \langle J_\beta^{E'}, F \rangle$, F is ω-complete, $\bar{N}$ is a d-premouse with $J_\alpha^{E^{N_i}} = J_\alpha^{E'}$ and $\bar{N}$ is a d-premouse with $\alpha = d^+_{\bar{N}}$, where $\gamma < \alpha < \gamma^+$ in N_i

proof.
Set $\delta = \delta(\alpha, i)$ (as in the section on arrays in §2). Then $\mu_{i\delta} = \mu_{ii}$ for $i < \delta$, where $\delta \leq i$, and $N_\delta = M_\delta = \langle J_\alpha^{E^{N_i}}, \emptyset \rangle$, and $\mu_{\delta, i} = \alpha$. Hence if $\bar{N}, \bar{N}'$ satisfy the above and are of the same type, then $\langle \bar{N}, \bar{N}', \alpha \rangle$ is a bicephalus and $\bar{N} = \bar{N}$. At γ is

a successor cardinal in N_i, then the only possible type is B. Hence $\bar{N}$ is unique.

Now let γ be a limit cardinal in N_i. We show that the only possible type is A, whence the uniqueness follows. Suppose not. Let $\bar{N}$ be the unique example of type B. Then $\bar{N} = N_{\delta+1}$. But $\omega\rho^{\omega}_{N_{\delta+1}} \leq \gamma$. $<$ is impossible, since otherwise $\mu_{\delta i} < \gamma$ and hence $\mu_{\delta i} \leq \gamma$, since γ is a cardinal in N_δ. Hence $\gamma = \omega\rho^{\omega}_{N_{\delta+1}}$. It follows that $\mu_{\delta+2,i} = \mathrm{ht}(N_{\delta+2}) > \mathrm{ht}(N_{\delta+1}) = \beta$. But $J^{E^{N_{\delta+2}}}_{\mu_{\delta+2,i}} = J^{E^{N_i}}_{\mu_{\delta+2,i}}$. Hence $\bar{N} = N_i \| \beta \in$ $\in N_i$. Hence $\bar{N}_\xi \in J^{E^{N_i}}_\gamma = J^{E^{\bar{N}}}_\gamma$ for sufficiently large $\xi < \gamma$. Hence $\bar{N}$ is of type A, since $\gamma = \alpha_{\bar{N}}$.

QED (Lemma 6.2)

In the following we shall make frequent tacit use of this Lemma.

We recall that a d-premouse is a D-premouse satisfying two conditions (a), (b). We can, in fact, show:

<u>Lemma 6.3</u> Let $M_i = \langle J_\alpha^E, \emptyset \rangle$. Let $N = \langle J_\beta^{E'}, F \rangle$ s.t. F is ω-complete, $J_\alpha^E = J_\beta^{E'}$, $\alpha = d_N^+$, and N is a D-premouse satisfying (a). Then N is a d-premouse.

proof.

Suppose not. Clearly $\rho_{M_i}^\omega = \alpha$. Hence $M_i = N_i$.

Since N_i is a d-premouse and there is $\pi : J_\tau^E \to_F J_\beta^{E'}$ $(\tau = \tau_N)$, it follows that $N \| \nu$ is a d-premouse for $\nu < \alpha$. Hence there must be a limit cardinal γ in N s.t. $\gamma^+ \leq s$ in N and there is no $\mu \in (\gamma, \gamma^+)$ in N s.t. $E_\mu \neq \emptyset$ and $\mathrm{crit}(E_\mu) = \kappa_N$. Let γ be the least such. Then $s_N \leq d_N < d_N^+ = \alpha$, where $\gamma^{+N} \leq s_N$. Hence γ^+ is a cardinal in N_i.

<u>Case 1</u> There is no $\mu \in (\kappa, \gamma]$ s.t. $o(\mu) > \gamma$ in N_i. $(\kappa = \kappa_N)$

Let: $\pi : J_\tau^E \to_F J_\beta^{\bar{E}}$, $\bar{\pi} : J_\tau^E \to_{F|\gamma} J_{\bar{\beta}}^{\bar{E}}$.

