1.4 Barwise Theory

Jon Barwise worked out the syntax and model theory of certain infinitary (but M–finite) languages in countable admissible structures M. In so doing, he created a powerful and flexible tool for set theory, which we shall utilize later in this book. In this chapter we give an introduction to Barwise's work.

1.4.1 Syntax

Let M be admissible. Barwise developed a first order theory in which arbitrary M–finite conjunction and disjunction are allowed. The predicates, however, have only a (genuinely) finite number of argument places and there are no infinite strings of quantifiers. In order that the notion "M–finite" have a meaning for the symbols in our language, we must "arithmetize" the language — i.e. identify its symbols with objects in M. There are many ways of doing this. For the sake of definitness we adopt a specific arithmetization of M–finitary first order logic:

Predicates: For each $x \in M$ and each n such that $1 \leq n < \omega$ we appoint an n–ary predicate $P^n_x =: \langle 0, \langle n, x\rangle\rangle$.

Constants: For each $x \in M$ we appoint a constant $c_x =: \langle 1, x\rangle$.

Variables: For each $x \in M$ we appoint a variable $v_x =: \langle 2, x\rangle$.

Note The set of variables must be M–infinite, since otherwise a single formula might exhaust all the variables.

We let P^2_0 be the identity predicate $\dot{=}$ and also reserve P^2_1 as the $\in$–predicate ($\dot{\in}$).

By a *primitive formula* we mean $Pt_1 \dots t_n =: \langle 3, \langle P, t_1, \dots, t_n\rangle\rangle$ where P is an n–ary predicate and $t_1, \dots, t_n$ are variables or constants.

We then define:

$$\neg\varphi =: \langle 4, \varphi\rangle, (\varphi \vee \psi) =: \langle 5, \langle \varphi, \psi\rangle\rangle,$$
$$(\varphi \wedge \psi) =: \langle 6, \langle \varphi, \psi\rangle\rangle, (\varphi \to \psi) =: \langle 7, \langle \varphi, \psi\rangle\rangle,$$
$$(\varphi \leftrightarrow \psi) =: \langle 8, \langle \varphi, \psi\rangle\rangle, \bigwedge v\varphi = \langle 9, \langle v, \varphi\rangle\rangle,$$
$$\bigvee v\varphi = \langle 10, \langle v, \varphi\rangle\rangle.$$

The infinitary conjunctions and disjunctions are

$$\bigwedge\!\!\!\bigwedge f =: \langle 11, f\rangle, \bigvee\!\!\!\bigvee f =: \langle 12, f\rangle.$$

The set Fml of *first order M–formulae* is then the smallest set X which contains all primitive formulae, is closed under $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, and such that

- If v is a variable and $\varphi \in X$, then $\bigwedge v\varphi \in X$ and $\bigvee v\varphi \in X$.
- If $f = \langle \varphi_i | i \in I\rangle \in M$ and $\varphi_i \in X$ for $i \in I$, then $\bigwedge\!\!\!\bigwedge f \in X$ and $\bigvee\!\!\!\bigvee f \in X$.

(In this case we also write:

$$\bigwedge\!\!\!\bigwedge_{i\in I} \varphi_i =: \bigwedge\!\!\!\bigwedge f, \bigvee\!\!\!\bigvee_{i\in I} \varphi_i =: \bigwedge\!\!\!\bigwedge f.$$

If $B \in M$ is a set of formulae we may also write: $\bigwedge\!\!\!\bigwedge B$ for $\bigwedge\!\!\!\bigwedge_{\varphi\in B} \varphi$.)

It turns out that the usual syntactical notions are $\Delta_1(M)$, including: Fml, $Const$ (set of constants), Vbl (set of variables), $Sent$ (set of all sentences), as are the functions:

> $Fr(\varphi) =$ The set of free variables in φ
> $\varphi({}^v/t) \simeq$ the result of replacing occurences of the variable v by t (where $t \in Vbl \cup Const$), as long as this can be done without a new occurence of t being bound by a quantifier in φ (if t is a variable).

That Vbl, $Const$ are Δ_1 (in fact Σ_0) is immediate. The characteristic function X of Fml is definable by a recursion of the form:

$$X(x) = G(x, \langle X(z)|z \in TC(x))$$

where $G : M^2 \to M$ is Δ_1. (This is an instance of the recursion schema in §1 Lemma 1.1.16. We are of course using the fact that any proper subformula of φ lies in $TC(\varphi)$.)

