etc. One solution is to employ the theory of *rudimentary functions* in an auxiliary role. These functions, which were discovered by Gandy and Jensen, are exactly the functions which are generated by the schemata for primitive recursive functions when the recursion schema is omitted. (Cf. the remark following chapter 1, §2, Lemma 1.1.4). If $\mathrm{rn}(x_i) < \gamma$ for $i = 1, \ldots, n$ and f is rudimentary, then $\mathrm{rn}(f(x_1, \ldots, x_n)) < \gamma + \omega$. All reasonable "elementary" set theoretic functions are rudimentary. If α is a limit ordinal, then L_α is closed under rudimentary functions. If α is a successor, then closing L_α under rudimentary functions yields a transitive structure L^*_α of rank $\alpha+\omega$. It then turns out that every $\underline{\Sigma}_\omega(L^*_\alpha)$ definable subset of L_α is already $\underline{\Sigma}_\omega(L_\alpha)$, and conversely. Hence we can, in effect, replace the rather weak definability theory of L_α by the rather nice definability theory of L^*_α. (This method was used in [JH], except that L^*_α was given a different but equivalent definition, since the rudimentary functions were not yet known.) It turns out that if N is transitive and rudimentarily closed, and $\mathrm{Rud}(N)$ is defined to be the closure of $N \cup \{N\}$ under rudimentary functions, then $\mathbb{P}(N) \cap \mathrm{Rud}(N) = \mathrm{Def}(N)$. This suggests an alternative version of the constructible hierarchy in which every level is rudimentarily closed. We shall index this hierarchy by the class Lm of limit ordinals, setting:

$$J_\omega = H_\omega = \mathrm{Rud}(\emptyset)$$

$$J_{\alpha+\omega} = \mathrm{Rud}(J_\alpha) \text{ for } \alpha \in \mathrm{Lm}$$

$$J_\lambda = \bigcup_{\nu<\lambda} J_\nu \text{ for } \lambda \text{ a limit p.t. of Lm.}$$

Note. Setting $J = \bigcup\limits_{\alpha} J_\alpha$, we have: $J = L$ in fact $J_\alpha = L_\alpha$ whenever α is pr closed.

Note. This indexing was introduced by Sy Friedman. In [FSC] we indexed by *all* ordinals, so that our $J_{\omega\alpha}$ corresponds to the J_α of [FSC]. The usage in [FSC] has been followed by most authors. Nonetheless we here adopt Friedman's usage, which seems to us more natural, since we then have: $\alpha = \mathrm{rn}(J_\alpha) = \mathrm{On} \cap J_\alpha$.

In the following section we develop the theory of rudimentary functions.

2.2 Rudimentary Functions

Definition 2.2.1. $f : V^n \to V$ is a *rudimentary* (rud) *function* iff it is generated by successive applications of schemata (i) – (v) in the definition of *primitive recursive* in chapter 1, §2.

A relation $R \subset V^n$ is rud iff there is a rud function f such that: $R\vec{x} \leftrightarrow f(\vec{x}) = 1$. In chapter 1, §1.2 we established that:

Lemma 2.2.1. *Lemmas 1.2.1 – 1.2.4 of chapter 1, §1.2 hold with 'rud' in place of 'pr'.*

Note. Our definition of 'rud function', like the definition of 'pr function' is ostensibly in second order set theory, but just as in chapter 1, §1.2 we can work in ZFC by talking about rud *definitions*. The notion of rud definition is defined like that of pr definition, except that instances of schema (vi) are not allowed. As before, we can assign to each rud definition s a rud function $F_s : V^n \to V$ with the property that $F_s^M = F_s \upharpoonright M$ whenever M is admissible and $F_s^M : M^n \to M$ is the function on M defined by s. But then if M is transitive and closed under rud functions, it follows by induction on the length of s that there is a unique $F_s^M = F_s \upharpoonright M$.

A rudimentary function can raise the rank of its arguments by at most a finite amount:

Lemma 2.2.2. *Let $f : V^n \to V$ be* rud. *Then there is $p < \omega$ such that*

$$f(\vec{x}) \subset \mathbb{P}^p(TC(x_1 \cup \ldots \cup x_n)) \text{ for all } x_1, \ldots, x_n.$$

(Hence $\mathrm{rn}(f\vec{x}) \leq \max\{\mathrm{rn}(x_1), \ldots, \mathrm{rn}(x_n)\} + p$ *and* $\bigcup^p f(\vec{x}) \subset TC(x_1 \cup \ldots \cup x_n)$*.)*

Proof: Call any such p *sufficient* for f. Then if p is sufficient, so is every $q \geq p$. By induction on the defining schemata for f, we prove that f has a sufficient p. If f is given by an initial schema, this is trivial. Now let $f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_m(\vec{x}))$. Let p be sufficient for h and q be sufficient for $g_i (i = 1, \ldots, m)$. It follows easily that $p + q$ is sufficient for f. Now let $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$, where p is sufficient for g. It follows easily that p is sufficient for f. QED

By lemma 2.2.1 and chapter 1 lemma 1.2.3 (i) we know that every Σ_0 relation is rud. We now prove the converse. In fact we shall prove a stronger result. We first define:

Definition 2.2.2. $f : V^n \to V$ is *simple* iff whenever $R(z, \vec{y})$ is a Σ_0 relation, then so is $R(f(\vec{x}), \vec{y})$.