There is $\sigma : J_{\bar{\beta}}^{\bar{E}} \to_{\Sigma_0} J_\beta^E$ cofinally defined by $\sigma(\bar{\pi}(f)(\alpha)) = \pi(f)(\alpha)$ $(\alpha < \gamma)$

Then $\sigma \upharpoonright J_\gamma^{\bar{E}} = \mathrm{id}$, $J_\gamma^{\bar{E}} = J_\gamma^E$. Set:

$\bar{N} = \langle J_{\bar{\beta}}^{\bar{E}}, \bar{F} \rangle$, where $\bar{F} = \bar{\pi} \upharpoonright \mathbb{P}(\kappa)$. Then $s_{\bar{N}} \leq \gamma$. By the minimality of γ it follows that $\bar{N}$ is a d-premouse.

<u>Claim</u> Let $\tilde{\gamma} := \gamma^{+\bar{N}}$. Then $J_{\tilde{\gamma}}^{\bar{E}} = J_{\tilde{\gamma}}^E$.

proof.

Let $\xi \in (\gamma, \tilde{\gamma})$ s.t. $\omega\rho_{\bar{N}\|\xi}^\omega = \gamma$. Then

$\sigma \upharpoonright \bar{N}\|\xi : \bar{N}\|\xi \xrightarrow[\Sigma_\omega]{} N\|\sigma(\xi)$, where $\sigma \upharpoonright \gamma = \gamma$. A mild use of the condensation lemma tells us that $\bar{N}\|\xi = N\|\xi$. QED (Claim)

$\bar{F}$ is easily seen to be ω-complete, since $F|\gamma$ is. (Let $U \subset \mathbb{P}(\kappa) \cup lh(\bar{F})$ be countable. For $\alpha \in lh(\bar{F}) \cap U$ choose f_α, ξ_α s.t. $\xi_\alpha < \gamma$, $\pi(f_\alpha)(\xi_\alpha) = \alpha$, where $f_\alpha : \kappa \to J_\kappa^E$. For $X \in U \cap \mathbb{P}(\kappa)$, $\alpha_1, \ldots, \alpha_n \in U \cap lh(\bar{F})$ set: $X_{\alpha_1 \ldots \alpha_n}$ = the set of $\langle \xi_1, \ldots, \xi_n \rangle < \kappa$ s.t. $\langle f_{\alpha_1}(\xi_1), \ldots, f_{\alpha_n}(\xi_n) \rangle \in X$. Let U' be the set of all ξ_α and $X_{\vec{\alpha}}$. Let $g' : U' \cap \gamma \to \kappa$ verify the completeness of $F|\gamma$ wrt. U'. Set: $g(\alpha) = g'(\xi_\alpha)$. Then g verifies the ω-completeness of $\bar{F}$ wrt. U.)

Let $d^+ = d^+_{\bar{N}}$ (hence $d^+ \leq \gamma + \bar{N}$). Then d^+ is cardinally absolute in N_i and $J_{d^+}^{E^{N_i}} = J_{d^+}^{E^{\bar{N}}}$. Set: $\delta = \delta(d^+, i)$ (as in the section on arrays in §2). Then $\mu_{j\delta} = \mu_{ji}$ for $j < \delta$, $\delta \leq i$, and $N_\delta = M_\delta = \langle J_{d^+}^E, \emptyset \rangle$ and $\mu_{\delta i} = d^+$.

η is a cardinal in $\bar N$ and

Hence $N_{\delta+1} = \bar N$. But $\omega\rho^{\omega}_{N_{\delta+1}} \le \eta$, where $\eta^{+\bar N} = d^{+}$. $<$ is impossible, since otherwise $\mu_{\delta i} < \eta$ and hence $\mu_{\delta i} \le \eta$, since η is a cardinal in N_δ. Hence $\eta = \omega\rho^{\omega}_{N_{\delta+1}}$. It follows that $\mu_{\delta+2,i} =$

$= ht(N_{\delta+2}) > ht(N_{\delta+1}) = \bar\beta$. But

$$J^{E^{N_{\delta+2}}}_{\mu_{\delta+2,i}} = J^{E^{N_i}}_{\mu_{\delta+2,i}}.$$

Hence $\bar N = N_i \| \bar\beta$.

