2.4 J–models

We can add further unary predicates to the structure $J_\alpha^{\vec{A}}$. We call the structure:

$$M = \langle J_\alpha^{A_1,\dots,A_n}, B_1, \dots, B_m \rangle$$

a *J–model* if it is amenable in the sense that $x \cap B_i \in J_\alpha^{\vec{A}}$ whenever $x \in J_\alpha^{\vec{A}}$ and $i = 1, \dots, m$. The B_i are again taken as unary predicates. The *type* of M is $\langle n, m \rangle$. (Thus e.g. J_α has type $\langle 0, 0 \rangle$, J_α^A has type $\langle 1, 0 \rangle$, and $\langle J_\alpha, B \rangle$ has type $\langle 0, 1 \rangle$.) By an abuse of notation we shall often fail to distinguish between M and the associated structure:

$$\hat{M} = \langle J_\alpha[\vec{A}], A'_1, \dots, A'_n, B_1, \dots, B_m \rangle$$

where $A'_i = A_i \cap J_\alpha[\vec{A}]$ $(i = 1, \dots, n)$.
We may for instance write $\Sigma_1(M)$ for $\Sigma_1(\hat{M})$ or $\pi : N \to_{\Sigma_n} M$ for $\pi : \hat{N} \to_{\Sigma_n} \hat{M}$. (However, we cannot unambignously identify M with $\hat{M}$, since e.g. for $M = \langle J_\alpha^A, B \rangle$ we might have: $\hat{M} = J_\alpha^{A,B}$.)

In practice we shall usually deal with J models of type $\langle 1, 1 \rangle$, $\langle 1, 0 \rangle$, or $\langle 0, 0 \rangle$. In any case, following the precedent in earlier section, when we prove general theorem about J–models, we shall often display only the proof for type $\langle 1, 1 \rangle$ or $\langle 1, 0 \rangle$, since the general case is then straightforward.

Definition 2.4.1. If $M = \langle J_\alpha^{\vec{A}}, \vec{B} \rangle$ is a J–model and $\beta \leq \alpha$ in Lm, we set:

$$M|\beta =: \langle J_\beta^{\vec{A}}, B_1 \cap J_\beta^{\vec{A}}, \dots, B_n \cap J_\beta^{\vec{A}} \rangle.$$

In this section we consider $\Sigma_1(M)$ definability over an arbitrary $M = \langle J_\alpha^{\vec{A}}, \vec{B} \rangle$. If the context permits, we write simply Σ_1 instead of $\Sigma_1(M)$. We first list some properties which follow by rud closure alone:

- $\models_M^{\Sigma_1}$ is uniformly Σ_1, by corollary 2.2.18 (**Note** 'Uniformly' here means that the Σ_1 definition is the same for any two M having the same type.)
- If $R(y, x_1, \dots, x_n)$ is a Σ_1 relation, then so is $\bigvee y R(y, x_1, \dots, x_n)$ (since $\bigvee y \bigvee z P(yz, \vec{x}) \leftrightarrow \bigvee u \bigvee y, z \in u P(y, z, \vec{x})$ where $R(y, \vec{x}) \leftrightarrow \bigvee z P(y, z, \vec{x})$ and P is Σ_0).

  By an n–ary $\Sigma_1(M)$ *function* we mean a partial function on M^n which is $\Sigma_1(M)$ as an $n+1$–ary relation.
- If R, R' are n–ary Σ_1 relations, then so are $R \cap R'$, $R \cup R'$. (Since e.g.

$$(\bigvee y P(y, \vec{x}) \wedge \bigvee P'(y, \vec{x})) \leftrightarrow \\ \bigvee y y' (P(y, \vec{x}) \wedge P'(y', \vec{x})).)$$

- If $R(y_1, \ldots, y_m)$ is an n–ary Σ_1 relation and $f_i(\vec{x})$ is an n–ary Σ_1 function for $i = 1, \ldots, m$, then so is the n–ary relation

$$R(\vec{f}(\vec{x})) \leftrightarrow: \bigvee y_1, \ldots, y_m (\bigwedge_{i=1}^{m} y_i = f_i(\vec{x}) \wedge R(\vec{y})).$$

- If $g(y_1, \ldots, y_m)$ is an m–ary Σ_1 function and $f_i(\vec{x})$ is an n–ary Σ_1 function for $i = 1, \ldots, m$ then $h(\vec{x}) \simeq g(\vec{f}(\vec{x}))$ is an n–ary Σ_1 function. (Since $z = h(\vec{x}) \leftrightarrow \bigvee y_1, \ldots, y_m (\bigwedge_{i=1}^{m} y_i = f_i(\vec{x}) \wedge z = g(\vec{y}))$.)