The simple functions are obviously closed under composition. The simplicity of a function f is equivalent to the conjunction of the two conditions:

(i) $x \in f(\vec{y})$ is Σ_0

(ii) If $A(z, \vec{u})$ is Σ_0, then $\bigwedge z \in f(\vec{x}) A(z, \vec{u})$ is Σ_0,

for given these we can verify by induction on the Σ_0 definition of R that $R(f(\vec{x}), \vec{y})$ is Σ_0.
But then:

Lemma 2.2.3. *All* rud *functions are simple.*

Proof: Using the above facts we verify by induction on the defining schemata of f that f is simple. The proof is left to the reader. QED

In particular:

Corollary 2.2.4. *Every* rud *function f is Σ_0 as a relation. Moreover $f \restriction U$ is uniformly $\Sigma_0(U)$ whenever U is transitive and* rud *closed.*

Corollary 2.2.5. *Every* rud *relation is Σ_0.*

In chapter 1, §2 we relativized the concept 'pr' to 'pr in $A_1, \ldots, A_n$'. We can do the same thing with 'rud'.

Definition 2.2.3. Let $A_i \subset V (i = 1, \ldots, m)$. $f : V^n \to V$ is *rudimentary in* $A_1, \ldots, A_n$ (rud *in* $A_1, \ldots, A_n$) iff it is obtained by successive applications of the schemata (i) – (v) and:

$$f(x) = \chi_A(x) \ (i = 1, \ldots, n)$$

where χ_A is the characteristic function of A.

Lemma 1.1.1 and 1.1.2 obviously hold with 'rud in $A_1, \ldots, A_n$' in place of 'rud'. Lemma 2.2.3 and its corollaries do *not* hold, however, since e.g. the relation $\{x\} \in A$ is not Σ_0 in A.

However, we do get:

Lemma 2.2.6. *If f is* rud *in $A_1, \ldots, A_n$, then*

$$f(\vec{x}) = f_0(\vec{x}, A_1 \cap f_1(\vec{x}), \ldots, A_n \cap f_n(\vec{x}))$$

where $f_0, f_1, \ldots, f_n$ are rud *functions.*

Proof: We display the proof for the case $n = 1$. Let f be rud in A. By induction on the defining schemata for f we show:

$$f(\vec{x}) = f_0(\vec{x}, A \cap f_1(\vec{x})) \text{ where } f_0, f_1 \text{ are rud}.$$

Case 1 f is given by schemata (i) – (iii). This is trivial.

Case 2 $f(x) = \chi_A(x)$. Then

$$f(x) = \left\{ \begin{array}{l} 1 \text{ if } A \cap \{x\} \neq \emptyset \\ 0 \text{ if not} \end{array} \right\} = f'(x, A \cap \{x\})$$

where f' is rud. QED (Case 2)

Case 3 $f(\vec{x}) = g(h^1(\vec{x}), \ldots, h^m(\vec{x}))$. Let

$$g(\vec{z}) = g_0(\vec{z}, A \cap g_1(\vec{z}))$$
$$h^i(\vec{x}) = h_0^i(\vec{x}, A \cap h_1^i(\vec{x}))(i = 1, \ldots, m)$$

where g_0, g_1, h_0^i, h_1^i are rud. Set:

$$\tilde{g}(\vec{z}, u) = g_0(\vec{z}, u \cap g_1(\vec{z}))$$
$$\tilde{h}^i(\vec{x}, u) = h_0^i(\vec{x}, u \cap h_1^i(\vec{x}))$$
$$\tilde{f}(\vec{x}, u) = \tilde{g}(\tilde{h}^1(\vec{x}, u), \ldots, \tilde{h}^m(\vec{x}, u), u)$$
$$k(\vec{x}) = g_1(\vec{h}_1(\vec{x})) \cup \bigcup_{i=1}^{m} h_1^i(\vec{x}).$$

Then $f(\vec{x}) = \tilde{f}(\vec{x}, A \cap k(\vec{x}))$, where $\tilde{f}, k$ are rud. This follows from the facts:

$$\tilde{h}^i(\vec{x}, A \cap v) = h_0^i(\vec{x}, A \cap h_1^i(\vec{x})) = h^i(\vec{x}) \text{ if } h_1^i(\vec{x}) \subset v$$
$$\tilde{g}^i(\vec{z}, A \cap v) = g_0(\vec{z}, A \cap z) \text{ if } g_1(\vec{z}) \subset v.$$

QED (Case 3)

Case 4 $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$. Let $g(z, \vec{x}) = g_0(z, \vec{x}, A \cap g_1(z, \vec{x}))$. Set

$$\tilde{g}(z, \vec{x}, u) = g_0(z, \vec{x}, u \cap g_1(z, \vec{x}))$$
$$\tilde{f}(y, \vec{x}, u) = \bigcup_{z \in y} \tilde{g}(z, \vec{x}, u)$$
$$k(y, \vec{x}) = \bigcup_{z \in y} g_1(z, \vec{x})$$

Then $f(y, \vec{x}) = \tilde{f}(y, \vec{x}, A \cap k(y, \vec{z}))$ where $\tilde{f}, k$ are rud.

QED (Lemma 2.2.6)

Definition 2.2.4. X is *rudimentarily closed* (rud closed) iff it is closed under rudimentary functions. $\langle M, A_1, \ldots, A_n \rangle$ is rud closed iff M is closed under functions rudimentary in $A_1, \ldots, A_n$.

