Chapter 1

Transfinite Recursion Theory

1.1 Admissibility

Some fifty years ago Kripke and Platek brought out about a wide ranging generalization of recursion theory — which dealt with "effective" functions and relations on ω — to transfinite domains. This, in turn, gave the impetus for the development of fine structure theory, which became a basic tool of inner model theory. We therefore begin with a discussion of Kripke and Platek's work, in which ω is replaced by an arbitrary "admissible" structure.

1.1.1 Introduction

Ordinary recursion theory on ω can be developed in three different ways. We can take the notion of *algorithm* as basic, defining a recursive function on ω to be one given by an algorithm. Since, however, we have no definition for the general notion of algorithm, this approach involves defining a special class of algorithms and then convincing ourselves that "Church's thesis" holds — i.e. that every function generated by an algorithm is, in fact, generated by one which lies in our class. Alternatively we can take the notion of *calculus* as basic, defining an n–ary relation R on ω to be recursively enumerable (r.e.) if for some calculus involving statements of the form "$R(i_1, \ldots, i_n)$" $(i_1, \ldots, i_n < \omega)$, R is the set of tuples $\langle i_1, \ldots, i_n \rangle$ such that "$R(i_1, \ldots, i_n)$" is provable. R is then recursive if both it and its complement are r.e. A function defined on ω is recursive if it is recursive as a relation. But again, since we have no general definition of calculus, this involves specifying a special class of calculi and appealing to the appropriate form of Church's thesis.

A third alternative is to base the theory on *definability*, taking the r.e. relation as those which are definable in elementary number theory by one of a certain class of formulae. This approach has often been applied, but characterizing the class of defining formula tends to be a bit unnatural. The situation changes radically, however, if we replace ω by the set $H = H_\omega$ of heredetarily finite sets. We consider definability over the structure $\langle H, \in\rangle$, employing the familiar Levy hierarchy of set theoretic formulae:

$\Pi_0 = \Sigma_0 =$: formulae in which all quantifiers are bounded

$\Sigma_{n+1} =$: formulae $\bigvee x\varphi$ where φ is Π_n

$\Pi_{n+1} =$: formulae $\bigwedge x\varphi$ where φ is Σ_n.

We then call a relation on H r.e. (or H–r.e.) iff it is definable by a Σ_1 formula. Recalling that $\omega \subset H$ it then turns out that a relation on ω is H–r.e. iff it is r.e. in the classical sense. Moreover, there is an H–recursive map $\pi : H \leftrightarrow \omega$ such that $A \subset H$ is H–r.e. iff $\pi''A$ is r.e. in the classical sense.

This suggests a very natural way of relativizing recursion theory to transfinite domains. Let $N = \langle |N|, \in, A_1, \ldots, A_n\rangle$ be any transitive structure. We first define:

Definition 1.1.1. A relation on N is $\Sigma_n(N)$ (in the parameters $p_1, \ldots, p_n \in N$) iff it is N–definable (in $\vec{p}$) by a Σ_n formula. It is $\Delta_n(N)$ (in $\vec{p}$) if both it and its completement are $\Sigma_n(N)$ (in $\vec{p}$). It is $\underline{\Sigma}_n(N)$ iff it is $\Sigma_n(N)$ in some parameters. Similarly for $\underline{\Delta}_n(N)$.

Following our above example of $N = \langle H, \in\rangle$, it is natural to define a relation on N as being *N–r.e.* iff it is $\underline{\Sigma}_1(N)$, and *N–recursive* iff it is $\underline{\Delta}_1(N)$. A partial function F on N is *N–r.e.* iff it is N–r.e. as a relation. F is *N–recursive* as a function iff it is N–r.e. and $\mathrm{dom}(F)$ is $\underline{\Delta}_1(N)$.

(Note that $\underline{\Sigma}_1(\langle H, \in\rangle) = \Sigma_1(\langle H, \in\rangle)$, which will not hold for arbitrary N.)

However, this will only work for an N satisfying rather strict conditions since, when we move to transfinite structures N, we must relativize not only the concepts "recursive" and "r.e.", but also the concept "finite". In the theory of H the finite sets were simply the elements of H.

Correspondingly we define:

u is *N–finite* iff $u \in N$.

But there are certain basic properties which we expect any recursion theory to have. In particular:

- If A is recursive and u is finite, then $A \cap u$ is finite.
- If u is finite and $F : u \to N$ is recursive, then $F''u$ is finite.

Those transitive structures $N = \langle |N|, \in A_1, \ldots, A_n \rangle$ which yield a satisfactory recursion theory are called *admissible*. An ordinal α is then called *admissible* iff L_α is admissible. The admissible structures were characterized by Kripke and Platek as those transfinite structures which satisfy the following axioms:

(1) $\emptyset, \{x, y\}, \bigcup x$ are sets

(2) *The Σ_0 axiom of subsets:*

$$x \cap \{z | \varphi(z)\} \text{ is a set}$$

(where φ is any Σ_0–formula)

(3) *The Σ_0 axiom of collection:*

$$\bigwedge x \in u \bigvee y\ \varphi(x, y) \to \bigvee v \bigwedge x \in u \bigvee y \in v\ \varphi(x, y),$$

(where φ is any Σ_0–formula).