Now let $h(\varphi)$ be the set of immediate subformulae of φ (e.g. $h(\neg\varphi) = \{\varphi\}$, $h(\bigwedge\!\!\!\bigwedge_{i\in I}\varphi_i) = \{\varphi_i | i \in I\}$, $h(\bigwedge v\varphi) = \{\varphi\}$ etc.) Then h satisfies the condition in §1 Lemma 1.1.16. It is fairly easy to see that

$$Fr(\varphi) = G(\varphi, \langle F(x)|x \in h(\varphi)\rangle)$$

where G is a Σ_1 function defined on Fml. Then $Sent = \{\varphi | Fr(\varphi) = \emptyset\}$.

To define $\varphi({}^v/t)$ we first define it on primitive formulae, which is straightforward. We then set:

$(\varphi \wedge \psi)(^v/t) \simeq (\varphi(^v/t) \wedge \psi(^v/t))$ (similarly for $\wedge, \rightarrow, \leftrightarrow$)

$\neg\varphi(^v/t) \simeq \neg(\varphi(^v/t))$

$(\bigwedge\!\!\!\bigwedge_{i \in I} \varphi_i)(^v/t) \simeq \bigwedge\!\!\!\bigwedge_{i \in I}(\varphi_i(^v/t))$ similarly for $\bigvee\!\!\!\bigvee$.

$$(\bigwedge u\varphi)(^v/t) \simeq \begin{cases} \bigwedge u\varphi \text{ if } u = v \\ \bigwedge u(\varphi(^v/t)) \text{ if } u \neq v, t \quad \text{(similarly for } \bigvee) \\ \text{otherwise undefined} \end{cases}$$

This has the form:

$$\varphi(^v/t) \simeq G(\varphi, v, t \langle X(^v/t) | X \in h(\varphi) \rangle),$$

where G is $\Sigma_1(M)$. The domain of the function $f(\varphi, v, t) = \varphi(^v/t)$ is $\Delta_1(M)$, however, so f is M–recursive.

(We can then define:

$$\varphi(^{v_1,\ldots,v_n}/t_1, \ldots, t_n) = \varphi(^{v_1}/w_1) \ldots (^{v_n}/w_n)(^{w_1}/t_1) \ldots (^{w_n}/t_n)$$

where $v_1, \ldots, v_n$ is a sequence of distinct variables and $w_1, \ldots, w_n$ is any sequence of distinct variables which are different from $v_1, \ldots, v_n$, $t_1, \ldots, t_n$ and do not occur bound or free in φ. We of cours follow the usual conventions, writing $\varphi(t_1, \ldots, t_n)$ for $\varphi(^{v_1,\ldots,v_n}/t_1, \ldots, t_n)$, taking $v_1, \ldots, v_n$ as known.)

M–finite predicate logic has the axioms:

- all instances of the usual propositional logic axiom schemata (enough to derive all tautologies with the help of modus ponens).
- $\bigwedge\!\!\!\bigwedge_{i \in U} \varphi_i \rightarrow \varphi_j,\ \varphi_j \rightarrow \bigvee\!\!\!\bigvee_{i \in U} \varphi_i\ (j \in U \in M)$
- $\bigwedge x\varphi \rightarrow \varphi(^x/t),\ \varphi(^x/t) \rightarrow \bigvee x\varphi$
- $x \doteq y \rightarrow (\varphi(x) \leftrightarrow \varphi(y))$

The *rules of inference* are:

- $\frac{\varphi, \varphi \rightarrow \psi}{\psi}$ (modus ponens)
- $\frac{\varphi \rightarrow \psi}{\varphi \rightarrow \bigwedge x\psi}$ if $x \notin Fr(\varphi)$
- $\frac{\psi \rightarrow \varphi}{\bigvee x\psi \rightarrow \varphi}$ if $x \notin Fr(\varphi)$
- $\frac{\varphi \rightarrow \psi_i (i \in u)}{\varphi \rightarrow \bigwedge\!\!\!\bigwedge \psi_i}$ $(u \in M)$

- $\frac{\psi_i \to \varphi (i \in u)}{\bigvee\!\!\!\bigvee \psi_i \to \varphi}$ $(u \in M)$

We say that φ is *provable* from a set of sentences A iff φ is in the smallest set which contains A and the axioms and is closed under the rules of inference. We write $A \vdash \varphi$ to mean that φ is provable from A. $\vdash \varphi$ means the same as $\emptyset \vdash \varphi$.

However, this definition of provability cannot be stated in the 1st order language of M and rather misses the point which is that a provable formula should have an M–finite proof. This, as it turns out, will be the case whenever A is $\underline{\Sigma}_1(M)$. In order to state and prove this, we must first formalize the notion of proof. Because we have not assumed the axiom of choice to hold in our admissible structure M, we adopt a somewhat unorthodox concept of proof:

Definition 1.4.1. By a *proof from* A we mean a sequence $\langle p_i | i < \alpha \rangle$ such that $\alpha \in \mathrm{On}$ and for each $i < \alpha$, $p_i \subset Fml$ and whenever $\psi \in p_i$, then either $\psi \in A$ or ψ is an axiom or ψ follows from $\bigcup\limits_{h<i} p_h$ by a single application of one of the rules.