If $M = \langle |M|, A_1, \ldots, A_n \rangle$ is transitive and rud closed, then it is amenable, since it is closed under $f(x) = x \cap A$. By lemma 2.2.6 we then have:

Corollary 2.2.7. *Let $M = \langle |M| A_1, \ldots, A_n \rangle$ be transitive. M is rud closed iff it is amenable and $|M|$ is rud closed.*

Corresponding to corollary 2.2.4 we have:

Corollary 2.2.8. *Every function f which is* rud *in A is Σ_1 in A as a relation. Moreover $f \upharpoonright U$ is $\Sigma_1(\langle U, A \cap U \rangle)$ by the same Σ_1 definition whenever $\langle U, A \cap U \rangle$ is transitive and rud closed. (Similarly for "*rud *in $A_1, \ldots, A_n$".)*

Proof: Let $f(\vec{x}) = f_0(\vec{x}, A \cap f_1(\vec{x}))$ where f_0, f_1 are rud. Then:

$$y = f(\vec{x}) \leftrightarrow \bigvee u \bigvee z (y = f_0(\vec{x}, z) \wedge u = f_1(\vec{x}) \wedge z = A \cap u).$$

QED (Corollary 2.2.8)

In chapter 1 §2.2 we extended the notion of "pr definition" so as to deal with functions pr in classes $A_1, \ldots, A_n$. We can do the same for rudimentary functions:

We appoint new designated function variables $\dot{a}_1, \ldots, \dot{a}_n$ and define the set of rud *definition in* $a_1, \ldots, a_n$ exactly as before, except that we omit the schema (vi). Given $A_1, \ldots, A_n$ we can, exactly as before, assign to each rud definition s in $\dot{a}_1, \ldots, \dot{a}_n$ a function $F_s^{A_1, \ldots, A_n}$ are then exatly the functions rud in $A_1, \ldots, A_n$. Since lemma 2.2.6 (and with it corollary 2.2.8) is proven by induction on the defining schemata, its proof implicitly defines an algorithm which assigns to each s as Σ_1 formula φ_s which defines $F_s^{\vec{A}}$.

Corresponding to chapter 1 §1 Lemma 1.1.13 we have:

Lemma 2.2.9. *Let f be* rud *in $A_1, \ldots, A_n$, where each A_i is* rud *in $B_1, \ldots, B_m$. Then f is* rud *in $B_1, \ldots, B_m$.*

The proof is again by induction on the defining schemata. It shows, in fact that f is *uniformly* rud in $\vec{B}$ in the sense that its rud definition from $\vec{B}$ depends only on its rud definition from $\vec{A}$ and the rud definition of A_i from $\vec{B}$ $(i = 1, \ldots, n)$.

We also note:

Lemma 2.2.10. *Let $\pi : \overline{M} \to_{\Sigma_0} M$, where $\overline{M}, M$ are* rud *closed. Then π preserves rudimentarily in the following sense: Let $\overline{f}$ be defined from the predicates of $\overline{M}$ by the* rud *definition s. Let f be defined from the predicates of M by s. Then $\pi(\overline{f}(\vec{x})) = f(\pi(\vec{x}))$ for $x_1, \ldots, x_n \in \overline{M}$.*

Proof: Let φ_s be the canonical Σ_1 definition. Then $\overline{M} \models \varphi_s[y, \vec{x}] \rightarrow M \models \varphi_s[\pi(y), \pi(\vec{x})]$ by Σ_0–preservation. QED (Lemma 2.2.10)

We now define:

Definition 2.2.5.
$\text{rud}(U) =:$ The closure of U under rud functions
$\text{rud}_{A_1,\dots,A_n}(U) =:$ The closure of U under functions rud in $A_1, \dots, A_n$

(Hence $\text{rud}(U) = \text{rud}_{\emptyset}(U)$.)

Lemma 2.2.11. *If U is transitive, then so is* $\text{rud}(U)$.

Proof: Let $W = \text{rud}(U)$. Let $Q(x)$ mean: $TC(\{x\}) \subset W$. By induction on the defining schemata of f we show:

$$(Q(x_1) \wedge \dots \wedge Q(x_n)) \rightarrow Q(f(x_1, \dots, x_n))$$

for $x_1, \dots, x_n \in W$. The details are left to the reader. But $x \in U \rightarrow Q(x)$ and each $z \in W$ has the form $f(\vec{x})$ where f is rud and $x_1, \dots, x_n \in U$. Hence $TC(\{z\}) \subset W$ for $z \in W$. QED

The same proof shows:

Corollary 2.2.12. *If U is transitive, then so is* $\text{rud}_{\vec{A}}(U)$.

Using Corollary 2.2.12 and Lemma 2.2.3 we get:

Lemma 2.2.13. *Let U be transitive and $W = \text{rud}(U)$. Then the restriction of any $\underline{\Sigma}_0(W)$ relation to U is $\underline{\Sigma}_0(U)$.*

Proof: Let R be $\underline{\Sigma}_0(W)$. Let $R(\vec{x}) \leftrightarrow R'(\vec{x}, \vec{p})$ where R' is $\Sigma_0(W)$ and $p_1, \dots, p_n \in W$. Let $p_i = f_i(\vec{z})$, where f_i is rud and $z_1, \dots, z_n \in U$. Then for $x_1, \dots, x_m \in U$:

$$\begin{aligned} R(\vec{x}) &\leftrightarrow R'(\vec{x}, \vec{f}(\vec{z})) \\ &\leftrightarrow R''(\vec{x}, \vec{z}) \end{aligned}$$

where R'' is $\Sigma_0(U)$, by lemma 2.2.3. QED (Lemma 2.2.13)