Note. *Kripke–Platek set theory* (KP) consists of the above axioms together with the axiom of extensionality and the full axiom of foundation (i.e. for all formulae, not just the Σ_0 ones). This axiom can be stated as:

$$\bigwedge y (\bigwedge x \in y \varphi(x) \longrightarrow \varphi(y)) \longrightarrow \bigwedge y \varphi(y)$$

and is also known as the axiom of *induction*.

Note. Although the definability approach is the one most often employed in transfinite recursion theory, the approaches via algorithms and calculi have also been used to define the class of admissible ordinals.

1.1.2 Properties of admissible structures

We now show that admissible structures satisfy the two criteria stated above. In the following let $M = \langle |M|, \in A_a, \ldots, A_n \rangle$ be admissible.

Lemma 1.1.1. *Let $u \in M$. Let A be $\underline{\Delta}_1(M)$. Then $A \cap u \in M$.*

Proof: Let $Ax \leftrightarrow \bigvee y A_0 yx; \neg Ax \leftrightarrow \bigvee y A_1 yx$, where A_0, A_1 are $\underline{\Sigma}_0(M)$. Then $\bigwedge x \in u \bigvee y (A_0 yx \vee A_1 yx)$. Hence there is $v \in M$ such that $\bigwedge x \in u \bigvee y \in v (A_0 yx \vee A_1 yx)$. QED

Before verifying the second criterion we prove:

Lemma 1.1.2. *M satisfies:*

$$\bigwedge x \in u \bigvee y_1 \dots y_n \varphi(x, \vec{y}) \to \bigvee u \bigwedge x \in u \bigvee y_1 \dots y_n \in v \varphi(x, \vec{y})$$

for Σ_0–formulae φ.

Proof. Assume $\bigwedge x \in u \bigvee y_1 \dots y_n \varphi(x, \vec{y})$. Then

$$\bigwedge x \in u \bigvee w \underbrace{\bigvee y_1 \dots y_n \in w \varphi(x, \vec{y})}_{\Sigma_0}.$$

Hence there is $v' \in M$ such that $\bigwedge x \in u \bigvee w \in v' \bigvee y_1 \dots y_n \in w \varphi(x, \vec{y})$. Take $v = \bigcup v'$. QED (Lemma 1.1.2)

We now verify the second criterion:

Lemma 1.1.3. *Let $u \in M, u \subset \mathrm{dom}(F)$, where F is a $\underline{\Sigma}_1(M)$ function. Then $F''u \in M$.*

Proof. Let $y = F(x) \leftrightarrow \bigvee z F' zyx$, where F' is a $\underline{\Sigma}_0(M)$ relation. Then $\bigwedge x \in u \bigvee z, y F' zyx$. Hence there is $v \in M$ such that $\bigwedge x \in u \bigvee z, y \in v F' zyx$. Hence $F''u = v \cap \{y | \bigvee x \in u \bigvee z \in v F' zxy\}$. QED (Lemma 1.1.3)

Assuming the admissibility of M, we immediately get from Lemma 1.1.2:

Lemma 1.1.4. *Let $\varphi(y, \vec{x})$ be a Σ_1–formula. Then $\bigvee y \varphi(y, \vec{x})$ is uniformly Σ_1 in M.*

Note. "Uniformly" is a word which recursion theorists like to use. Here it means that $M \models \bigvee y \varphi(y, \vec{x}) \leftrightarrow \Psi(\vec{x})$ for a Σ_1 formula Ψ which depends only on φ and not on the choice of M.

Lemma 1.1.5. *Let $\varphi(y, \vec{x})$ be Σ_1. Then $\bigwedge y \in u \varphi(y, \vec{x})$ is uniformly Σ_1 in M.*

Proof. Let $\varphi(y, \vec{x}) = \bigvee z \varphi'(z, y, x)$, where φ' is Σ_0. Then

$$\bigwedge y \in u \varphi(y, \vec{x}) \leftrightarrow \bigvee v \underbrace{\bigwedge y \in u \bigvee z \in v \varphi'(z, y, x)}_{\Sigma_0}$$

in M. QED (Lemma 1.1.5)

Lemma 1.1.6. *Let $\varphi_0(\vec{x}), \varphi_1(\vec{x})$ be Σ_1. Then $(\varphi_0(\vec{x}) \wedge \varphi_1(\vec{x})), (\varphi_0(\vec{x}) \vee \varphi_1(\vec{x}))$ are uniformly Σ_1 in M.*