  Since $f(x_1, \ldots, x_n) = x_i$ is Σ_1 function, we have:

- If $R(x_1, \ldots, x_n)$ is Σ_1 and $\sigma : n \to m$, then

$$P(z_1, \ldots, z_m) \leftrightarrow: R(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$$

  is Σ_1.

- If $f(x_1, \ldots, x_n)$ is a Σ_1 function and $\sigma : n \to m$, then the function:

$$g(z_1, \ldots, z_m) \simeq: f(z_{\sigma(1)}, \ldots, z_{\sigma n})$$

  is Σ_1.

J–models have the further property that every binary Σ_1 relation is uniformizable by a Σ_1 function. We define

Definition 2.4.2. A relation $R(y, \vec{x})$ is *uniformized* by the function $F(\vec{x})$ iff the following hold:

- $\bigvee y R(y, \vec{x}) \to F(\vec{x})$ is defined
- If $F(\vec{x})$ is defined, then $R(F(\vec{x}), \vec{x})$

We shall, in fact, prove that M has a uniformly Σ_1 definable *Skolem function*. We define:

Definition 2.4.3. $h(i, x)$ is a Σ_1*–Solem function* for M iff h is a $\Sigma_1(M)$ partial map from $\omega \times M$ to M and, whenever $R(y, x)$ is a $\Sigma_1(M)$ relation, there is $i < \omega$ such that h_i uniformizes R, where $h_i(x) \simeq h(i, x)$.

Lemma 2.4.1. *M has a Σ_1–Skolem function which is uniformly $\Sigma_1(M)$.*

Proof: $\models_M^{\Sigma_1}$ is uniformly Σ_1. Let $\langle\varphi_i | i<\omega\rangle$ be a recursive enumeration of the Σ_1 formulae in which at most the two variables v_0, v_1 occur free. Then the relation:

$$T(i,y,x) \leftrightarrow: \models_M^{\Sigma_1} \varphi_i[y,x]$$

is uniformly Σ_1. But then for any Σ_1 relation R there is $i<\omega$ such that

$$R(y,x) \leftrightarrow T(i,y,x).$$

Since T is Σ_1, it has the form:

$$\bigvee z T'(z,i,y,x)$$

where T' is Σ_0. Writing $<_M$ for $<_\alpha^{\vec{A}}$, we define:

$$\begin{aligned} y = h(i,x) \leftrightarrow \bigvee z(\langle z,y\rangle \text{ is the } <_M \text{–least} \\ \text{pair } \langle z',y'\rangle \text{ such that } T'(z',i,y',x). \end{aligned}$$

Recalling that the function $f(x)=\{z|z<_M x\}$ is Σ_1, we have:

$$\begin{aligned} y = & h(i,x) \leftrightarrow \bigvee z \bigvee u(T'(z,i,y,x)\wedge \\ & \wedge u = \{w|w <_M \langle z,y\rangle\}\wedge \\ & \wedge \bigwedge \langle z',y'\rangle \in u \neg T'(z,i,y,x)) \end{aligned}$$

QED 2.4.1

We call the function h defined above the *canonical* Σ_1 *Skolem function* for M and denote it by h_M. The existence of h implies that every $\Sigma_1(M)$ relation is uniformizable by a $\Sigma_1(M)$ function:

Corollary 2.4.2. *Let $R(y,x_1,\ldots,x_n)$ be Σ_1. R is uniformizable by a Σ_1 function.*

Proof: Let h_i uniformize the binary relation

$$\{\langle y,z\rangle | \bigvee x_1 \ldots x_n(R(y,\vec{x}) \wedge z = \langle x_1,\ldots,x_n\rangle)\}.$$

Then $f(\vec{x}) \simeq: h_i(\langle\vec{x}\rangle)$ uniformizes R. QED

We say that a $\Sigma_1(M)$ function has a *functionally absolute* definition if it has a Σ_1 definition which defines a function over every J–model of the same type.