Since γ is a cardinal in N_i it follows that $\gamma = \eta$. Since $\omega\rho^{\omega}_{N_i\|\bar\beta} \le \gamma$, we have: $\bar\beta < \gamma^{+N_i}$. But $E^{N_i}_{\bar\beta} \ne \emptyset$, and $\kappa = crit(E^{N_i}_{\bar\beta})$, since $E^{N_i}_{\bar\beta} = F$.

Contra!

QED (Case 1)

Case 2 Case 1 fails.

Let $\nu =$ the least $\nu > \gamma$ s.t. $E^{N_i}_\nu \ne \emptyset$ and $crit(E^{N_i}_\nu) \in (\kappa, \gamma)$. Then $\nu < \gamma^{+N_i}$, $\nu \in N_i$ since N_i is a d-premouse. ν is applicable in N_i, since otherwise there is $\nu' \in N_i$ with $d^{+}_{\nu'} < d^{+}_{\nu} < \nu < \nu'$. Clearly $\gamma^{+} \le d^{+}_{\nu'}$ in N_i, since γ^{+} is a cardinal in N_i. ~~It follows~~ But $s_\nu \ge d^{+}_{\nu'} \ge \gamma^{+}$.

Hence there is $\bar{\nu} \in (\gamma, \gamma^+)$ s.t. $E_{\bar{\nu}}^{N_i} \neq \emptyset$ and $\mathrm{crit}(E_\nu) = \mathrm{crit}(E_{\bar{\nu}})$ in N_i. This contradicts the minimality of ν. Contr!

Let $\pi_0 : J_\tau^E \underset{F}{\longrightarrow} J_\beta^{E'}$ and let $\pi_1 : J_\beta^{E'} \underset{E_\nu}{\longrightarrow} J_{\tilde{\beta}}^{\tilde{E}}$. Set:

$\tilde{F} = \pi_1 \pi_0 \upharpoonright \mathcal{P}(\kappa)$. (Hence $\tilde{\pi}_1 : N \underset{E_\nu}{\longrightarrow} \tilde{N} = \langle J_{\tilde{\beta}}^{\tilde{E}}, \tilde{F} \rangle$). Let

$\bar{\pi} : J_\tau^E \underset{\tilde{F}|\gamma}{\longrightarrow} J_{\bar{\beta}}^{\bar{E}}$, $\bar{F} = \bar{\pi} \upharpoonright \mathcal{P}(\kappa)$.

It follows as before that $\bar{N} = \langle J_{\bar{\beta}}^{\bar{E}}, \bar{F} \rangle$ is a d^+-premouse. Note that $\tilde{F}|\gamma$ is ω-complete, hence so is $\bar{F}$. (Let U be countable, $\kappa_1 = \mathrm{crit}(E_\nu)$. Let $g_1 : U \cap \gamma \longrightarrow \kappa_1$ verify the ω-completeness of E_ν w.r.t. $U_1 = \{\pi_0(X) \cap \kappa_1 \mid X \in \mathcal{P}(\kappa) \cap U\} \cup (U \cap \gamma)$. Let $g_0 : U \cap \gamma \longrightarrow \kappa_1$ verify the ω-completeness of $F|\gamma$ w.r.t. U. Then $g_0 g_1 : U \cap \gamma \longrightarrow \kappa$ verifies the ω-completeness of F w.r.t. U.) By a virtual repetition of the proof in Case 1 we show: $\bar{N} = N_i \| \bar{\beta}$, where $\mathrm{crit}(E_{\bar{\beta}}^{N_i}) = \kappa$. Contr!

QED (Lemma 6.3)

<u>Def</u> Let F be an extender at κ on V. Let $\beta > \kappa$. F is <u>strong at</u> β (or β-<u>strong</u>) iff $V_\beta \subset W$ where $\pi : V \xrightarrow[F]{} W$.