We now define:

Definition 2.2.6. Let U be transitive.

$$\begin{aligned} \text{Rud}(U) &=: \text{rud}(U \cup \{U\}) \\ \text{Rud}_{\vec{A}}(U) &=: \text{rud}_{\vec{A}}(U \cup \{U\}) \end{aligned}$$

Then $\text{Rud}(U)$ is a proper transitive extension of U. By Lemma 2.2.13:

Corollary 2.2.14. Def$(U) = \mathbb{P}(U) \cap$ Rud(U) *if* $U \neq \emptyset$ *is transitive.*

Proof: If $A \in$ Def(U), then A is $\underline{\Sigma}_0(U \cup \{U\})$. Hence $A \in$ Rud(U). Conversely, if $A \in$ Rud(U), then A is $\underline{\Sigma}_0(U \cup \{U\})$ by lemma 1.1.7. It follows easily that $A \in$ Def(U). QED (Corollary 2.2.14)

Note. To see that $A \in$ Def(U), consider the $\in$–language augmented by a new constant $\dot{U}$ which is interpreted by U. We assign to every Σ_0 formula φ in this language a first order formula φ' not containing $\dot{U}$ such that for all $x_1, \ldots, x_n \in U$:

$$U \cup \{U\} \models \varphi[\vec{x}] \leftrightarrow U \models \varphi'[\vec{x}].$$

(Here x_i is taken to interpret v_i where $v_1, \ldots, v_n$ is an arbitrarily chosen sequence of distinct variables, including all variables which occur free in φ.) We define φ' by induction on φ. For primitive formulae we set first:

$$(v \in w)' = v \in w, (v \in \dot{U})' = v = v,$$
$$(\dot{U} \in v)' = v \neq v, (\dot{U} \in \dot{U}) = \bigvee v\ v \neq v.$$

For sentential combinations we do the obvious thing:

$$(\varphi \wedge \psi)' = (\varphi' \wedge \psi'), (\neg\varphi)' = \neg\varphi',$$

etc. Quantifiers are treated as follows:

$$(\bigwedge v \in w\varphi)' = \bigwedge v \in w\varphi'$$
$$(\bigwedge v \in \dot{U}\varphi)' = \bigwedge v\varphi'$$

Given finitely many rud functions $s_1, \ldots, s_p$ we say that they constitute a *basis* for the rud function iff every rud function is obtainable by successive application of the schemata:

- $f(x_1, \ldots, x_n) = x_j$ $(j = 1, \ldots, n)$
- $f(\vec{x}) = s_i(g_1(\vec{x}), \ldots, g_m(\vec{x}))$ $(i = 1 \ldots, p)$

Note that if $s_1, \ldots, s_p$ is a basis, then rud(U) is simply the closure of U under the finitely many functions $s_1, \ldots, s_p$. We shall now prove the *Basis Theorem*, which says that the rud functions possess a finite basis. We first define:

Definition 2.2.7. $(x, y) =: \{\{x\}, \{x, y\}\}; (x) = x,$
$(x_1, \ldots, x_n) = (x_1, (x_2, \ldots, x_n))$ for $n \geq 2$.

(Note: Our "official" notation for n–tuples is $\langle x_1, \ldots, x_n \rangle$. However, we have refrained from specifying its definition. Thus we do not know whether $(\vec{x}) = \langle \vec{x} \rangle$.)

We also set:

Definition 2.2.8.

$$\begin{aligned} &x \otimes y = \{(z,w) | z \in x \wedge w \in y\} \\ &\mathrm{dom}^*(x) = \{z | \bigvee y (y,z) \in x\} \\ &x^* z = \{y | (y,z) \in x\} \end{aligned}$$

Theorem 2.2.15. *The following functions form a basis for the* rud *function:*

$$\begin{aligned} &F_0(x,y) = \{x,y\} \\ &F_1(x,y) = x \setminus y \\ &F_2(x,y) = x \otimes y \\ &F_3(x,y) = \{(u,z,v) | z \in x \wedge (u,v) \in y\} \\ &F_4(x,y) = \{(u,v,z) | z \in x \wedge (u,v) \in y\} \\ &F_5(x,y) = \bigcup x \\ &F_6(x,y) = \mathrm{dom}^*(x) \\ &F_7(x,y) = \{(z,w) | z,w \in x \wedge z \in w\} \\ &F_8(x,y) = \{x^* z | z \in y\} \end{aligned}$$

Proof: The proof stretches over several subclaims. Call a function f *good* iff it is obtainable from $F_0, \ldots, F_8$ by successive applications of the above schemata. Then every good function is rud. We must prove the converse. We first note:

Claim 1 The good functions are closed under composition — i.e. if $g, h_1, \ldots, h_n$ are good, then so is $f(\vec{x}) = g(\vec{h}(\vec{x}))$.