Corollary 2.4.3. *Every $\Sigma_1(M)$ function g has functionally absolute definition.*

Proof: Apply the construction in Corollary 2.4.2 to $R(y, \vec{x}) \leftrightarrow y = g(\vec{x})$. Then $f(x) \simeq: h_i(\langle\vec{x}\rangle)$ is functionally absolute since h_i is.

QED (Corollary 2.4.2)

Lemma 2.4.4. *Every $x \in M$ is $\Sigma_1(M)$ in parameters from* $\mathrm{On} \cap M$.

Proof: We must show: $x = f(\xi_1, \ldots, \xi_n)$ where f is $\Sigma_1(M)$. If $M = \langle J_\alpha^{\vec{A}}, \vec{B}\rangle$, it obviously suffices to show it for the model $M' = J_\alpha^{\vec{A}}$. For the sake of simplicity we display the proof for J_α^A. (i.e. M has type $\langle 1, 0\rangle$). We proceed by induction on $\alpha \in \mathrm{Lm}$.

Case 1 $\alpha = \omega$.

Then $J_\alpha^A = \mathrm{Rud}(\emptyset)$ and $x = f(\{0\})$ where f is rudimentary.

Case 2 $\alpha = \beta + \omega,\ \beta \in \mathrm{Lm}$.

Then $x = f(z_1, \ldots, z_n, J_\beta^A)$ where $z_1, \ldots, z_n \in J_\beta^A$ and f is rud in A. (This is meant to include the case: $n = 0$ and $x = f(J_\beta^A)$.) By the induction hypothesis there are $\vec{\xi} \in \beta$ such that $z_i = g_i(\vec{\xi})$ $(i = 1, \ldots, n)$ and g_i is $\Sigma_1(J_\beta^A)$. For each i pick a functionally absolute Σ_1 definition for g_i and let g_i' be $\Sigma_1(J_\alpha^A)$ by the same definition. Then $z_i = g_i'(\vec{\xi})$ since the condition is Σ_1. Hence $x = f'(\vec{\xi}, \beta) = f(\vec{g}'(\xi, J_\beta^A)$ where f' is Σ_1.

QED (Case 2)

Case 3 $\alpha \in \mathrm{Lm}^*$.

Then $x \in J_\beta^A$ for a $\beta < \alpha$. Hence $x = f(\vec{\xi})$ where f is $\Sigma_1(J_\beta^A)$. Pick a functionally absolute Σ_1 definition of f and let f' be $\Sigma_1(J_\alpha^A)$ by the same definition. Then $x = f'(\vec{\xi})$.

QED (Lemma 2.4.4)

But being Σ_1 in parameters from $\mathrm{On} \cap M$ is the same as being Σ_1 in a finite subset of $\mathrm{On} \cap M$:

Lemma 2.4.5. *Let $x = f(\vec{\xi})$ where f is $\Sigma_1(M)$. Let $a \subset \mathrm{On} \cap M$ be finite such that $\xi_1, \ldots, \xi_n \in a$. Then $x = g(a)$ for a $\Sigma_1(M)$ function g.*

Proof: Set:

$$k_i(a) = \begin{cases} \text{the } i\text{–th element of } a \text{ in order} \\ \text{of size if } a \subset \mathrm{On} \text{ is finite} \\ \text{and } \mathrm{card}(a) > i, \\ \text{undefined if not.} \end{cases}$$

Then k_i is $\Sigma_1(M)$ since:

$$y = k_i(a) \leftrightarrow \bigvee f \bigvee n < \omega(f : n \leftrightarrow a \wedge \bigwedge i, j < n(f(i) < f(j) \leftrightarrow i < j) \\ \wedge a \subset \mathrm{On} \wedge y = f(i))$$

Thus $x = f(k_{i_1}(a), \ldots, k_{i_n}(a))$ where $\xi_l = k_{i_l}(a)$ for $l = 1, \ldots, n$.