F is <u>A-strong at</u> β (or β, A-<u>strong</u>) for $A \subset V$ iff $\langle V_\beta, A \cap V_\beta \rangle \subseteq \langle W, \bar{A} \rangle$, where $\pi : \langle V, A \rangle \xrightarrow[F]{} \langle W, \bar{A} \rangle$

F is <u>Σ_n-strong at</u> β iff $V_\beta \prec_{\Sigma_n} V$ and $V_\beta \prec_{\Sigma_n} W$, where $\pi : V \xrightarrow[F]{} W$.

<u>Def</u> κ is <u>strong</u> iff for every β there is F at κ which is β-strong.

κ is A-<u>strong</u> iff for every β there is F at κ which is A, β-strong.

κ is Σ_n-strong iff there are arbitrarily large β s.t. there is F at κ which is Σ_n-strong at β.

<u>Note</u> If $W = J^E_\infty$ is a weasel and $E_\nu \neq \emptyset$, and $\beta < \nu$ is a cardinal in W, then E_ν is E-strong at β in W. Hence, if $o(\kappa) = \infty$, then κ is E-strong in W.

<u>Lemma 7.1</u> Let $K^d = J^E_\infty$. Let κ be Σ_2-strong. Then κ is E-strong.

proof of Lemma 7.1

Let $V_\beta \prec V$ and $V_\beta \prec_{\Sigma_2} W$, where $\pi : V \to_F W$.

Let $\pi : \langle V, E \rangle \to_F \langle W, \bar{E} \rangle$.

<u>Claim</u> $E \cap V_\beta = \bar{E} \cap W_\beta$.

prf.

By Lemma 6.2, $\langle N_i \mid i < \beta \rangle = \langle N_i \mid i < \beta \rangle_W$.

Hence $N_\beta = N_\beta^W$. But $\mathrm{rn}(N_\beta) = \beta$,
since otherwise $N_\beta = N \in V_\beta$ and the
statement $\bigvee \gamma\ N = N_\gamma$ is Σ_2 in N.
(This is by Lemma 6.2 and the fact that
$\Sigma_2 = (\Sigma_1$ in $\langle V_\xi \mid \xi < \infty \rangle)$.) Hence we
would have $N_\gamma = N_\beta$ for an $\gamma < \beta$. Contr!
But then $N_\beta = \langle J_\beta^E, \emptyset \rangle = K^d \| \beta$,
since β is a cardinal and because
otherwise $\omega\rho^\omega_{N_\gamma} < \beta$ for an $\gamma \geq \beta$. Hence
the statement:
$M = M_\gamma \in V_\beta$. But the statement:
$\bigvee \gamma\ M_\gamma = M$ is Σ_2 in M. Hence $M_\gamma =$
$= M_\delta$ for a $\delta < \beta$. Hence $N_{\gamma+1} = N_{\delta+1}$.
Contr! Since $V_\beta \prec W$, we can carry
out the same proof in W to get:
$N_\beta = N_\beta^W = K_W^d \| \beta$. QED (Lemma 7.1)

Lemma 7.2 Let κ be E-strong. Then $o(\kappa) = \infty$ in K^d. (Hence κ is E-strong in K^d.)

proof. Suppose not.

Let $o(\kappa) < \beta < \gamma$ s.t. $V_\beta \prec_{\Sigma_2} V_\gamma \prec_{\Sigma_2} V$.

As before, β is a limit cardinal and $N_\beta = \langle J_\beta^E, \emptyset \rangle = K^d \| \beta$. Similarly for γ.

Let F^* be an extender at κ which is γ-strong. Let

$$\pi : \langle V, E \rangle \xrightarrow[F^*]{} \langle W, \bar{E} \rangle.$$

Case 1 There is no $\mu \in (\kappa, \beta)$ s.t. $o(\mu) > \beta$ in K^d.