Proof: Set $G =$ the set of good function $g(y_1, \ldots, y_v)$ such that whenever $h_i(\vec{x})$ is good for $i = 1, \ldots, r$, then so is $f(\vec{x}) = g(\vec{h}(\vec{x}))$. By a straightforward induction on the defining schemata it is easily shown that all good functions are in G. QED (Claim 1)

Claim 2 The following functions are good:

$$\begin{aligned} &\{x,y\}, x \setminus y, x \otimes y, x \cup y = \bigcup \{x,y\}, \\ &x \cap y = x \setminus (x \setminus y), \{x_1, \ldots, x_n\} = \{x_1\} \cup \ldots \cup \{x_n\}, \\ &C_n(u) = u \cup \bigcup u \cup \ldots \cup \overbrace{\bigcup \ldots \bigcup}^{n} u, (x_1, \ldots, x_n) \end{aligned}$$

(since $(x_1, \ldots, x_n)$ is obtained by iteration of F_0.) By an $\in$*–formula* we mean a first order formula containing only $\dot{\in}$ as a non logical predicate. If

$\varphi = \varphi(v_1, \ldots, v_n)$ is any $\in$–formula in which at most the distinct variables $(v_1, \ldots, v_n)$ occur free, set:

$$t_\varphi(u) =: \{(x_1, \ldots, x_n) | \vec{x} \in u \wedge \langle u, \in \rangle \models \varphi[\vec{x}]\}.$$

Note. We follow the usual convention of suppressing the list of variables. We should, of course, write: $t_{\varphi, v_1, \ldots, v_n}(u)$.

Note. Recall our convention that $\vec{x} \in u$ means that $x_i \in u$ for $i = 1, \ldots, n$.

Then t_φ is *rud*. We claim:

Claim 3 t_φ is good for every $\in$–formula φ.

Proof:

(1) It holds for $\varphi = v_i \in v_j$ $(1 \leq i < j \leq n)$

Proof: For $i = 2, 3$ set:

$$F_i^0(u, w) = w, \; F_i^{m+1}(u, w) = F_i(u, F_i^m(u, w))$$

then F_i^m is good for all m. For $m \geq 1$ we have:

$$F_2^m(u, w) = \{(x_1, \ldots, x_m, z) | \vec{x} \in u \wedge z \in w\}$$
$$F_3^m(u, w) = \{(y, x_1, \ldots, x_m, z) | \vec{x} \in u \wedge (y, z) \in w\}$$

We also set

$$u^{(m)} = \{(x_1, \ldots, x_m) | \vec{x} \in u\} = F_2^{m-1}(u, u)$$

If $j = n$, then

$$t_\varphi(u) = \{(x_1, \ldots, x_n) | \vec{x} \in u \wedge x_i \in x_j\} = F_2^{i-1}(u, F_3^{n-i-1}(u, F_7(u, u))).$$

Now let $n > j$. Noting that:

$$F_4(u^{(m)}, w) = \{(y, z, x_1, \ldots, x_m) | \vec{x} \in u \wedge (y, z) \in w\},$$

we have:

$$t_\varphi(u) = F_2^{i-1}(u, F_3^{j-n-1}(u, F_4(u^{(n-j)}, F_7(u, u)))).$$

QED (1)

(2) It holds for $\varphi = v_i \in v_i$.

Proof: $t_\varphi(w) = \emptyset = w \setminus w$.

(3) If it holds for $\varphi = \varphi(v_1, \ldots, v_n)$, then for $\neg\varphi$.

Proof:

$$t_{\neg\varphi}(w) = (w^{(n)} \setminus t_\varphi(w)).$$

QED (3)

(4) If it holds for φ, ψ, then for $\varphi \wedge \psi$, $\varphi \vee \psi$. (Hence for $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$ by (3).)

Proof:

$$t_{\varphi\vee\psi}(w) = t_\varphi(w) \cup t_\psi(w) = \bigcup\{t_\varphi(w), t_\psi(w)\}$$
$$t_{\varphi\wedge\psi}(w) = t_\varphi(w) \cap t_\psi(w), \text{ where } x \cap y = (x \setminus (x \setminus y)).$$

QED (4)

(5) If it holds for $\varphi = \varphi(u, v_1, \ldots, v_n)$, then for $\bigwedge u\varphi$, $\bigvee_u \varphi$.

Proof:

$$t_{\bigvee u\varphi}(w) = F_6(t_\varphi(\omega), t_\varphi(\omega)) \text{ hence}$$
$$t_{\bigwedge u\varphi}(w) = t_{\neg\bigvee u\neg\varphi}(w) \text{ by (3)}$$

QED (5)

(6) It holds for $\varphi = v_i = v_j$ $(i, j \leq n)$.

Proof: Let $\psi(v_1, \ldots, v_n) = \bigwedge z(z \in v_i \leftrightarrow z \in v_j)$. Then for $(\vec{x}) \in U^{(n)}$ we have:

$$(\vec{x}) \in t_\psi(u \cup \bigcup u) \leftrightarrow x_i = x_j,$$

since $x_i, x_j \subset (u \cup \bigcup u)$. Hence

$$t_\varphi(u) = u^{(n)} \cap t_\psi(u \cup \bigcup u).$$

QED (6)

(7) It holds for $\varphi = v_j \in v_i$ $(i < j)$

Proof:

$$v_j \in v_i \leftrightarrow \bigvee u(u = v_j \wedge u \in v_i).$$

We apply (6), (5) and (4). QED (7)

But then if $\varphi(v_1, \ldots, v_n) = Qu_1, \ldots Qu_n\psi(\vec{u}, \vec{v})$ is any formula in prenex normal form, we apply (1), (2), (6), (7) and (3), (4) to see that t_ψ is good. But then t_φ is good by iterated applications of (5). QED (Claim 3)

In our application we shall use the function t_φ only for Σ_0 formulae φ. We shall make strong use of the following well known fact, which can be proven by induction on n.