Proof. Let $\varphi_i(\vec{x}) = \bigvee y_i \varphi'_i(y_i, \vec{x})$ where without loss of generality $y_0 \neq y_1$. Then

$$(\varphi_0(\vec{x}) \wedge \varphi_1(\vec{x})) \leftrightarrow \bigvee y_0 \bigvee y_1(\varphi'_0(y_0, x) \wedge \varphi'_1(y_1, x)).$$

Similarly for $\vee$. QED (Lemma 1.1.6)

Putting this together:

Lemma 1.1.7. *Let $\varphi_1, \ldots, \varphi_n$ be Σ_1–formulae. Let Ψ be formed from $\varphi_1, \ldots, \varphi_n$ using only conjunction, disjunction, existence quantification and bounded universal quantification. Then $\Psi(x_1, \ldots, x_m)$ is uniformly $\Sigma_1(M)$*

An immediate consequence of Lemma 1.1.7 is:

Lemma 1.1.8. *$R \subset M^n$ is $\Sigma_1(M)$ in the parameter $\emptyset$ iff it is $\Sigma_1(M)$ in no parameter.*

Proof. Let $R(\vec{x}) \leftrightarrow R'(\emptyset, \vec{x})$. Then

$$R(\vec{x}) \leftrightarrow \bigvee z(R'(z, \vec{x}) \wedge \bigwedge y \in zy \neq y).$$

QED (Lemma 1.1.8)

Note. R is in fact *uniformly* $\Sigma_1(M)$ in the sense that its Σ_1 definition depends only on the original Σ_1 definition of R from $\emptyset$, and not on M.

Lemma 1.1.9. *Let $R(y_1, \ldots, y_n)$ be a relation which is $\Sigma_1(M)$ in the the parameter p. For $i = 1, \ldots, n$ let $f_i(x_1, \ldots, x_m)$ be a partial function on M which (as a relation) is $\Sigma_1(M)$ in p. Then the following relation is uniformly $\Sigma_1(M)$ in p:*

$$R(f_1(\vec{x}), \ldots, f_n(\vec{x})) \leftrightarrow: \bigvee y_1 \ldots y_n(\bigwedge_{i=1}^{n} y_i = f_i(\vec{x}) \wedge R(\vec{y})).$$

This follows by Lemma 1.1.7. ("Uniformly" again means that the Σ_1 definition depends only on the Σ_1 definition of $R, f_1, \ldots, f_n$.)

Similarly:

Lemma 1.1.10. *Let $f(y_1, \ldots, y_n), g_i(x_1, \ldots, x_m)(i = 1, \ldots, n)$ be partial functions which are $\Sigma_1(M)$ in p, then the function $h(\vec{x}) \simeq f(g(\vec{x}))$ is uniformly $\Sigma_1(M)$ in p.*

Proof.

$$z = h(\vec{x}) \leftrightarrow \bigvee y_1 \ldots y_n(\bigwedge_{i=1}^{n} y_i = g_i(\vec{x}) \wedge z = f(\vec{y})).$$

QED (Lemma 1.1.10)

Lemma 1.1.11. *Let $f_i(\vec{x})$ be a function which is $\Sigma_1(M)$ in $p (i = 1, \ldots, n)$. Let $R_i(\vec{x})(i = 1, \ldots, n)$ be mutually exclusive relations which are $\Sigma_1(M)$ in p. Then the function*

$$f(\vec{x}) \simeq f_i(\vec{x}) \text{ if } R_i(\vec{x})$$

is uniformly $\Sigma_1(M)$ in p.

Proof.

$$y = f(\vec{x}) \leftrightarrow \bigvee_{i=1}^{n} (y = f_i(\vec{x}) \wedge R_i(\vec{x})).$$

QED (Lemma 1.1.11)

Using these facts, we see that the restrictions of many standard set theoretic functions to M are $\Sigma_1(M)$.

Lemma 1.1.12. *The following functions are uniformly $\Sigma_1(M)$:*

(a) $f(x) = x, f(x) = \cup x, f(x, y) = x \cup y, f(x, y) = x \cap y, f(x, y) = x \setminus y$ (set difference)

(b) $f(x) = C_n(x)$, where $C_0(x) = x, C_{n+1}(x) = C_n(x) \cup \bigcup C_n(x)$

(c) $f(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$

(d) $f(x) = i$ (where $i < \omega$)

(e) $f(x_1, \ldots, x_n) = \langle x_1, \ldots, x_n \rangle$

(f) $f(x) = \mathrm{dom}(x)$, $f(x) = \mathrm{rng}(x)$, $f(x, y) = x''y, f(x, y) = x \upharpoonright y$, $f(x) = x^{-1}$

(g) $f(x_1, \ldots, x_n) = x_1 \times x_2 \times \ldots \times x_n$

(h) $f(x) = (x)_i^n$ where $(\langle z_0, \ldots, z_{n-1} \rangle)_i^n = z_i$ and $(u)_i^n = \emptyset$ in all other cases