QED (Lemma 2.4.5)

We now show that for every J–model M there is a $\underline{\Sigma}_1(M)$ partial map of $\mathrm{On} \cap M$ onto M. As a preliminary we prove:

Lemma 2.4.6. *There is a partial* $\underline{\Sigma}_1(M)$ *map of* $\mathrm{On} \cap M$ *onto* $(\mathrm{On} \cap M)^2$.

Proof: Order the class of pairs On^2 by setting: $\langle \alpha, \beta \rangle <^* \langle \gamma, \delta \rangle$ iff $\langle \max(\alpha, \beta), \alpha, \beta \rangle$ is lexicographically less than $\langle \max(\gamma, \delta), \gamma, \delta \rangle$. This ordering has the property that the collection of predecessors of any pair form a set. Hence there is a function $p : \mathrm{On} \to \mathrm{On}^2$ which enumerates the pairs in order $<^*$.

Claim 1 $p \restriction \mathrm{On}_M$ is $\Sigma_1(M)$.

Proof: If $M = \langle J_\alpha^{\vec{A}}, \vec{B} \rangle$, it suffices to prove it for $J_\alpha^{\vec{A}}$. To simplify notation, we assume: $M = J_\alpha^A$ for an $A \subset M$ (i.e. M is of type $\langle 1, 0 \rangle$.)

We know:

$$y = p(\nu) \leftrightarrow \bigvee f (\varphi(f) \wedge y = f(\nu))$$

where φ is the Σ_0 formula:

$$\begin{aligned}
&f \text{ is a function } \wedge \mathrm{dom}(f) \in \mathrm{On} \wedge \\
&\wedge \bigwedge u \in \mathrm{rng}(f) \bigvee \beta, \gamma \in C_n(u) u = \langle \beta, \gamma \rangle \wedge \\
&\wedge \bigwedge \nu, \tau \in \mathrm{dom}(f)(\nu < \tau \leftrightarrow f(\nu) <^* f(\tau)) \\
&\wedge \bigwedge u \in \mathrm{rng}(f) \bigwedge \mu, \xi \leq \max(u)(\langle \mu, \xi \rangle <^* u \to \langle \mu, \xi \rangle \in \mathrm{rng}(f)).
\end{aligned}$$

Thus it suffices to show that the existence quantifier can be restricted to J_α^A — i.e. that $p \restriction \xi \in J_\alpha^A$ for $\xi < \alpha$. This follows by induction on α in the usual way (cf. the proof of Lemma 2.3.14). QED (Claim 1)

We now proceed by induction on $\alpha = \mathrm{On}_M$, considering three cases:

Case 1 $p(\alpha) = \langle 0, \alpha \rangle$.

Then $p \restriction \alpha$ maps α onto

$$\{u | u <_* \langle 0, \alpha \rangle\} = \alpha^2$$

and we are done, since $p \restriction \alpha$ is $\Sigma_1(J_\alpha^A)$. (Note that ω satisfies Case 1.)

Case 2 $\alpha = \beta + \omega, \beta \in \mathrm{Lm}$ and Case 1 fails.

There is a $\Sigma_1(J_\alpha^A)$ bijection of β onto α defined by:

$$\begin{aligned}
&f(2n) = \beta + n \text{ for } n < \omega \\
&f(2n+1) = n \text{ for } n < \omega \\
&f(\nu) = \nu \text{ for } \omega \leq \nu < \beta
\end{aligned}$$

Let g be a $\underset{\sim}{\Sigma}_1(J^A_\beta)$ partial map of β onto β^2. Set $(\langle\gamma_0,\gamma_1\rangle)_i = \gamma_i$ for $i = 0, 1$.

$$g_i(\nu) \simeq (g(\nu))_i (i = 0, 1).$$

Then $\tilde{f}(\nu) \simeq \langle fg_0(\nu, fg_1(\nu)\rangle$ maps β onto α^2. QED (Case 2)

Case 3 The above cases fail.