Definition 1.4.2. $p = \langle p_i | i < \alpha \rangle$ is a *proof of* φ *from* A iff p is a proof from A and $\varphi \in \bigcup\limits_{i<\alpha} p_i$.

(**Note** that this definition does *not* require a proof to be M–finite.)

It is straightforward to show that φ is provable iff it has a proof. However, we are more interested in M–finite proofs. If A is $\Sigma_1(M)$ in a parameter q, it follows easily that $\{p \in M | p \text{ is a proof from } A\}$ is $\Sigma_1(M)$ in the same parameter. A more interesting conclusion is:

Lemma 1.4.1. *Let A be $\underline{\Sigma}_1(M)$. Then $A \vdash \varphi$ iff there is an M–finite proof of φ from A.*

Proof: $(\leftarrow)$ trivial. We prove $(\rightarrow)$

Let $X =$ the set of φ such that there is $p \in M$ which proves φ from A.

Claim: $\{\varphi | A \vdash \varphi\} \subset X$.

Proof: We know that $A \subset X$ and all axioms lie in X. Hence it suffices to show that X is closed under the rules of proof. This must be demonstrated rule by rule. As an example we show:

Claim: Let $\varphi \to \psi_i$ be in X for $i \in u$. Then $\varphi \to \bigwedge\!\!\!\bigwedge_{i \in u} \psi_i \in X$.

Proof: Let $P(p,\varphi)$ mean: p is a proof of φ from A. Then P is $\underline{\Sigma}_1(M)$. We have assumed:

(1) $\bigwedge i \in u \bigvee_P \; P(p, \varphi \to \psi_i)$.
Now let $P(p,x) \leftrightarrow \bigvee z P'(z,p,x)$ where P' is Σ_0. We then have:

(2) $\bigwedge i \in u \bigvee p \bigvee z P'(z,p,\varphi \to \psi_i)$.
Hence there is $v \in M$ with:

(3) $\bigwedge i \in u \bigvee p, z \in v P'(z,p,\varphi \to \psi_i)$.
Set: $w = \{p \in v | \bigvee i \in u \bigvee z \in v P'(z,p,\varphi \to \psi_i)\}$
Set: $\alpha = \bigcup\limits_{p \in w} \mathrm{dom}(p)$. For $i < \alpha$ set:

$$q_i = \bigcup \{p_i | p \in w \wedge i \in \mathrm{dom}(p)\}$$

Then $q = \langle q_i | i < \alpha \rangle \in M$ is a proof.

? But then $q^\frown\{\varphi \longrightarrow \bigwedge\!\!\!\bigwedge_{i \in U} \psi_i\}$ is a proof of $\varphi \longrightarrow \bigwedge\!\!\!\bigwedge_{i \in U} \psi_i$. Hence $\varphi \longrightarrow \bigwedge\!\!\!\bigwedge_{i \in U} \psi_i \in X$.

QED (Lemma 1.4.1)

From this we get the *M–finiteness lemma*:

Lemma 1.4.2. *Let A be $\underline{\Sigma}_1(M)$. Then $A \vdash \varphi$ iff there is $a \subset A$ such that $a \in M$ and $a \vdash \varphi$.*

Proof: $(\leftarrow)$ is trivial. We prove $(\rightarrow)$. Let $p \in M$ be a proof of φ from A. Set:

$a =$ the set of ψ such that for some $i \in \mathrm{dom}(p)$, $\psi \in p_i$ and ψ is neither an axiom nor follows from $\bigcup\limits_{l<i} p_l$ by an application of a single rule.

Then $a \subset A$, $a \in M$, and p is a proof of φ from a. QED (Lemma 1.4.2)

Another consequence of Lemma 1.4.1 is:

Lemma 1.4.3. *Let A be $\Sigma_1(M)$ in q. Then $\{\varphi | A \vdash \varphi\}$ is $\Sigma_1(M)$ in the same parameter (uniformly in the Σ_1 definition of A).*

Proof: $\{\varphi | A \vdash \varphi\} = \{\varphi | \bigvee p \in M \; p$ proves φ from $A\}$.