(i) $f(x, z) = x[z] = \begin{cases} x(z) \text{ if } x \text{ is a function} \\ \text{and } z \in \mathrm{dom}(x) \\ \emptyset \text{ otherwise.} \end{cases}$

Proof. We display sample proofs. (a) is straightforward. (b) follows by induction on n. To see (c), $y = \{x_1, \ldots, x_n\}$ can be expressed by the Σ_0–statement

$$x_1, \ldots, x_n \in y \wedge \bigwedge z \in y (z = x_1 \vee \ldots \vee z = x_n).$$

(d) follows by induction on i, since

$$0 = \emptyset, i + 1 = i \cup \{i\}.$$

The proof of (e) depends on the precise definition of $\langle x_1, \dots x_n \rangle$. If we want each tuple to have a unique length, then the following definition recommends itself: First define a notion of ordered pair by: $(x, y) =: \{\{x\}, \{x, y\}\}$ Then (x, y) is a Σ_1 function. Then if $\langle x_1, \dots, x_n \rangle =: \{(x_1, 0), \dots, (x_n, n-1)\}$, the conclusion is immediate.

For (f) we display the proof that $\mathrm{dom}(x)$ is a Σ_1 function. Note that $x, y \in C_n(\langle x, y \rangle)$ for a sufficient n. But since every element of $\mathrm{dom}(x)$ is a component of a pair lying in x, it follows that $\mathrm{dom}(x) \subset C_n(x)$ for a sufficient n. Hence $y = \mathrm{dom}(x)$ can be expressed as:

$$\bigwedge z \in y \bigvee w \langle w, z \rangle \in x \wedge \bigwedge z, w \in C_n(x)(\langle w, z \rangle \in x \to z \in y).$$

To see (g), note that $y = x_1 \times \ldots \times x_n$ can be expressed by:

$$\begin{array}{l} \bigwedge z_1 \in x_1 \ldots \bigwedge z_n \in x_n \langle z_1, \ldots, z_n \rangle \in y \\ \wedge \bigwedge w \in y \bigvee z_1 \in x_1 \ldots \bigvee z_n \in x_n w = \langle z_1, \ldots, z_n \rangle. \end{array}$$

To see (h) note that, for sufficiently large $m, y = (x)^n_i$ can be expressed by:

$$\begin{array}{l} \bigvee z_0 \ldots z_{n-1}(x = \langle z_0, \ldots, z_{n-1} \rangle \wedge y = z_i) \\ \vee (y = \emptyset \wedge \bigwedge z_0 \ldots z_{n-1} \in C_m(x) x \neq \langle z_0, \ldots, z_{n-1} \rangle) \end{array}$$

(i) is similarly straightforward. QED (Lemma 1.1.12)

The *recursion theorem* of classical recursion theory says that if $g(n, m)$ is recursive on ω and $f : \omega \to \omega$ is defined by:

$$f(0) = k, f(n+1) = g(n, f(n)),$$

then f is recursive. The point is that the value of f at any n is determined by its values at smaller numbers. Working with H instead of ω we can express this in the elegant form:

> Let $g : \omega \times H \to \omega$ be Σ_1.
> Then $f : \omega \to \omega$ is Σ_1, where $f(n) = g(n, f \restriction n)$.

If we take $g : H^2 \to H$, then f will be Σ_1 where $f(x) = g(x, f \restriction x)$ for $x \in H$. We can even take g as being a partial function on H^2. Then f is Σ_1 where:

$$f(x) \simeq g(x, \langle f(z) | z \in x \rangle).$$

(This means that $f(x)$ is defined if and only if $f(z)$ is defined for $z \in x$ and g is defined at $\langle x, f \upharpoonright x\rangle$, in which case the above equality holds.)

We now prove the same thing for an arbitrary admissible M. If f is a partial $\underline{\Sigma}_1$ function and $x \subset \mathrm{dom}(f)$, we know by Lemma 1.1.3 that $f''x \in M$. But then $f \upharpoonright x \in M$, since $f^*(z) \simeq \langle f(z), z\rangle$ is a $\underline{\Sigma}_1$ function with $x \subset \mathrm{dom}(f^*)$, and $f^{*\prime\prime}x = f \upharpoonright x$. The *recursion theorem for admissibles* $M = \langle |M|, \in, A_1, \ldots, A_n\rangle$ then reads:

Lemma 1.1.13. *Let $G(y, \vec{x}, u)$ be a $\Sigma_1(M)$ function in the parameter p. Then there is exactly one function $F(y, \vec{x})$ such that*

$$F(y, \vec{x}) \simeq G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle).$$

Moreover, F is uniformly $\Sigma_1(M)$ in p (i.e. the Σ_1 definition depends only on the Σ_1 definition of G.)

