Manuscript on fine structure, inner model theory, and the core model below one Woodin cardinal

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Preface

Here are the first three chapters of a prospective book. It is intended to provide a detailed introduction to fine structure theory, ultimately leading up to a proof of the Covering Lemma for the Core Model under the assumption that there is no inner model with a Woodin cardinal.

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Chapter 0

Preliminaries

(1) Throughout the book we assume ZFC. We use "virtual classes", writing $\{x|\varphi(x)\}$ for the class of x such that $\varphi(x)$. We also write:

$$\{t(x_1,\dots,x_n)|\varphi(x_1,\dots,x_n)\},\ \text{(where e.g.}$$
$$t(x_1,\dots,x_n)=\{y|\psi(y,x_1,\dots,x_n)\})$$

for:

$$\{y|\bigvee x_1,\dots,x_n(y=t(x_1,\dots,x_n)\wedge\varphi(x_1,\dots,x_n))\}$$

We also write

$$\mathbb{P}(A)=\{z|z\subset A\}, A\cup B=\{z|z\in A\vee z\in B\}$$
$$A\cap B=\{z|z\in A\wedge z\in B\},\neg A=\{z|\notin A\}$$

(2) Our notation for ordered n–tuples is $\langle x_1,\dots,x_n\rangle$. This can be defined in many ways and we don't specify a definition.

(3) An *n–ary relation* is a class of n–tuples. The following operations are defined for all classes, but are mainly relevant for binary relations:

$$\text{dom}(R)=:\{x|\bigvee y\langle y,x\rangle\in R\}$$
$$\text{rng}(R)=:\{y|\bigvee x\langle y,x\rangle\in R\}$$
$$R\circ P=\{\langle y,x\rangle|\bigvee z|\langle y,z\rangle\in R\wedge\langle z,x\rangle\in P\}$$
$$R\restriction A=\{\langle y,x\rangle|\langle y,x\rangle\in R\wedge x\in A\}$$
$$R^{-1}=\{\langle y,x\rangle|\langle x,y\rangle\in R\}$$

We write $R(x_1,\dots,x_n)$ for $\langle x_1,\dots,x_n\rangle\in R$.

(4) A *function* is identified with its *extension* or *field* — i.e. an n–ary function is an $n+1$–ary relation F such that

$$\bigwedge x_1\dots x_n\bigwedge z\bigwedge w((F(z,x_1,\dots,x_n)\wedge F(w,x_1,\dots,x_n))\rightarrow$$
$$\rightarrow z=w)$$

$F(x_1, \ldots, x_n)$ then denotes the value of F at $x_1, \ldots, x_n$.

(5) "*Functional abstraction*" $\langle t_{x_1,\ldots,x_n} | \varphi(x_1, \ldots, x_n)\rangle$ denotes the function which is defined and takes value $t_{x_1,\ldots,x_n}$ whenever $\varphi(x_1, \ldots, x_n)$ and $t_{x_1,\ldots,x_n}$ is a set:

$$\langle t_{x_1,\ldots,x_n} | \varphi(x_1, \ldots, x_n)\rangle =: \\ \{\langle y, x_1, \ldots, x_n\rangle | y = t_{x_1,\ldots,x_n} \wedge \varphi(x_1, \ldots, x_n)\},$$

where e.g. $t_{x_1,\ldots,x_n} = \{z|\psi(z, x_1, \ldots, x_n)\}$.

(6) *Ordinal numbers* are defined in the usual way, each ordinal being identified with the set of its predecessors: $\alpha = \{\nu|\nu < \alpha\}$. The *natural numbers* are then the finite ordinals: $0 = \emptyset, 1 = \{0\}, \ldots, n = \{0, \ldots, n-1\}$. On is the class of all ordinals. We shall often employ small greek letters as variables for ordinals. (Hence e.g. $\{\alpha|\varphi(\alpha)\}$ means $\{x|x \in \mathrm{On} \wedge \varphi(x)\}$.) We set:

$$\sup A =: \bigcup(A \cap \mathrm{On}),\ \inf A =: \bigcap(A \wedge \mathrm{On})$$
$$\mathrm{lub}\, A =: \sup\{\alpha + 1|\alpha \in A\}.$$

(7) *A note on ordered n–tuples.* A frequently used definition of ordered pairs is:

$$\langle x, y\rangle =: \{\{x\}, \{x, y\}\}.$$

One can then define n–tuples by:

$$\langle x\rangle =: x,\ \langle x_1, x_2, \ldots, x_n\rangle =: \langle x_1, \langle x_1, \ldots, x_n\rangle\rangle.$$