Set: $F = (F^* | \beta) \upharpoonright N_\beta$. Let:

$$\pi' : J_\tau^E \xrightarrow[F]{} J_\nu^{E'} \quad \text{where } \tau = \kappa^{+K^d}$$

There is $\sigma : J_\nu^{E'} \xrightarrow[\Sigma_0]{} \pi(N_\beta)$ defined by $\sigma(\pi'(f)(\alpha)) = \pi(f)(\alpha)$ $(\alpha < \beta)$. Then $\sigma \upharpoonright \beta = \mathrm{id}$. Set $F' = \pi' \upharpoonright \mathbb{P}(\kappa)$. Then $Q = \langle J_\nu^{E'}, F' \rangle$ is a ppm and $\langle J_\nu^{E'}, \emptyset \rangle$ is a d-premouse. Clearly $s_Q \leq \beta$. Hence $s_Q = \beta$, since β is a cardinal.

(1) Let $\tilde{\beta} = \beta^+$ Then $J_{\tilde{\beta}}^{E} = J_{\tilde{\beta}}^{E}$.

prf of (1).

Set $N^* = \pi(N_\beta) = N^w_{\pi(\beta)}$, N^* internally satisfies condensation. Let $\xi \in (\beta, \tilde\beta)$ s.t. $\omega\rho^\omega_{Q\|\xi} = \beta$. Then $\sigma \upharpoonright (Q\|\xi) : Q\|\xi \to N^*\|\sigma(\xi)$

is an element of N^*, since:

$$\sigma \upharpoonright (Q\|\xi) : Q\|\xi \xleftrightarrow{\sim} h^\omega_{N^*\|\sigma(\xi)}(\beta \cup \sigma(p_{N\|\xi})).$$

By condensation: $Q\|\xi = N^*\|\xi = K^d\|\xi$, since $K^d\|\gamma = N^*\|\gamma$, $\gamma > \xi$.

QED(1)

(2) Let $\mu \in (\kappa, \beta)$. Then $o(\mu) \leq \beta$ in Q.

proof.

Suppose not. $\langle J^{E^Q}_\nu, \emptyset \rangle$ is a d-premouse and $o(\mu) > \beta$ in $J^{E^Q}_\nu$. Hence there is $\bar\nu \in (\beta, \tilde\beta)$ s.t. $E^Q_{\bar\nu} \neq \emptyset$, $crit(E^Q_{\bar\nu}) = \mu$.

But $Q\|\nu = K^d_\nu\|\nu$ by (1). Hence $o(\mu) > \beta$ in K^d. Contr! QED(2)

Hence:

(3) $\beta = d_Q \in T_Q$

(4) Let $\eta < \beta$ s.t. $\eta = d_{Q_\eta}$. Then $\forall \bar\nu \; Q_\eta = Q\|\bar\nu$.

proof.

Let η be the least counterexample. Then for $\mu \in (\kappa, \eta)$ we have:

$o(\mu)_Q \in rng\, \sigma_\eta^Q$, hence $o(\mu) <_{Q_\eta} \lambda_{Q_\eta}$.

Hence $\sup_{\mu \in (\kappa, \eta)} O(\mu) < \lambda_{Q_\eta}$ (since $J^{E^{Q_\eta}}_{\lambda_{Q_\eta}}$ is a ZFC - model). Hence there is $\gamma \in T_{Q_\eta}$ s.t. $\gamma \geq \eta \geq \kappa_{Q_\eta}$. Hence $\eta = d_{Q_\eta} \in T_{Q_\eta}$. By the minimality of η, Q_η is a D-premouse. Since η is a limit cardinal in Q_η, it is a limit cardinal in Q, hence in K^d (since $\eta < \beta$, $Q \| \beta = K^d \| \beta$). Let $\tilde{\eta} = \eta^{+Q}$. We can repeat the proof of (1) to show: $Q \| \tilde{\eta} = Q_\eta \| \tilde{\eta}$. Hence $Q_\eta \| \tilde{\eta} = K^d \| \tilde{\eta}$. Set $\delta = \delta(\tilde{\eta}, \infty)$.