Proof. We first show existence. Set:

$$\begin{aligned} \Gamma_{\vec{x}} =: \ & \{f \in M | f \text{ is a function } \wedge \mathrm{dom}(f) \text{ is} \\ & \text{transitive } \wedge \bigwedge y \in \mathrm{dom}(f) f(y) = G(y, \vec{x}, f \upharpoonright y)\} \end{aligned}$$

Set $F_{\vec{x}} = \bigcup \Gamma_{\vec{x}}; F = \{\langle y, \vec{x}\rangle | y \in F_{\vec{x}}\}$. Then F is $\Sigma_1(M)$ in p uniformly.

(1) F is a function.

Proof. Suppose not. Then for some $\vec{x}$ there are $f, f' \in \Gamma_{\vec{x}}$, $y \in \mathrm{dom}(f) \cap \mathrm{dom}(f')$ such that $f(y) \neq f'(y)$. Let y be $\in$–minimal with this property. Then $f \upharpoonright y = f' \upharpoonright y$. But then $f(y) = G(y, \vec{x}, f \upharpoonright y) = G(y, \vec{x}, f' \upharpoonright y) = f'(y)$. Contradiction! QED (1)

Hence $F(y, \vec{x}) = f(y)$ if $y \in \mathrm{dom}(f)$ and $f \in \Gamma_{\vec{x}}$.

(2) Let $\langle y, \vec{x}\rangle \in \mathrm{dom}(F)$. Then $y \subset \mathrm{dom}(F_{\vec{x}}), \langle y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle\rangle \in \mathrm{dom}(G)$ and

$$F(y, \vec{x}) = G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle).$$

Proof. Let $y \in \mathrm{dom}(f), f \in \Gamma_{\vec{x}}$. Then

$$\begin{aligned} F(y, \vec{x}) = f(y) \ &= G(y, \vec{x}, f \upharpoonright x) \\ &= G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle). \end{aligned}$$

QED (2)

(3) Let $y \subset \mathrm{dom}(F_{\vec{x}}), \langle y, \vec{x}, F_{\vec{x}} \upharpoonright y\rangle \in \mathrm{dom}(G)$. Then $y \in \mathrm{dom}(F_{\vec{x}})$.

Proof. By our assumption: $\bigwedge z \in y \bigvee f (f \in \Gamma_{\vec{x}} \wedge z \in \mathrm{dom}(f))$. Hence there is $u \in M$ such that

$$\bigwedge z \in y \bigvee f \in u (f \in \Gamma_{\vec{x}} \wedge z \in \mathrm{dom}(f)).$$

Set: $f' = \bigcup(u \cap \Gamma_{\vec{x}})$. Then $f' \in \Gamma_{\vec{x}}$ and $y \subset \mathrm{dom}(f')$. Moreover $f' \upharpoonright y = F_{\vec{x}} \upharpoonright y$. Set $f'' = f' \cup \{\langle G(y, \vec{x}, f' \upharpoonright y), y\rangle\}$. Then $f'' \in \Gamma_{\vec{x}}$ and $y \in \mathrm{dom}(f'')$, where $f'' \subset F_{\vec{x}}$. QED (3)

This proves existence. To show uniqueness, we virtually repeat the proof of (1): Let F^* satisfy the same condition. Set $F^*_{\vec{x}}(y) \simeq F^*(y, \vec{x})$. Suppose $F^* \neq F$. Then $F^*_{\vec{x}}(y) \not\simeq F_{\vec{x}}(y)$ for some $\vec{x}, y$. Let y be $\in$–minimal such that $F^*_{\vec{x}}(y) \not\simeq F_{\vec{x}}(y)$. Then $F^*_{\vec{x}} \upharpoonright y = F_{\vec{x}} \upharpoonright y$. Hence

$$\begin{aligned} F^*_{\vec{x}}(y) \quad &\simeq G(y, \vec{x}, \langle F^*_{\vec{x}}(z) | z \in y\rangle) \\ &\simeq G(y, \vec{x}, \langle F_{\vec{x}}(z) | z \in y\rangle) \\ &\simeq F_{\vec{x}}(y). \end{aligned}$$

Contradiction! QED (Lemma 1.1.13)

We recall that the transitive closure $TC(x)$ of a set x is recursively definable by: $TC(x) = x \cup \bigcup_{z \in x} TC(z)$. Similarly, the rank $rn(x)$ of a set is definable by $rn(x) = \mathrm{lub}\{rn(z) | z \in x\}$. Hence:

Corollary 1.1.14. *TC, rn are uniformly $\Sigma_1(M)$.*

The successor function $s\alpha = \alpha + 1$ on the ordinals is defined by:

$$sx = \begin{cases} x \cup \{x\} \text{ if } x \in On \\ \text{undefined if not} \end{cases}$$

which is Σ_1. The function $\alpha + \beta$ is defined by:

$$\begin{aligned} &\alpha + 0 = \alpha \\ &\alpha + s\nu = s(\alpha + \nu) \\ &\alpha + \lambda = \textstyle\bigcup_{\nu < \lambda} \alpha + \nu \text{ for limit } \lambda. \end{aligned}$$