However, this has the disadvantage that every $n+1$–tuple is also an n–tuple. If we want each tuple to have a fixed length, we could instead identify the n–tuples with *vecton of length n* — i.e. functions with domain n. This would be circular, of course, since we must have a notion of ordered pair in order to define the notion of "function". Thus, if we take this course, we must first make a "preliminary definition" of ordered pairs — for instance:

$$(x, y) =: \{\{x\}, \{x, y\}\}$$

and then define:

$$\langle x_0, \ldots, x_{n-1}\rangle = \{(x_0, 0), \ldots, (x_{n-1}, n-1)\}.$$

If we wanted to form n–tuples of proper classes, we could instead identify $\langle A_0, \ldots, A_{n-1}\rangle$ with:

$$\{\langle x, i\rangle|(i = 0 \wedge x \in A_0) \vee \ldots \vee (i = n-1 \wedge x \in A_{n-1})\}.$$

(8) *Overhead arrow notation.* The symbol $\vec{x}$ is often used to donate a vector $\langle x_1, \ldots, x_n \rangle$. It is not surprising that this usage shades into what I shall call the *informal mode* of overhead arrow notation. In this mode $\vec{x}$ simply stands for a string of symbols $x_1, \ldots, x_n$. Thus we write $f(\vec{x})$ for $f(x_1, \ldots, x_n)$, which is different from $f(\langle x_1, \ldots, x_n \rangle)$. (In informal mode we would write the latter as $f(\langle \vec{x} \rangle)$.) Similarly, $\vec{x} \in A$ means that each of $x_1, \ldots, x_n$ is an element of A, which is different from $\langle \vec{x} \rangle \in A$. We can, of course, combine several arrows in the same expression. For instance we can write $f(\vec{g}(\vec{x}))$ for $f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$. Similarly we can write $f(\overrightarrow{g(\vec{x})})$ or $f(\vec{g}(\vec{x}))$ for

$$f(g_1(x_{1,1}, \ldots, x_{1,p_1}), \ldots, g_m(x_{m,1}, \ldots, x_{m,p_m})).$$

The precise meaning must be taken from the context. We shall often have recourse to such abbreviations. To avoid confusion, therefore, we shall use overhead arrow notation *only* in the informal mode.

(9) A *model* or *structure* will for us normally mean an $n{+}1$–tuple $\langle D, A_1, \ldots, A_n \rangle$ consisting of a domain D of individuals, followed by relations on that domain. If φ is a first order formula, we call a sequence $v_1, \ldots, v_n$ of distinct variables *good for* φ iff every free variable of φ occurs in the sequence. If M is a model, φ a formula, $v_1, \ldots, v_n$ a good sequence for φ and $x_1, \ldots, x_n \in M$, we write: $M \models \varphi(v_1, \ldots, v_n)[x_1, \ldots, x_n]$ to mean that φ becomes true in M if v_i is interpreted by x_i for $i = 1, \ldots, n$. This is the *satisfaction relation.* We assume that the reader knows how to define it. As usual, we often suppress the list of variables, writing only $M \models \varphi[x_1, \ldots, x_n]$. We may sometimes indicate the variables being used by writing e.g. $\varphi = \varphi(v_1, \ldots, v_n)$.

(10) *$\in$–models.* $M = \langle D, E, A_1, \ldots, A_n \rangle$ is an *$\in$–model* iff E is the restriction of the $\in$–relation to D^2. Most of the models we consider will be $\in$–models. We then write $\langle D, \in, A_1, \ldots, A_n \rangle$ or even $\langle D, A_1, \ldots, A_n \rangle$ for $\langle D, \in \cap D^2, A_1, \ldots, A_n \rangle$. M is *transitive* iff it is an $\in$–model and D is transitive.

(11) *The Levy hierarchy.* We often write $\bigwedge x \in y \varphi$ for $\bigwedge x(x \in y \to \varphi)$, and $\bigvee x \in y \varphi$ for $\bigvee x(x \in y \land \varphi)$. Azriel Levy defined a hierarchy of formulae as follows:

A formula is Σ_0 (or Π_0) iff it is in the smallest class Σ of formulae such that every primitive formula is in Σ and $\bigwedge v \in u\varphi$, $\bigvee v \in u\varphi$ are in Σ whenever φ is in Σ and v, u are distinct variables.