Since $\tilde{\eta}$ is cardinally absolute in K^d, we conclude by the array facts in §2 that $\langle J^E_{\tilde{\eta}}, \emptyset \rangle = N_\delta$ is a point in the array. Moreover $\mu_{i,\delta} = \mu_{i,\infty}$ for $i < \delta$. Since $F|\eta$ is ω-complete, so is the top extender of Q_η and Q_η is a candidate for $N_{\delta+1}$. But Q_η satisfies (a) in the defn. of d-premouse. Hence by Lemma 6.3 it satisfies (b) as well and we conclude: $Q_\eta = N_{\delta+1}$. It follows exactly as in the proof of Lemma 6.3 that $Q_\eta = K^d \| \bar{\nu}$ for some $\bar{\nu}$. Hence $Q_\eta = Q \| \bar{\nu}$.

It follows easily that:

(5) Q is a D-premouse.

Since $d_Q = \beta \in T_Q$, we know that Q is satisfies (a) in the def. of d-premouse. As in the proof of (4) we set: $\delta = \delta(\tilde{\beta}, \infty)$ ($\tilde{\beta} = \beta + Q$). Then $\langle J^E_{\tilde{\beta}}, \emptyset \rangle = N_\delta$, as before, and Q is a candidate for $N_{\delta+1}$. Hence, as before, Lemma 6.3 tells us that Q is a d-premouse and $Q = N_{\delta+1}$.

It follows as before that $Q = K^d \| \nu$ for some $\nu > \beta > o(\kappa)$. But then $E_\nu \neq \emptyset$, $crit(E_\nu) = \kappa$. Hence $\nu < o(\kappa)$. Contr!

QED (Case 1)

<u>Case 2</u> Case 1 fails.

Let ν be least s.t. $E_\nu \neq \emptyset$ and $crit(E_\nu) \in (\kappa, \beta)$. Then E_ν is ω-complete. Let $\pi_1 : K^d_W \xrightarrow[E_\nu]{} K^*$, (Note that $K^d_W \| \gamma = K^d \| \gamma$ and $E_\nu = E^{K^d_W}_\nu$.) Set $\pi^* = \pi_1 \pi$. Then

$\pi^* : K^d \longrightarrow K^*$.

Set: $F = (\pi^* \upharpoonright \mathbb{P}(\kappa)) \mid \beta$ — i.e.

$F(X) = \beta \cap \pi^*(X)$ for $X \in \mathbb{P}(\kappa) \cap K^d$.

Then $F(X)$ is an extender on N_γ and we set: $\pi': J_\kappa^E \to_F J_\nu^{E'}$ as before.

There is, as before, $\sigma: J_\nu^{E'} \to_{\Sigma_0} \pi^*(N_\beta)$

defined by $\sigma(\pi'(f)(\alpha)) = \pi^*(f)(\alpha)$,

As before, $\sigma \upharpoonright \beta = \mathrm{id}$. We again set $F' = \pi' \upharpoonright \mathbb{P}(\kappa)$. $Q = \langle J_{\nu'}^{E'}, F' \rangle$ is again a ppm. We then repeat the above proof (using $\pi^*(N_\beta)$ instead of $\bar\pi(N_\beta)$) to show that $Q = K^d \| \nu$. The details are left to the reader.

QED (Lemma 7.2)

We can improve these results slightly

<u>Def</u> "κ is strong at level α" is defined by ind. on α.

- Every κ is strong at level 0
- Let F be an extender at κ. F is β-strong at level α iff F is β-strong and κ is strong in W at all levels $\gamma < \alpha$, where $\pi: V \to_F W$.
- κ is strong at level α iff for every β there is F at κ which is β-strong at level α

The notion "κ is A-strong at level α" and "κ is Σ_n-strong at level α" are defined analogously. (We leave it to the reader to devise reformulation of these definitions which can be written in the language of ZFC.)

By virtually the same proofs we then get:

Lemma 7.3 Let κ be Σ_2-strong at level α. Then κ is E-strong at level α, where $K^d = J_\infty^E$.

Lemma 7.4 Let κ be E-strong at level α, where $K^d = J_\infty^E$. Then κ has this property in K^d.

The details are left to the reader.

. . . . .

Schindler has worked out a full core model theory for his "mice below 0-handgrenade". We hope that this will also be possible for d-mice.