This has the form:

$$x + y \simeq G(y, x, \langle x + z | z \in y\rangle).$$

Similarly for the function $x \cdot y, x^y, \ldots$ etc. Hence:

Corollary 1.1.15. *The ordinal functions $\alpha + 1, \alpha + \beta, \alpha^\beta, \ldots$ etc. are uniformly $\Sigma_1(M)$.*

We note that there is an even more useful form of Lemma 1.1.13:

Lemma 1.1.16. *Let G be as in Lemma 1.1.13. Let $h : M \to M$ be $\Sigma_1(M)$ in p such that $\{\langle x, y\rangle | x \in h(y)\}$ is well founded. There is a unique F such that*

$$F(y, \vec{x}) \simeq G(y, \vec{x}, \langle F(z, \vec{x}) | x \in h(y)\rangle).$$

Moreover, F is uniformly[1] $\Sigma_1(M)$ *in* p.

The proof is exactly like that of Lemma 1.1.13, using minimality in the relation $\{\langle x,y\rangle | x \in h(y)\}$ in place of $\in$–minimality. We now consider the structure of "really finite" sets in an admissible M.

Lemma 1.1.17. *Let $u \in H_\omega$. The class u and the constant function $f(x) = u$ are uniformly $\Sigma_1(M)$.*

Proof. By $\in$–induction on u: Let $u = \{z_1, \dots, z_n\}$.

$$x \in u \leftrightarrow \bigvee_{i=1}^{n} x = z_i$$
$$x = u \leftrightarrow \bigwedge y \in x\, y \in u \wedge \bigwedge_{i=1}^{n} z_i \in x.$$

QED

$x \in \omega$ is clearly a Σ_0 condition. But then:

Lemma 1.1.18. *Let $\omega \in M$. Then the constant function $f(x) = \omega$ is uniformly $\Sigma_1(M)$.*

Proof.

$$x = \omega \leftrightarrow (\bigwedge z \in x z \in \omega \wedge \emptyset \in x \wedge \bigwedge z \in x z \cup \{z\} \in x)$$

(where '$z \in \omega$' is Σ_0) QED

Lemma 1.1.19. *The class* Fin *and the function $f(x) = \mathbb{P}_\omega(x)$ are uniformly $\Sigma_1(M)$, where* $\mathrm{Fin} = \{x \in M | \overline{\overline{x}} < \omega\}, \mathbb{P}_\omega(x) = \mathbb{P}(x) \cap \mathrm{Fin}$.

Proof.

$$\begin{array}{ll} x \in \mathrm{Fin} & \leftrightarrow \bigvee n \in \omega \bigvee f\, f : n \leftrightarrow x \\ y = \mathbb{P}_\omega(x) & \leftrightarrow \bigwedge u \in y (u \subset x \wedge u \in \mathrm{Fin}) \wedge \emptyset \in y \wedge \\ & \wedge \bigwedge z \in x \{z\} \in y \wedge \bigwedge u, v \in y u \cup v \in y. \end{array}$$

We must show that $\mathbb{P}_\omega(x) \in M$. If $\omega \notin M$, then $rn(x) < \omega$ for all $x \in M$, Hence $M = H_\omega$ is closed under $\mathbb{P}_\omega$. If $\omega \in M$, there is $\underline{\Sigma}_1(M)$ f defined by

$$f(0) = \{\{z\} | z \in x\}, f(n+1) = \{u \cup v | \langle u, v\rangle \in f(n)^2\}.$$

Then $\mathbb{P}_\omega(x) = \bigcup f''\omega \in M$. QED (Lemma 1.1.19)

But then:

[1]("uniformly" meaning, of course, that the Σ_1 definition of F depends only on the Σ_1 definition of G, h.)

Lemma 1.1.20. *If $\omega \in M$, then $H_\omega \in M$ and the constant function $f(x) = H_\omega$ is uniformly $\Sigma_1(M)$.*

Proof. $H_\omega \in M$, since there is a $\Sigma_1(M)$ function g defined by $g(0) = \emptyset, g(n+1) = \mathbb{P}_\omega(g(n))$. Then $H_\omega = \bigcup g''\omega \in M$ and $f(x) = H_\omega$ is $\Sigma_1(M)$ since g and the constant function ω are $\Sigma_1(M)$. QED (Lemma 1.1.20)

1.1.3 The constructible hierarchy

We recall Gödel's definition of the *constructible hierarchy* $\langle L_r | r \in \mathrm{On}\rangle$:

$$\begin{aligned}&L_0 = \emptyset\\ &L_{\nu+1} = \mathrm{Def}(L_\nu)\\ &L_\lambda = \bigcup_{\nu<\lambda} L_\nu \text{ for limit } \lambda,\end{aligned}$$

where $\mathrm{Def}(u)$ is the set of all $z \subset u$ which are $\langle u, \in\rangle$–definable in parameters from u (taking $\mathrm{Def}(\emptyset) = \{\emptyset\}$). (Note that if u is transitive, then $u \subset \mathrm{Def}(u)$ and $\mathrm{Def}(u)$ is transitive.) Gödel's *constructible universe* is then $L =: \bigcup\limits_{\nu\in\mathrm{On}} L_\nu$.