(Alternatively we could introduce $\bigwedge v \in u$, $\bigvee v \in u$ as part of the primitive notation. We could then define a formula as being Σ_0 iff it contains no unbounded quantifiers.)

The Σ_{n+1} formulae are then the formulae of the form $\bigvee v\varphi$, where φ is Π_n. The Π_{n+1} formulae are the formulae of the form $\bigwedge v\varphi$ when φ is Σ_n.

If M is a transitive model, we let $\Sigma_n(M)$ denote the set of realations on M which are definable by a Σ_n formula. Similarly for $\Pi_n(M)$. We say that a relation R is $\Sigma_n(M)(\Pi_n(M))$ *in parameters* $p_1, \ldots, p_m$ iff

$$R(x_1, \ldots, x_n) \leftrightarrow R'(x_1, \ldots, x_n, p_1, \ldots, p_m)$$

and R' is $\Sigma_n(M)(\Pi_n(M))$. $\underline{\Sigma}_1(M)$ then denotes the set of relations which are $\Sigma_1(M)$ in some parameters. Similarly for $\underline{\Pi}_1(M)$.

(12) *Kleene's equation sign.* An equation '$L \simeq R$' means: 'The left side is defined if and only if everything on the right side is defined, in which case the sides are equal'. This is of course not a strict definition and must be interpreted from case to case.

$F(\vec{x}) \simeq G(H_1(\vec{x}), \ldots, H_n(\vec{x}))$ obviously means that the function F is defined at $\langle x_1, \ldots, x_n\rangle$ iff each of the H_i is defined at $\langle\vec{x}\rangle$ and G is defined at $\langle H_1(\vec{x}), \ldots, H_n(\vec{x})\rangle$, in which case equality holds.

The recursion schema of set theory says that, given a function G, there is a function F with:

$$F(y, \vec{x}) \simeq G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle).$$

This says that F is defined at $\langle y, \vec{x}\rangle$ iff $F$is defined at $\langle z, \vec{x}\rangle$ for all $z \in y$ and G is defined at $\langle y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle\rangle$, in which case equality holds.

(13) By the recursion theorem we can define:

$$TC(x) = x \cup \bigcup_{z \in x} TC(z)$$

(the transitive closure of x)

$$\mathrm{rn}(x) = \mathrm{lub}\{\mathrm{rn}(z) | z \in x\}$$

(the rank of x).

(14) By a *normal ultrafilter on* κ we mean an ultrafilter U on $\mathbb{P}(\kappa)$ with the property that whenever $f : \kappa \to \kappa$ is regressive modulo U (i.e. $\{\nu | f(v) < \nu\} \in U$), then there is $\alpha < \kappa$ such that $\{\nu | f(\nu) < \nu\} \in U$. Each normal ultrafilter determines an elementary embedding π of V into an inner model W. Letting

$$D = \text{ the class of functions } f \text{ with domain } \kappa,$$

we can characterize the pair $\langle W, \pi\rangle$ uniquely by the conditions:

- $\pi : V \prec W$ and write $(\pi) = \kappa$
- $W = \{\pi(f)(\nu) | \kappa \in D\}$
- $\pi(f)(\nu) \in \pi(g)(\kappa) \leftrightarrow \{\nu | f(\nu) \in g(\nu)\} \in U$.

U can then be recovered from π by:

$$U = \{x \subset \kappa | \kappa \in \pi(x)\}.$$

We shall call $\langle W, \pi \rangle$ the *extension of V by U*. W can be defined from U by the well known *ultrapower construction*: We first define a "term model" $\mathbb{D} = \langle D, \cong, \tilde{\in} \rangle$ by:

$$f \cong g \leftrightarrow: \{\nu | f(\nu) = g(\nu)\} \in U$$
$$f \tilde{\in} g \leftrightarrow: \{\nu | f(\nu) = g(\nu)\} \in U.$$

$\mathbb{D}$ is an *equality model* in the sense that $\cong$ is not the identity relation but rather a congruence relation for $\mathbb{D}$. We can then factor $\mathbb{D}$ by $\cong$, getting an identity model $\mathbb{D} \backslash \cong$, whose are the equivalence classes:

$$[x] = \{y | y \cong x\}$$

$\mathbb{D} \backslash \cong$ turns out to be isomorphic to an inner model W. If σ is the isomorphism, we can define π by:

$$\pi(x) = \sigma([\mathrm{const}_x])$$

where const_x is the constant function x defined on κ. W is then called the *ultrapower of V by U*. π is called the *canonical embedding*.

