T-Mice

Let Θ be a Mahlo cardinal. In these notes we develop a K^c-type model K^e with the following properties:

(a) K^e is an inner model of V_Θ

(b) If κ is Σ_2-strong in V_Θ, then it is strong in K^e

(c) If λ is a limit of Woodin cardinals in V_Θ, then it has the same property in K^e.

Thus K^e can have strong limits of Woodin cardinals, strong limits of strong limits of Woodin cardinals etc. We can improve on this somewhat, but the method will not yield a model for a Woodin limit of Woodins. The construction is closely related to that given in [ANS], but brings a twofold improvement: We obtain larger cardinals in our model and do so with weaker assumptions on V_Θ (the authors of [ANS] needed a superstrong cardinal in V_Θ). The construction in [ANS] is, in turn, closely related to Schindler's construction of a "K^c-model" in [ALI]. We obtain some improvement of that result as well. As in Schindler's case, we construct a model K^d using only ω-completeness as "background condition" for adding new extenders to the sequence. K^d is an inner model of V and, again, has the property that if κ is Σ_2-strong in V, then it is strong in K^d. (Thus K^d can have strong limits of strongs, strong limits of strong limits of strongs etc.)

In both cases the essential new element is the appropriate choice of what we may call "iteration indices". The premice themselves will all be of the type described in [NFS]. Thus if $M = \langle J^E_\alpha, E_{\omega\alpha} \rangle$ is a premouse and $F = E_\nu$ is an extender on M's sequence, then ν is the successor of F's "full length" in the ultraproduct. (I.e. if $\kappa = \mathrm{crit}(F)$, $\tau = \kappa^{+J^E_\nu}$, then $\pi : J^E_\tau \longrightarrow_F J^E_\nu$, $\lambda = \pi(\kappa) = lh(F)$ is called the "full length" of F.) In the premice of [MS] on the other hand F is indexed by the successor of the natural length in the ultrapower. One might call the former indexing "maximal" and the latter minimal". The [MS] indexing turned out to be advantageous in some model constructions. We therefore assigned to the extenders $F = E_\nu$ in our premice an auxiliary "iteration index" $s^+ = s^+_\nu$

defined by: $s^+ = s^{+J_\nu^E}$, where $s = s_\nu$ is the natural length of F. The index s^+ identifies F as surely as ν does, since it can be shown that $\nu \neq \nu' \to s_\nu^+ \neq s_{\nu'}^+$. We then defined normal s-iterations based on s_ν, s_ν^+ in place of λ, ν. If $\mathcal{I}$ is such an iteration with indices ν_i (in the old sense), we set: $s_i = s_{\nu_i}$, $s_i^+ = s_{\nu_i}^+$ (in M_i) and require that E_{ν_i} be applied to the smallest M_ξ s.t. $\kappa_i = \mathrm{crit}(E_{\nu_i}) < s_\xi$. In these iterations we then have: $i < j \to s_i < s_j$, but not necessarily $\lambda_i < \lambda_j$. We call M s-iterable (or an s-mouse) iff it has a successful strategy for s-iterations. s-mice behave very much like the mice of [MS] and are advantageous in the same model constructions. It turns out, however, that for many model constructions it is best to choose an iteration index $t^+ = t_\nu^+ = t^{+J_\nu^E}$, where the "length" $t = t_\nu$ is intermediate between between the "minimal" length s and the "maximal" length λ. A systematic choice of t_ν gives rise to the notion of t-iteration and t-<u>mouse</u>. In extending Schindler's result we define a class of premice M and fix the "length" $d = d_\nu$ and "iteration index" $d^+ = d_\nu^+ = d + J_\nu^{E^M}$ in such a way that if $\mathcal{I}$ is an iteration with indices ν_i, then, setting $d_i = d_{\nu_i}$, we have: If $i < j$, then $\kappa_j = \mathrm{crit}(E_{\nu_j}) \notin (\kappa_i, d_i)$. This is actually an old idea first developed by Tony Dodd. Schindler's construction can be regarded as the best realization of Dodd's idea using length λ and index ν. The Mitchell-Steel indices are unsuitable for this purpose. In proving the result mentioned at the outset, we adopt a somewhat less restricted class of premice and define our "length" e and "index" e^+ in such a way that if $i < j$ <u>and</u> κ_i is a limit of Woodins in $J_{\nu_i}^{E_{\nu_i}}$, then $\kappa_j \notin (\kappa_i, d_i)$. This was the central idea of [ANS], using the indices

λ, ν.

The theory of s-iteration was developed only sketchily (and in part inaccurately) in [MOI]. We therefore devote §1 to s-mice. In particular, we develop the initial segment condition for s-premice (a point left open in [MOI]). In §2 we then develop a more general theory of T-premice with indices t_ν^+ intermediate between s_ν^+ and ν. In §3, §4 we apply this theory to get the abovementioned results. In §3 we define the class of "Dodd-premice", which realize Dodd's idea more fully than Schindler's premice. We then define the model K^d. We suspect that there is still much to be found here. Schindler develops not only his "K^c-model" but also a full core model theory. We very much hope that a similar theory can be worked out for K^d. In §4 we extend the results in [ANS], obtaining the model K^c mentioned at the outset. It is a fact known to the cognoscenti that the results of [ANS] could have been obtained using a somewhat different iteration theory, based on normal iterations. Iterability is then established by a straightforward application of "Schindler realizability" on top of "Steel realizability". That is the approach we adopt.

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