By fairly standard methods one can show:

Lemma 1.1.21. *Let $\omega \in M$. Then the function $f(u) = \mathrm{Def}(u)$ is uniformly $\Sigma_1(M)$.*

We omit the proof, which is quite lengthy. It involves "arithmetizing" the language of first order set theory by identifying formulae with elements of ω or H_ω, and then showing that the relevant syntactic and semantic concepts are M–recursive.

By the recursion theorem we can of course conclude:

Corollary 1.1.22. *Let $\omega \in M$. The function $f(\alpha) = L_\alpha$ is uniformly $\Sigma_1(M)$.*

The constructible hierarchy *over* a set u is defined by:

$$\begin{aligned}&L_0(u) = TC(\{u\})\\ &L_{\nu+1}(u) = \mathrm{Def}(L_\nu(u))\\ &L_\lambda(u) = \bigcup_{\nu<\lambda} L_\nu(u) \text{ for limit } \lambda.\end{aligned}$$

Obviously:

Corollary 1.1.23. *Let $\omega \in M$. The function $f(u,\alpha) = L_\alpha(u)$ is uniformly $\Sigma_1(M)$.*

The constructible hierarchy *relative to* classes $A_1, \dots, A_n$ is defined by:

$$\begin{aligned} &L_0[\vec{A}] = \emptyset \\ &L_{\nu+1}[\vec{A}] = \mathrm{Def}(L_\nu[\vec{A}], \vec{A}) \\ &L_\lambda[\vec{A}] = \bigcup_{\nu<\lambda} L_\nu[\vec{A}] \text{ for limit } \lambda, \end{aligned}$$

where $\mathrm{Def}(U, A_1, \dots, A_n)$ is the set of all $z \subset u$ which are $\langle u, \in, A_1 \cap u, \dots, A_n \cap u\rangle$–definable in parameters from u.

Much as before we have:

Lemma 1.1.24. *Let $\omega \in M$. Let $A_1, \dots, A_n$ be $\Delta_1(M)$ in the parameter p. Then the function $f(u) = \mathrm{Def}(u, A_1, \dots, A_n)$ is uniformly $\Sigma_1(M)$ in p.*

Corollary 1.1.25. *Let $\omega \in M$. Let $A_1, \dots, A_n$ be as above. Then the function $f(\alpha) = L_\alpha[\vec{A}]$ is uniformly $\Sigma_1(M)$ in p.*

(In particular, if $M = \langle |M|, \in, A_1, \dots, A_n\rangle$. Then $f(\alpha) = L_\alpha[\vec{A}]$ is uniformly $\Sigma_1(M)$.)

(One could, of course, also define $L_\alpha(u)[\vec{A}]$ and prove the corresponding results.)

Any well ordering r of a set u induces a well ordering of $\mathrm{Def}(u)$, since each element of $\mathrm{Def}(u)$ is defined over $\langle u, \in\rangle$ by a tuple $\langle \varphi, x_1, \dots, x_n\rangle$, where φ is a formula and $x_1, \dots, x_n$ are elements of u which interpret free variables of φ. If u is transitive (hence $u \subset \mathrm{Def}(u)$), we can also arrange that the well ordering, which we shall call $< (u, r)$, is an end extension of r. The function $< (u, r)$ is uniformly Σ_1. If we then set:

$$\begin{aligned} &<_0 = \emptyset, <_{\nu+1} = < (L_\nu, <_\nu) \\ &<_\lambda = \bigcup_{\nu<\lambda} <_\nu \text{ for limit } \lambda, \end{aligned}$$

it follows that $<_\nu$ is a well ordering of L_ν for all ν. Moreover $<_\alpha$ is an end extension of $<_\nu$ for $\nu < \alpha$.

Similarly, if A is $\Sigma_1(M)$ in p, there is a hierarchy $<_\nu^A$ ($\nu \in \mathrm{On} \cap M$) such that $<_\nu^A$ well orders $L_\nu[A]$ and the function $f(\nu) = <_\nu^A$ is $\Sigma_1(M)$ in p (uniformly relative to the Σ_1 definition of A).

By Corollary 1.1.25 we easily get:

Lemma 1.1.26. *Let $M = \langle |M|, \in, A_1, \ldots, A_n\rangle$ be admissible. Let $\alpha = \mathrm{On} \cap M$. Then $\langle L_\alpha[\vec{A}], \in, \vec{A}\rangle$ is admissible.*

Proof: Set: $L_\nu^{\vec{A}} = \langle L_\alpha[\vec{A}], \in, \vec{A}\rangle$. Axiom (1) holds trivially in $L_\nu^{\vec{A}}$.