Corollary 1.4.4. *Let A be $\Sigma_1(M)$ in q. Then "A is consistent" is $\Pi_1(M)$ in the same parameter (uniformly in the Σ_1 definition of A).*

"p proves φ from u" is uniformly $\Sigma_i(M)$. Hence:

Lemma 1.4.5. *$\{\langle u, \varphi\rangle | u \in M \wedge u \vdash \varphi\}$ is uniformly $\Sigma_1(M)$.*

Corollary 1.4.6. *$\{\langle u \in M | u$ is consistent$\}$ is uniformly $\Pi_1(M)$.*

Note. Call a proof p *strict* iff $\overline{\overline{p}}_i = 1$ for $i \in \mathrm{dom}(p)$. This corresponds to the more usual notion of proof. If M satisfies the axiom of choice in the form: Every set is enumerable by an ordinal, then Lemma 1.4.1 holds with "strict proof" in place of "proof". We leave this to the reader.

1.4.2 Models

We will not normally employ all of the predicates and constants in our M–finitary first order logic, but cut down to a smaller set of symbols which we intend to interpret in a model. Thus we define a *language* to be a set $\mathbb{L}$ of predicates and constants. By a *model* of $\mathbb{L}$ we mean a structure:

$$\mathbb{A} = \langle |\mathbb{A}|, \langle t^{\mathbb{A}} | t \in \mathbb{L}\rangle\rangle$$

such that $|\mathbb{A}| \neq \emptyset$, $P^{\mathbb{A}} \subset |\mathbb{A}|^n$ whenever P is an n–ary predicate, and $c^{\mathbb{A}} \in |\mathbb{A}|$ whenever c is a constant. By a *variable assignment* we mean a partial map of f of the variables into $\mathbb{A}$. The *satisfaction relation* $\mathbb{A} \models \varphi[f]$ is defined in the usual way, where $\mathbb{A} \models [f]$ means that the formula φ becomes true in $\mathbb{A}$ if the free variables of φ are interpreted by the assignment f. We leave the definition to the reader, remarking only that:

$$\mathbb{A} \models \bigwedge\!\!\!\bigwedge_{i \in u} \varphi_i[f] \leftrightarrow \bigwedge i \in u \ \mathbb{A} \models \varphi_i[f]$$
$$\mathbb{A} \models \bigvee\!\!\!\bigvee_{i \in u} \varphi_i[f] \leftrightarrow \bigvee i \in u \ \mathbb{A} \models \varphi_i[f]$$

We adopt the usual conventions of model theory, writing $\mathbb{A} = \langle |\mathbb{A}|, t_1^{\mathbb{A}}, \ldots\rangle$ if we think of the predicates and constants of $\mathbb{L}$ as being arranged in a fixed sequence $t_1, t_2, \ldots$. Similarly, if $\varphi = \varphi(v_1, \ldots, v_n)$ is a formula in which at most the variables $v_1, \ldots, v_n$ occur free, we write $\mathbb{A} \models \varphi[a_1, \ldots, a_n]$ for:

$$\mathbb{A} \models \varphi[f] \text{ where } f(v_i) = a_i \text{ for } i = 1, \ldots, n.$$

If φ is a sentence we write: $\mathbb{A} \models \varphi$. If A is a set of sentences, we write $\mathbb{A} \models A$ to mean: $\mathbb{A} \models \varphi$ for all $\varphi \in A$.

Proof: The *correctness theorem* says that if A is a set of $\mathbb{L}$ sentences and $\mathbb{A} \models A$, then A is consistent. (We leave this to the reader.)
Barwise's Completeness Theorem says that the converse holds whenever our admissible structure is countable:

Theorem 1.4.7. *Let M be a countable admissible structure. Let $\mathbb{L}$ be an M–language and let A be a set of statements in $\mathbb{L}$. If A is consistent in M–finite predicate logic, then $\mathbb{L}$ has a model $\mathbb{A}$ such that $\mathbb{A} \models A$.*

Proof: (Sketch)
We make use of the following theorem of Rasiowa and Sikorski: Let $\mathbb{B}$ be a Boolean algebra. Let $X_i \subset \mathbb{B}(i < \omega)$ be such that the Boolean union $\bigcup X_i = b_i$ exists in the sense of $\mathbb{B}$. Then $\mathbb{B}$ has an ultrafilter U such that

$$b_i \in U \leftrightarrow X_i \cap U \neq \emptyset \text{ for } i < \omega.$$

(Proof. Successively choose $c_i(i < \omega)$ by: $c_0 = 1$, $c_{i+1} = c_i \cap b \neq 0$, where $b \in X_i \cup \{\neg b_i\}$. Let $\overline{U} = \{a \in \mathbb{B} | \bigvee i(c_i \subset a)\}$. Then $\overline{U}$ is a filter and extends to an ultrafilter on $\mathbb{B}$.)