Fact Let $\varphi = \varphi(v_1, \ldots, v_m)$ be a Σ_0 formula in which at most n quantifiers occur. Let u be any set and let $x_1, \ldots, x_m \in u$. Then $V \models \varphi[\vec{x}] \leftrightarrow C_n(u) \models \varphi[\vec{x}]$.

Definition 2.2.9. Let $f : V^n \to V$ be rud. f is *verified* iff there is a good $f^* : V \to V$ such that $f''U^n \subset f^*(U)$ for all sets U. We then say that f^* *verifies* f.

Claim 4 Every verified function is good.

Proof: Let f be verified by f^*. Let φ be the Σ_0 formula: $y = f(x_1, \ldots, x_n)$. For sufficient m we know that for any set u we have:

$$y = f(\vec{x}) \leftrightarrow (y, \vec{x}) \in t_\varphi(C_m(u \cup f^*(u)))$$
$$\text{for } y, \vec{x} \in u \cup f^*(u).$$

Define a good function F by:

$$F(u) =: (f^*(u) \otimes u^{(n)}) \cap t_\varphi(C_m(u \cup f^*(u))).$$

Then $F(u)$ is the set of $(f(\vec{x}), \vec{x})$ such that $\vec{x} \in u$. In particular, if $u = \{x_1, \ldots, x_n\}$, then:

$$F_8(F(\{\vec{x}\}), \{(\vec{x})\}) = \{f(\vec{x})\}$$

and $f(\vec{x}) = \bigcup F_8(F(\{\vec{x}\}), \{(\vec{x})\})$. QED (Claim 4)

Thus it remains only to prove:

Claim 5 Every rud function is verified.

Proof: We proceed by induction on the defining schemata of f.

Case 1 $f(\vec{x}) = x_i$

Take $f^*(u) = u = u \setminus (u \setminus u)$.

Case 2 $f(\vec{x}) = x_i \setminus x_j$

Let φ be the formula $z \in x \setminus y$. Then for $z, x, y \in v$ we have

$$\begin{aligned} z \in x \setminus y &\leftrightarrow v \models \varphi[z, x, y] \\ &\leftrightarrow (z, x, y) \in t_\varphi(v). \end{aligned}$$

But $x, y \in u \to x \setminus y \subset \bigcup u$. Hence for all x, y, u and all z we have:

$$z \in x \setminus y \leftrightarrow (z, x, y) \in t_\varphi(u \cup \bigcup u).$$

Hence:

$$f''u^n \subset \{x \setminus y | x, y \in u\} = F_8(t_\varphi(u \cup \bigcup u), u^{(2)}).$$

QED (Case 2)

Case 3 $f(\vec{x}) = \{x_i, x_j\}$

Then $f''u^n = \{\{x, y\} | x, y \in u\} = \bigcup u^{(2)}$. QED (Case 3)

Case 4 $f(\vec{x}) = g(\vec{h}(\vec{x}))$

Let h_i^* verify h_i and g^* verify g. Then $f^*(u) = g^*(\bigcup_i h_i^*(u))$ verifies f.

QED (Case 4)

Case 5 $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$. Let g^* verify g. Let $\varphi = \varphi(u, y, \vec{x})$ be the Σ_0 formula: $\bigvee z \in y \; w \in g(z, \vec{x})$. For sufficient m we have:

$$\bigvee z \in y \; w \in g(z, \vec{x}) \leftrightarrow (w, y, \vec{x}) \in t_\varphi(C_m(u \cup \bigcup g^*(u)))$$

for all $w, y, \vec{x} \in u \cup \bigcup g^*(u)$.

Set $F(u) = t_\varphi(C_m(u \cup \bigcup g^*(u)))$. Then $g(z, \vec{x}) \subset \bigcup g^*(u)$ whenever $y, \vec{x} \in u$ and $z \in y$. Hence

$$F(u)^*(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$$

for $y, \vec{x} \in U$. Hence

$$f''u^{n+1} \subset F_8(F(u), u^{(n+1)}).$$

QED (Theorem 2.2.15)

Combining Theorem 2.2.15 with Lemma 2.2.6 we get:

Corollary 2.2.16. *Let $A_1, \ldots, A_n \subset V$. Then $F_0, \ldots, F_8$ together with the functions $a_i(x) = x \cap A_i (i = 1, \ldots, n)$ form a basis for the functions which are rudimentary in $A_1, \ldots, A_n$.*

Let $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$. '$\models_M$' denotes the satisfaction relation for M and '$\models_M^{\Sigma_n}$' denotes its restriction to Σ_n formulae. We can make good use of the basis theorem in proving:

Lemma 2.2.17. $\models_M^{\Sigma_0}$ *is uniformly $\Sigma_1(M)$ over transitive* rud *closed $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$.*

Proof: We shall prove it for the case $n = 1$, since the extension of our proof to the general case is then obvious. We are then given: $M = \langle |M|, \in, A \rangle$. By a *variable evaluation* we mean a function e which maps a finite set of variables of the M–language into $|M|$. Let E be the set of such evaluations. If $e \in E$, we can extend it to an evaluation e^* of all variables by setting:

$$e^*(v) = \begin{cases} e(v) \text{ if } v \in \text{dom}(e) \\ \emptyset \text{ if not} \end{cases}$$

$\models_M \varphi[e]$ then means that φ becomes true in M if each free variable v in φ is interpreted by $e^*(v)$.