To verify the Σ_0–axiom of subsets, let B be $\underline{\Sigma}_0(L_\alpha^{\vec{A}})$. Let $u \in L_\alpha^{\vec{A}}$.

Claim $u \cap B \in L_\alpha^{\vec{A}}$.

Proof: Pick $\nu < \alpha$ such that $u \in L_\nu^{\vec{A}}$ and B is $\underline{\Sigma}_0$ in parameters from $L_\nu^{\vec{A}}$. By $\underline{\Sigma}_0$–absoluteness we have:

$$u \cap B \in \mathrm{Def}(L_\nu^{\vec{A}}) = L_{\nu+1}^{\vec{A}} \subset L_\alpha^{\vec{A}}.$$

QED (Claim)

We now prove Σ_0–collection. Let Rxy be a $\underline{\Sigma}_0$–relation. Let $u \in L_\alpha^{\vec{A}}$ such that $\bigwedge x \in u \bigvee y Rxy$.

Claim $\bigvee v \in L_\alpha^{\vec{A}} \bigwedge x \in u \bigvee y \in v Rxy$.

For each $x \in u$ let $g(x)$ be the least $\nu < \alpha$ such that $x \in L_\nu^{\vec{A}}$. Then g is in $\underline{\Sigma}_1(M)$ and $u \subset \mathrm{dom}(g)$. Hence $\delta = \sup g''u < \alpha$ and

$$\bigwedge x \in u \bigvee y \in L_\delta^{\vec{A}} Rxy.$$

QED (Lemma 1.1.26)

Definition 1.1.2. Let α be an ordinal.

- α is *admissible* iff L_α is admissible
- α is *admissible in* $A_1, \ldots, A_n \subset$ iff $L_\alpha^{\vec{A}} =: \langle L_\alpha[\vec{A}], \in \vec{A}\rangle$ is admissible
- $f : \alpha^n \to \alpha$ is *α–recursive* (in $\vec{A}$) iff f is $\underline{\Sigma}_1(L_\alpha)(\underline{\Sigma}_1(L_\alpha^{\vec{A}}))$
- $R \subset \alpha^n$ is *r.e.* (in $\vec{A}$) iff R is $\Sigma_1(L_\alpha)(\Sigma_1(L_\alpha^{\vec{A}}))$.

Note. The theory of α–recursive functions and relations on an admissible α has been built up without references to L_α, using a formalized notion of α–bounded calculus (Kripke) or α–bounded algorithm (Platek).

Similarly for α–recursiveness in $A_1, \ldots, A_n$, taking the A_i as "oracles".

A transitive structure $M = \langle |M|, \in \vec{A} \rangle$ is called *strongly admissible* iff, in addition to the Kripke–Platek axioms, it satisfies the Σ_1 *axiom of subsets*:

$$x \cap \{z | \varphi(z)\} \text{ is a set (for } \Sigma_1 \text{ formulae } \varphi).$$

Kripke defines the *projectum* δ_α of an admissible ordinal α to be the least δ such that $A \cap \delta \notin L_\alpha$ for some $\underline{\Sigma}_1(L_\alpha)$ set A. He shows that $\delta_\alpha = \alpha$ iff α is strongly admissible. He calls α *projectible* iff $\delta_\alpha < \alpha$. There are many projectible admissibles — e.g. $\delta_\alpha = \omega$ if α is the least admissible greater than ω. He shows that for every admissible α there is a $\underline{\Sigma}_1(L_\alpha)$ injection f_α of L_α into δ_α.

The definition of projectum of course makes sense for *any* $\alpha \geq \omega$. By refinements of Kripke's methods it can be shown that f_α exists for every $\alpha \geq \omega$ and that $\delta_\alpha < \alpha$ whenever $\alpha \geq \omega$ is not strongly admissible. We shall — essentially — prove these facts in chapter 2 (except that, for technical reasons, we shall employ a modified version of the constructible hierarchy).

1.2 Primitive Recursive Set Functions

1.2.1 PR Functions

The *primitive recursive set functions* comprise a collection of functions

$$f : V^n \to V$$

which form a natural analogue of the primitive recursive number functions in ordinary recursion theory. As with admissibility theory, their discovery arose from the attempt to generalize ordinary recursion theory. These functions are ubiquitous in set theory and have very attractive absoluteness properties. In this section we give an account of these functions and their connection with admissibility theory, though — just as in §1 — we shall suppress some proofs.

Definition 1.2.1. $f : V^n \to V$ is a *primitive recursive (pr) function* iff it is generated by successive application of the following schemata:

(i) $f(\vec{x}) = x_i$ (here $\vec{x}$ is $x_1, \ldots, x_n$)

(ii) $f(\vec{x}) = \{x_i, x_j\}$

(iii) $f(\vec{x}) = x_i \setminus x_j$

(iv) $f(\vec{x}) = g(h_1(\vec{x}), \ldots, h_m(\vec{x}))$