Extend the language $\mathbb{L}$ by adding an M–infinite set C of new constants. Call the extended language $\mathbb{L}^*$. Set:

$$[\varphi] =: \{\psi | A \vdash (\psi \leftrightarrow \varphi)\}$$

for $\mathbb{L}^*$–sentences φ. Then

$$\mathbb{B} =: \{[\varphi] | \varphi \in Sent_{\mathbb{L}^*}\}$$

is the Lindenbaum algebra of $\mathbb{L}^*$ with the defining equations:

$$[\varphi] \cup [\psi] = [\varphi \vee \psi], [\varphi] \cap [\psi] = [\varphi \wedge \psi], \neg[\varphi] = [\neg\varphi]$$
$$\bigcup_{i \in U} [\varphi_i] = [\bigwedge\!\!\!\bigwedge_{i \in U} \varphi_i](i \in u), \bigcap_{i \in U} [\varphi_i] = [\bigwedge\!\!\!\bigwedge_{i \in U} \varphi_i](i \in u)$$
$$\bigcup_{c \in C} [\varphi(c)] = [\bigvee v\varphi(v)], \bigcap_{c \in C} [\varphi(c)] = [\bigwedge v\varphi(v)].$$

The last two equations hold because the constants in C, which do not occur in the axiom A, behave like free variables. By Rasiowa and Sikorski there is then an ultrafilter U on $\mathbb{B}$ which respects the above operations. We define a model $\mathbb{A} = \langle |\mathbb{A}|, \langle t^{\mathbb{A}} | t \in \mathbb{L} \rangle \rangle$ as follows: For $c \in C$ set $[c] =: \{c' \in C | [c = c'] \in U\}$. If $P \in \mathbb{L}$ is an n–place predicate, set:

$$P^{\mathbb{A}}([c_1], \ldots, [c_n]) \leftrightarrow: [Pc_1, \ldots, c_n] \in U.$$

If $t \in \mathbb{L}$ is a constant, set:

$$t^{\mathbb{A}} = [c] \text{ where } c \in C, [t = c] \in U.$$

A straightforward induction then shows:

$$\mathbb{A} \models \varphi[[c_1], \ldots, [c_n] \leftrightarrow [\varphi(c_1, \ldots, c_n)] \in U$$

for formulae $\varphi = \varphi(v_1, \ldots, v_n)$ with at most the free variables $v_1, \ldots, v_n$. In particular, $\mathbb{A} \models \varphi \leftrightarrow [\varphi] \in U$ for $\mathbb{L}^*$–statements φ. Hence $\mathbb{A} \models A$.

QED (Theorem 1.4.7)

Combining the completeness theorem with the M–finiteness lemma, we get the well known *Barwise compactness theorem*:

Corollary 1.4.8. *Let M be countable. Let $\mathbb{L}$ be a language. Let A be a $\underline{\Sigma}_1(M)$ set of sentences in $\mathbb{L}$. If every M–finite subset of $\mathbb{A}$ has a model, then so does A.*

1.4.3 Applications

Definition 1.4.3. By a *theory* or *axiomatized language* we mean a pair $\mathbb{L} = \langle \mathbb{L}_0, A\rangle$ such that $\mathbb{L}_0$ is a language and A is a set of $\mathbb{L}_0$–sentences. We say that $\mathbb{A}$ *models* $\mathbb{L}$ iff $\mathbb{A}$ is a model of $\mathbb{L}_0$ and $\mathbb{A} \models A$. We also write $\mathbb{L} \vdash \varphi$ for: ($\varphi \in Fml_{\mathbb{L}_0}$ and $A \vdash \varphi$). We say that $\mathbb{L} = \langle \mathbb{L}_0, A\rangle$ is $\Sigma_1(M)$ (in p) iff $\mathbb{L}_0$ is $\Delta_1(M)$ (in p) and A is $\Sigma_1(M)$ (in p). Similarly for: $\mathbb{L}$ is $\Delta(M)$ (in p).

We now consider the class of axiomatized languages containing a fixed predicate $\dot\in$, the special constants $\underline{x}(x \in M)$ (we can set e.g. $\underline{x} = \langle 1, \langle 0, x\rangle\rangle$), and the *basic axioms*:

- Extensionality
- $\bigwedge v(v \dot\in \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z \in x} v \dot= \underline{z})$ for $x \in M$.