We assume, of course, that the first order language of M has been "arithmetized" in a reasonable way — i.e. the syntactic objects such as formulae and variables have been identified with elements of H_ω in such a way that the basic syntactic relations and operations become recursive. (Without this the assertion we are proving would not make sense.) In particular the set Vbl of variables, the set Fml of formulae, and the set Fml_0 of Σ_0–formulae are all recursive (i.e. $\Delta_1(H_\omega)$). We first note that every $\Sigma_0(M)$ relation is rud, or equivalently:

(1) Let φ be Σ_0. Let $v_1, \ldots, v_n$ be a sequence of distinct variables containing all variables occuring free in φ. There is a function f uniformly rud in A such that

$$\models_M \varphi[e] \leftrightarrow f(e^*(v_1), \ldots e^*(v_n)) = 1$$

for all $e \in E$.

Proof: By induction on φ. We leave the details to the reader.

QED (1)

The notion A–*good* is defined like "good" except that we now add the function $F_9(x, y) = x \cap A$ to our basis. By Corollary 2.2.16 we know that every function rud in A is A–good. We now define in H_ω an auxiliary term language whose terms represent the A–good function. We first set: $\dot{F}_i(x, y) =: \langle i, \langle x, y\rangle\rangle$ for $i = 0, \ldots, 9$: $\dot{x} = \langle 10, x\rangle$. The set Tm of *Terms* is then the smallest set such that

- $\dot{v}$ is a term whenever $v \in Vbl$
- If t, t' are terms, then so is $\dot{F}_i(t, t')$ for $i = 0, \ldots, 9$.

Applying the methods of Chapter 1 to the admissible set H_ω it follows easily that the set Tm is recursive (i.e. $\Delta_1(H_\omega)$). Set

$C(t) \simeq$: The smallest set C such that the term $t \in C$ and C is closed under subterms (i.e. $\dot{F}_i(s, s') \in C \to s, s' \in C$).

Then $C(t) \in H_\omega$ for $t \in Tm$, and the function $C(t)$ is recursive (hence $\Delta_1(H_\omega)$). Since Vbl is recursive, the function
$Vbl(t) \simeq: \{v \in Vbl | \dot{v} \in C(t)\}$ is recursive.

We note that:

(2) Every recursive relation on H_ω is uniformly $\Sigma_1(M)$.

Proof: It suffices to note that: H_ω is uniformly $\Sigma_1(M)$, since

$$x \in H_\omega \leftrightarrow \bigvee f \bigvee u \bigvee n \varphi(f,u,n,x)$$

where φ is the Σ_0 formula: f is a function $\wedge$ u is transitive $\wedge n \in \omega \wedge f : n \leftrightarrow u \wedge x \in u$. QED (2)

Given $e \in E$ we recursively define an evaluation $\langle \overline{e}(t) | t \in Tm \rangle$ by:

$$\overline{e}(\dot{v}) = e^*(v) \text{ for } v \in Vbl$$
$$\overline{e}(\dot{F}_i(t,s)) = F_i(\overline{e}(t), \overline{e}(s)).$$

Then:

(3) $\{\langle y,e,t\rangle | e \in E \wedge t \in Tm \wedge y = \overline{e}(t)\}$ is uniformly $\Sigma_1(M)$.

Proof: Let $e \in E$, $t \in Tm$. Then $y = \overline{e}(t)$ can be expressed in M by:

$$\bigvee g \bigvee u \bigvee v (u = C(t) \wedge v = Vbl(t) \wedge \varphi(y,e,u,v,y,t))$$

where φ is the Σ_0 formula:

(g is a function $\wedge \operatorname{dom}(g) = u \wedge \bigwedge x \in v \; x \in u$

$$\begin{aligned} &\wedge \bigwedge x \in v((x \in \operatorname{dom}(e) \wedge g(\dot{x}) = e(x)) \vee \\ &\quad \vee (x \notin \operatorname{dom}(e) \wedge g(\dot{x}) = \emptyset)) \\ &\wedge \bigwedge_{i=0}^{9} \bigwedge t,s,i \in u (t = \dot{F}_i(s,s') \rightarrow \\ &\quad \rightarrow g(t) = F_i(g(s), y(s'')) \\ &\wedge y = g(t)) \end{aligned}$$

QED (3)

(4) Let $f(x_1, \ldots, x_n)$ be A–good. Let $v_1, \ldots, v_n'$ be any sequence of distinct variables. There is $t \in Tm$ such that

$$f(e^*(v_1), \ldots, e^*(v_n)) = \overline{e}(t)$$

for all $e \in E$.

Proof: By induction on the defining schemata of f. If $f(\vec{x}) = x_i$, we take $t = \dot{v}_i$. If $e^*(\vec{v})) = \overline{e}(s_i)$ for $e \in \mathbb{E}(i = 0, 1)$, and $f(\vec{x}) = F_i(g_0(\vec{x}), g_1(\vec{x}))$, we set $t = \dot{F}_i(s_0, s_1)$. Then

$$\overline{e}(t) = F_i(\overline{e}(s_0), \overline{e}(s_1)) = F_i(g_0(\vec{x}), g_1(\vec{x})) = f(\vec{x}).$$

QED (4)

But then:

(5) Let φ be a Σ_0 formula. There is $t \in Tm$ such that $M \models \varphi[e] \leftrightarrow \overline{e}(t) = 1$ for all $e \in E$.