(15) *(Extenders)* The normal ultrafilter is one way of coding an embedding of V into an inner model by a set. However, many embeddings cannot be so coded, since $\pi(\kappa) \leq 2^\kappa$ whenever $\langle W, \pi \rangle$ is the extension by U. If we wish to surmount this restriction, we can use *extenders* in place of ultrafilters. (The extenders we shall deal with are also known as "short extenders".)

An extender F at κ maps $\bigcup\limits_{n<\omega} \mathbb{P}(u^n)$ into $\bigcup\limits_{n<\omega} \mathbb{P}(\lambda^n)$ for $a\lambda > u$.

It engenders an embedding π of V into an inner model W characterized by:

- $\pi : V \prec W$ $\mathrm{crit}(\pi = \kappa)$
- Every element of W has the form $\pi(f)(\vec{\alpha})$ where $\alpha_1, \ldots, \alpha_n < \lambda$ and f is a function with domain κ^n
- $\pi(f)(\vec{\alpha}) \in \pi(g)(\vec{\alpha}) \leftrightarrow \langle \vec{\alpha} \rangle \in \pi(\{\langle \vec{\xi} \rangle | f(\vec{\xi}) \in g(\vec{\xi})\})$

F is then recoverable from $\langle W, \pi \rangle$ by:

$$F(X) = \pi(X) \cap \lambda^n \text{ for } X \subset \kappa^n.$$

The concept "F is an extender" can be defined in ZFC, but we defer that to Chapter 3. If $\langle W, \pi \rangle$ is as above, we call it the *extension of V by F*. We also call W the *ultrapower of V by F* and π the *canonical* embedding. $\langle W, \pi \rangle$ can be obtained from F by a "term model" construction analogous to that described above.

(16) *(Large Cardinals)*

Definition 0.0.1. We call a cardinal κ *strong* iff for all $\beta > \kappa$ there is an extender F such that if $\langle W, \pi \rangle$ is the extension of V by F, then $V_\beta \subset W$.

Definition 0.0.2. Let A be any class. κ is *A–strong* iff for all $\beta > \kappa$ there is F such that letting $\langle W, \pi \rangle$ be the extension of V by F, we have:

$$A \cap V_\beta = \pi(A) \cap V_\beta.$$

These concepts can of course be relativized to V_τ in place of V when τ is strongly inaccessible. We then say that κ is strong (or A–strong) *up to τ*.)

Definition 0.0.3. τ is *Woodin* iff τ is strongly inaccessible and for every $A \subset V_\tau$ there is $\kappa < \tau$ which is strong up to τ.

(17) *(Embeddings)*

Definition 0.0.4. Let M, M' be $\in$–structures and let π be a structure preserving embeddings of M into M'. We say that π is *Σ_n–preserving* (in symbols: $\pi : M \to_{\Sigma_n} M'$) iff for all Σ_n formulae we have:

$$M \models \varphi[a_1, \ldots, a_n] \leftrightarrow M' \models \varphi[\pi(a_1), \ldots, (a_n)]$$

for $a_1, \ldots, a_n \in M$. It is *elementary* (in symbols: $\pi : M \prec M'$ of $\pi : M \to_{\Sigma_\omega} M'$) iff the above holds for *all* formulae φ of the M–sprache. It is easily seen that π is elementary iff it is Σ_n–preserving for all $n < \omega$.

We say that π is *cofinal* iff $M' = \bigcup_{u \in M} \pi(u)$.

We note the following facts, which we shall occasionally use:

Fact 1 Let $\pi : M \to_{\Sigma_0} M'$ cofinally. Then π is Σ_1–preserving.

Fact 2 Let $\pi : M \to_{\Sigma_0} M'$ cofinally, where M is a ZFC^- model. Then M' is a ZFC^- model and π is elementary.

Fact 3 Let $\pi : M \to_{\Sigma_0} M'$ cofinally where M' is a ZFC^- model. Then M is a ZFC^- model and π is elementary.

We call an ordinal κ the *critical point* of an embedding $\pi : M \to M'$ (in symbols: $\kappa = \mathrm{crit}(\pi)$) iff $\pi \restriction \kappa = \mathrm{id}$ and $\pi(\kappa) > \kappa$.