(Further predicates, constants, and axioms are allowed of course.) We call any such theory an "$\in$–theory". Then:

Lemma 1.4.9. *Let $\mathbb{A}$ be a grounded model of an $\in$–theory $\mathbb{L}$. Then $\underline{x}^{\mathbb{A}} = x \in \mathrm{wfc}(\mathbb{A})$ for $x \in M$.*

In an $\in$–theory $\mathbb{L}$ we often adopt the set of axioms ZFC^- (or more precisely $\mathsf{ZFC}^-_{\mathbb{L}}$). This is the collection of all $\mathbb{L}$–sentences φ such that φ is the universal quantifier closure of an instance of the ZFC^- axiom schemata — but does *not* contain infinite conjunctions or disjunctions. (Hence the collection of all subformulae is finite.) (Similarly for ZF^-, ZFC, ZF.)

(**Note** If we omit the sentences containing constants, we get a subset $B \subset \mathsf{ZFC}^-$ which is equivalent to ZFC^- in $\mathbb{L}$. Since each element of B contain at most finitely many variables, we can restrict further to the subset B' of sentences containing only the variables $v_i (i < \omega)$. If $\omega \in M$ and the set of predicates in $\mathbb{L}$ is M–finite, then B' will be M–finite. Hence ZFC^- is equivalent in $\mathbb{L}$ to the statement $\bigwedge\!\!\!\bigwedge B'$.)

We now bring some typical applications of $\in$–theories. We say that an ordinal α is *admissible in* $a \subset \alpha$ iff $\langle L_\alpha[a], \in, a\rangle$ is admissible.

Lemma 1.4.10. *Let $\alpha > \omega$ be a countable admissible ordinal. Then there is $a \subset \omega$ such that α is the least ordinal admissible in a.*

This follows straightforwardly from:

Lemma 1.4.11. *Let M be a countable admissible structure. Let $\mathbb{L}$ be a consistent $\underline{\Sigma}_1(M)$ $\in$–theory such that $\mathbb{L} \vdash ZF^-$. Then $\mathbb{L}$ has a grounded model $\mathbb{A}$ such that $\mathbb{A} \neq \mathrm{wfc}(\mathbb{A})$ and $\mathrm{On} \cap \mathrm{wfc}(\mathbb{A}) = \mathrm{On} \cap M$.*

We first show that lemma 1.4.11 implies lemma 1.4.10. Take $M = L_\alpha$. Let $\mathbb{L}$ be the M–theory with:

Predicate: $\dot{\in}$

Constants: $\underline{x}(x \in M), \dot{a}$

Axioms: Basic axioms $+\mathsf{ZFC}^- + \underline{\beta}$ is not admissible in $\dot{a}(\beta \in M)$

Then $\mathbb{L}$ is consistent, since $\langle H_{\omega_1}, \in, a\rangle$ is a model, where a is any $a \subset \omega$ which codes a well ordering of type $\geq \alpha$. Let $\mathbb{L}$ be a grounded model of $\mathbb{L}$ such that $\mathrm{wfc}(\mathbb{A}) \neq \mathbb{A}$ and $\mathrm{On} \cap \mathrm{wfc}(\mathbb{A}) = \alpha$. Then $\mathrm{wfc}(\mathbb{A})$ is admissible by §3. Hence so is $L_\alpha[a]$ where $a = \dot{a}^{\mathbb{A}}$. QED

Note This is a very typical application in that Barwise theory hands us an ill founded model, but our interest is entirely concentrated on its well founded part.

Note Pursuing this method a bit further we can use lemma 1.4.11 to prove: Let $\omega < \alpha_0 < \ldots < \alpha_{n-1}$ be a sequence of countable admissible ordinals. There is $a \subset \omega$ such that α_i = the i–th $\alpha < \omega$ which is admissible in $a(1 = 0, \ldots, n-1)$.

We now prove lemma 1.4.11 by modifying the proof of the completeness theorem. Let $\Gamma(v)$ be the set of formulae: $v \in \mathrm{On}$, $v > \underline{\beta}(\beta \in \mathrm{On} \wedge M)$. Add an M–infinite (but $\underline{\Delta}_1(M)$) set E of new constants to $\mathbb{L}$. Let $\mathbb{L}'$ be $\mathbb{L}$ with the new constants and new axioms: $\Gamma(e)$ $(e \in E)$. Then $\mathbb{L}'$ is consistent, since any M–finite subset of the axioms can be modeled in an arbitrary grounded model $\mathbb{A}$ of $\mathbb{L}$ by interpreting the new constants as sufficiently large elements of α. As in the proof of completeness we then add a new class C of constants which is not M–finite. We assume, however, that C is $\Delta_1(M)$. We add no further axioms, so the elements of C behave like free variables. The so–extended language $\mathbb{L}''$ is clearly $\underline{\Sigma}_1(M)$.