Proof: Let $v_1, \ldots, v_n$ be a sequence of distinct variables containing all variables which occur free in φ. Then

$$M \models \varphi[e] \leftrightarrow M \models \varphi[e^*(v_1), \ldots, e^*(v_n)]$$

for all $e \in E$. Set

$$(*)\ f(\vec{x}) = \begin{cases} 1 \text{ if } M \models \varphi[\vec{x}] \\ 0 \text{ if not.} \end{cases}$$

Then f is rudimentary, hence A–good. Let $t \in Tm$ such that

$$(**)\ f(e^*(v_1), \ldots, e^*(v_n)) = \overline{e}(t).$$

Then: $M \models \varphi[e] \leftrightarrow \overline{e}(t) = 1$. QED (6)

(5) is, however, much more than an existence statement, since our proofs are *effective*: Clearly we can effectively assign to each Σ_0 formula φ a sequence $v(\varphi) = \langle v_1, \ldots, v_n \rangle$ of distinct variables containing all variables which occur free in φ. But the proof that the f defined by $(*)$ is rud in fact implicity defines a rud definition D_φ such that D_φ defines such an $f = f_{D_\varphi}$ over any rud closed $M = \langle M, \in, A \rangle$. The proof that f is A–good is by induction on the defining schemata and implicitly defines a term $t = T_\varphi$ which satisfies $(**)$ over any rud closed M. Thus our proofs implicitly describe an algorithm for the function $\varphi \mapsto T_\varphi$. Hence this function is recursive, hence uniformly $\Sigma_1(M)$. But then Σ_0 satisfaction can be defined over M by:

$$M \models \varphi[e] \leftrightarrow: \overline{e}(T_\varphi) = 1.$$

QED (Lemma 2.2.17)

Corollary 2.2.18. *Let $n \geq 1$. $\models^{\Sigma_n}_M$ is uniformly $\Sigma_n(M)$ for transitive* rud *closed structures $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$.*

(We leave this to the reader.)

2.2.1 Condensation

The *condensation lemma* for rud closed sets $U = \langle U, \in \rangle$ reads:

Lemma 2.2.19. *Let $U = \langle U, \in \rangle$ be transitive and* rud *closed. Let $X \prec_{\Sigma_1} U$. Then there is an isomorphism $\pi : \overline{U} \stackrel{\sim}{\longleftrightarrow} X$, where $\overline{U}$ is transitive and* rud *closed. Moreover, $\pi(f(\vec{x})) = f(\pi(\vec{x}))$ for all* rud *functions f.*

Proof: X satisfies the extensionality axiom. Hence by Mostowski's isomorphism theorem there is $\pi : \overline{U} \overset{\sim}{\longleftrightarrow} X$, where $\overline{U}$ is transitive. Now let f be rud and $x_1, \dots, x_n \in \overline{U}$. Then there is $y' \in X$ such that $y' = f(\pi(\vec{x}))$, since $X \prec_{\Sigma_1} U$. Let $\pi(y) = y'$. Then $y = f(\vec{x})$, since the condition '$y = f(\vec{x})$' is Σ_0 and π is Σ_1–preserving. QED (Lemma 2.2.19)

The condensation lemma for rud closed $M = \langle |M|, \in, A_1, \dots, A_n \rangle$ is much weaker, however. We state it for the case $n = 1$.

Lemma 2.2.20. *Let* $M = \langle |M|, \in, A \rangle$ *be transitive and* rud *closed. Let* $X \prec_{\Sigma_1} M$. *There is an isomorphism* $\pi : \overline{M} \overset{\sim}{\longleftrightarrow} X$, *where* $\overline{M} = \langle |\overline{M}|, \in, \overline{A} \rangle$ *is transitive and* rud *closed. Moreover:*

(a) $\pi(\overline{A} \cap x) = A \cap \pi(x)$

(b) Let f *be* rud *in* A. *Let* f *be characterized by:* $f(\vec{x}) = f_0(\vec{x}, A \cap f_1(\vec{x}))$, *where* f_0, f_1 *are* rud. *Set:* $\overline{f}(\vec{x}) =: f_0(\vec{x}, \overline{A} \cap f_1(\vec{x}))$. *Then:*

$$\pi(\overline{f}(\vec{x})) = f(\pi(\vec{x})).$$

The proof is left to the reader.

2.3 The J_α hierarchy

We are now ready to introduce the alternative to Gödel's constructible hierarchy which we had promised in §1. We index it by ordinals from the class Lm of limit ordinals.

Definition 2.3.1.

$$\begin{aligned} J_\omega &= \mathrm{Rud}(\emptyset) \\ J_{\beta+\omega} &= \mathrm{Rud}(J_\beta) \text{ for } \beta \in \mathrm{Lm} \\ J_\lambda &= \bigcup_{\gamma<\lambda} J_\gamma \text{ for } \lambda \text{ a limit point of Lm} \end{aligned}$$

It can be shown that $L = \bigcup_\alpha J_\alpha$ and, indeed, that $L_\alpha = J_\alpha$ for a great many α (for instance closed α). Note that $J_\omega = L_\omega = H_\omega$.

By §2 Corollary 2.2.14 we have:

$$\mathbb{P}(J_\alpha) \cap J_{\alpha+\omega} = \mathrm{Def}(J_\alpha),$$

which pinpoints the resemblance of the two hierarchies. However, we shall not dwell further on the relationship of the two hierarchies, since we intend to consequently employ the J–hierarchy in the rest of this book. As usual, we shall often abuse notation by not distinguishing between J_α and $\langle J_\alpha, \in \rangle$.