Now set:

$$\Delta(v) =: \{v \notin \mathrm{On}\} \cup \bigcup_{\beta \in M} \{v \leq \underline{\beta}\} \cup \bigcup_{e \in E} \{e < v\}.$$

Claim Let $c \in C$. Then $\bigcup\{[\varphi] | \varphi \in \Delta(c)\} = 1$ in the Lindenbaum algebra of $\mathbb{L}''$.

Proof: Suppose not. Then there is ψ such that $A \vdash \varphi \to \psi$ for all $\varphi \in \Delta(c)$ and $A \cup \{\neg\psi\}$ is consistent, where $\mathbb{L}'' = \langle \mathbb{L}''_0, A\rangle$. Pick an $e \in E$ which does not occur in ψ. Let A^* be the result of omitting the axioms $\Gamma(e)$ from A. Then $A^* \cup \{\neg\psi\} \cup \Gamma(e) \vdash c \leq e$. By the finiteness lemma there is $\beta \in M$ such that $A^* \cup \{\neg\psi\} \cup \{\underline{\beta} \leq e\} \vdash c \leq e$. But e behaves here like a free variable, so $A^* \cup \{\neg\psi\} \vdash c \leq \underline{\beta}$. But $A \supset A^*$ and $A \cup \{\neg\psi\} \vdash \underline{\beta} < c$. Hence $A \cup \{\neg\psi\} \vdash \underline{\beta} < \underline{\beta}$ and $A \cup \{\neg\psi\}$ is inconsistent.
Contradiction! QED (Claim)

Now let U be an ultrafilter on the Lindenbaum algebra of $\mathbb{L}''$ which respects both two operations listed in the proof of the completeness theorem and the unions $\bigcup\{[\varphi] | \varphi \in \Delta(c)\}$ for $c \in C$. Let $X = \{\varphi | [\varphi] \in U\}$. Then as before, $\mathbb{L}''$ has a grounded model $\mathbb{A}$, all of whose elementes have the form $c^{\mathbb{A}}$ for $c \in C$ and such that:

$$\mathbb{A} \models \varphi \text{ iff } \varphi \in X$$

for $\mathbb{L}''$–statements φ. But then for each $x \in A$ we have either $x \notin \mathrm{On}_{\mathbb{A}}$ or $x < \beta$ for a $\beta \in \mathrm{On} \cap M$ or $e^{\mathbb{A}} < v$ for all $e \in E$. In particular, if $x \in \mathrm{On}_{\mathbb{A}}$ and $x > \beta$ for all $\beta \in \mathrm{On} \cap M$, then there is $e^{\mathbb{A}} < x$ in $\mathbb{A}$. But $\beta < e^{\mathbb{A}}$ for all $\beta \in \mathrm{On} \cap M$. Hence $\mathrm{On}_{\mathbb{A}} \setminus \mathrm{On}_M$ has no minimal element in $\mathbb{A}$.
QED (Lemma 1.4.11)

Another typical application is:

Lemma 1.4.12. *Let W be an inner model of* ZFC. *Suppose that, in W, U is a normal measure on κ. Let $\tau > \kappa$ be regular in W. Set: $M = \langle H^W_\tau, U\rangle$. Assume that M is countable in V. Then for any $\alpha \leq \kappa$ there is $\overline{M} = \langle \overline{H}, \overline{U}\rangle$ such that*

- $\overline{M} \models \overline{U}$ *is a normal measure on $\overline{\kappa}$ for a $\overline{\kappa} \in \overline{M}$*
- $\overline{M}$ *iterates to M in α many steps.*

(Hence $\overline{M}$ is iterable, since M is.)

Proof: The case $\alpha = 0$ is trivial, so assume $\alpha > 0$. Let δ be least such that $L_\delta(M)$ is admissible. Let $\mathbb{L}$ be the $\in$–theory on $L_\delta(M)$ with:

Predicate: $\dot{\in}$

Constants: $\underline{x}(x \in L_\delta(M)), \dot{M}$

Axiom:
- Basic axioms $+\mathsf{ZFC}^-$
- $\dot{M} = \langle \dot{H}, \dot{U} \rangle \models (\mathsf{ZFC}^- + \dot{U}$ is a normal measure on a $\kappa < \dot{H})$
- $\dot{M}$ iterates to $\underline{M}$ in $\underline{\alpha}$ many steps.

It will suffice to show:

Claim $\mathbb{L}$ is consistent.

We first show that the claim implies the theorem. Let $\mathbb{A}$ be a grounded model of $\mathbb{L}$. Then $\mathbb{L}_\delta(M) \subset \mathrm{wfc}(\mathbb{A})$. Hence $M, \overline{M} \in \mathrm{wfc}(\mathbb{A})$, where $\overline{M} = \dot{M}^{\mathbb{A}}$. But then in $\mathbb{A}$ there is an iteration $\langle \overline{M}_i | i \leq \alpha \rangle$ of $\overline{M}$ to M. By absoluteness $\langle \overline{M}_i | i \leq \alpha \rangle$ really is such an iteration. QED

We now prove the claim.

Case 1 $\alpha < \kappa$
Iterate $\langle W, U \rangle$ α many times, getting $\langle W_i, U_i \rangle (i \leq \alpha)$ with iteraton maps $\pi_{i,j}$. Then $\pi_{0,\alpha}(\alpha) = \alpha$. Set $M_i = \pi_{0,i}(M)$. Then $\langle M_i | i \leq \alpha \rangle$ is an iteration of M with iteration maps $\pi_{i,j} \upharpoonright M_i$. But $M_\alpha = \pi_{0,\alpha}(M)$. Hence $\langle H_{\kappa^+}, M \rangle$ models $\pi_{0,\alpha}(\mathbb{L})$. But then $\pi_{0,\alpha}(\mathbb{L})$ is consistent. Hence so is $\mathbb{L}$. QED

Case 2 $\alpha = \kappa$
Iterate $\langle W, U \rangle$ β many times, where $\pi_{0,\beta}(\kappa) = \beta$. Then $\langle M_i | i \leq \beta \rangle$ iterates M to M_β in β many steps. Hence $\langle H_{\kappa^+}, M \rangle$ models $\pi_{0,\beta}(\mathbb{L})$. Hence $\pi_{0,\beta}(\mathbb{L})$ is consistent and so is $\mathbb{L}$. QED (Lemma 1.4.12)

Barwise theory is useful in situations where one is given a transitive structure Q and wishes to find a transitive structure $\overline{Q}$ with similar properties inside an inner model. Another tool, which is often used in such situations, is Schoenfield's lemma, which, however, requires coding Q by a real. Unsurprizingly, Schoenfield's lemma can itself be derived from Barwise theory. We first note the well known fact that every Σ^1_2 condition on a real is equivalent to a $\Sigma_1(H_{\omega_1})$ condition, and conversely. Thus it suffices to show:

Lemma 1.4.13. *Let $H_{\omega_1} \models \varphi[a], a \subset \omega$, where φ is Σ_1. Then:*

$$H_{\omega_1} \models \varphi[a] \text{ in } L(a).$$

Proof: Let $\varphi = \bigvee z\psi$, where ψ is Σ_0. Let $H_{\omega_1} \models \psi[z,a]$ where $\mathrm{rn}(z) = \delta < \alpha < \omega_1$ and α is admissible in a. Let $\mathbb{L}$ be the language on $L_\alpha(a)$ with:

Predicate: $\dot\in$

Constants: $\underline{x}(x \in L_\alpha(a))$

Axioms: Basic axioms $+\mathsf{ZFC}^- + \bigvee z(\psi(z,\underline{a}) \wedge \mathrm{rn}(z) = \underline{\delta})$.

Then $\mathbb{L}$ is consistent, since $\langle H_{\omega_1}, a\rangle$ is a model. We cannot necessarily chose α such that it is countable in $L(a)$, however. Hence, working in $L(a)$, we apply a Skolem–Löwenheim argument to $L_\alpha(a)$, getting countable $\overline{\alpha}, \overline{\delta}, \pi$ such that $\pi : L_{\overline{\alpha}}(a) \prec L_\alpha(a)$ and $\pi(\overline{\delta}) = \delta$. Let $\overline{\mathbb{L}}$ be defined from $\overline{\delta}$ over $L_{\overline{\alpha}}(a)$ as $\mathbb{L}$ was defined from δ over $L_\alpha(a)$. Then $\overline{\mathbb{L}}$ is consistent by corollary 1.4.4. Since $L_{\overline{\alpha}}(a)$ is countable in $L(a)$, $\overline{\mathbb{L}}$ has a grounded model $\mathbb{A} \in L(a)$. But then there is $z \in \mathbb{A}$ such that $\mathbb{A} \models \psi[z,a]$ and $rn^{\mathbb{A}}(z) = \overline{\delta}$. Thus $rn(z) = \overline{\beta} \in \mathrm{wfc}(\mathbb{A})$ and $z \in \mathrm{wfc}(\mathbb{A})$. Thus $\mathrm{wfc}(\mathbb{A}) \models \psi[z,a]$, where $\mathrm{wfc}(\mathbb{A}) \subset H_{\omega_1}$ in $L(a)$. Hence $H_{\omega_1} \models \varphi[a]$ in $L(a)$. QED