Then $p(\alpha) = \langle\nu,\tau\rangle$, where $\nu,\tau < \alpha$. Let $\gamma \in \mathrm{Lm}$ such that $\max(\nu,\tau) < \gamma < \alpha$. Let g be a partial $\underset{\sim}{\Sigma}_1(J^A_\alpha)$ map of γ onto γ^2. Then $g \in M, p^{-1}$ is a partial map of γ^2 onto α; hence $f = p^{-1} \circ g$ is a partial map of γ onto α. Set: $\tilde{f}(\langle\xi,\delta\rangle) \simeq \langle f(\xi), f(\delta)\rangle$ for ξ,δ,γ. Then $\tilde{f}g$ is a partial map of γ onto α^2. QED (Lemma 2.4.6)

We can now prove:

Lemma 2.4.7. *There is a partial $\underset{\sim}{\Sigma}_1(M)$ map of On_M onto M.*

Proof: We again simplify things by taking $M = J^A_\alpha$. Let g be a partial map of α onto α^2 which is $\Sigma_1(J^A_\alpha)$ in the parameters $p \in J^A_\alpha$. Define "ordered pairs" of ordinals $< \alpha$ by:

$$(\nu,\tau) =: g^{-1}(\langle\nu,\tau\rangle).$$

We can then, for each $n \geq 1$, define "ordered n–tuples" by:

$$(\nu) =: \nu, (\nu_1,\ldots,\nu_n) = (\nu_1,(\nu_2,\ldots,\nu_n))(n \geq 2).$$

We know by Lemma 2.4.4 that every $y \in J^A_\alpha$ has the form: $y = f(\nu_1,\ldots,\nu_n)$ where $\nu_1,\ldots,\nu_n < \alpha$ and f is $\Sigma_1(J^A_\alpha)$. Define a function f^* by:

$$y = f^*(\tau) \leftrightarrow \bigvee \nu_1,\ldots,\nu_n(\tau = (\nu_1,\ldots,\nu_n)\wedge \\ \wedge y = f(\nu_1,\ldots,\nu_n)).$$

Then f^* is $\Sigma_1(J^A_\alpha)$ in p and $y \in f^{*\prime\prime}\alpha$. If we set: $h^*(i,x) \simeq h(i,\langle x,p\rangle)$, then each binary relation which is $\Sigma_1(J^A_\alpha)$ in p is uniformized by one of the functions $h^*_i(x) \simeq h^*(i,x)$. Hence $y = h^*(i,\gamma)$ for some $\gamma < \alpha$. Hence $J^A_\alpha = h^{*\prime\prime}(\omega\times\alpha)$. But, setting:

$$y = \hat{h}(\mu) \leftrightarrow \bigvee i,\nu(\mu = (i,\nu) \wedge y = h^*(i,\nu))$$

we see that $\hat{h}$ is $\Sigma_1(J^A_\alpha)$ in p and $y \in \hat{h}''\alpha$. Hence $J^A_\alpha = \hat{h}''\alpha$, where $\hat{h}$ is $\Sigma_1(J^A_\alpha)$ in p. QED (Lemma 2.4.7)

Corollary 2.4.8. *Let $x \in M$. There are $f,\gamma \in J^A_\alpha$ such that f maps γ onto x.*

Proof: We again prove it for $M = J_\alpha^A$. If $\alpha = \omega$ it is trivial since $J_\alpha^A = H_\omega$. If $\alpha \in \mathrm{Lm}^*$ then $x \in J_\beta^A$ for a $\beta < \alpha$ and there is $f \in J_\alpha^A$ mapping β onto J_β^A by Lemma 2.4.7. There remains only the case $\alpha = \beta + \omega$ where β is a limit ordinal. By induction on $n < \omega$ we prove:

Claim There is $f \in J_\alpha^A$ mapping β onto $S_{\beta+n}^A$. If $n = 0$ this follows by Lemma 2.4.7.

Now let $n = m + 1$.
Let $f : \beta \xrightarrow{\text{onto}} S_{\beta+m}^A$ and define f' by $f'(0) = S_{\beta+m}^A, f'(n+1) = f(n)$ for $n < \omega$, $f'(\xi) = f(\xi)$ for $\xi \geq \omega$. Then f' maps β onto $U = S_{\beta+m}^A \cup \{S_{\beta+m}^A\}$ and $S_{\beta+m}^A = \bigcup_{\delta=\beta}^{8} F_i''U^2 \cup \bigcup_{i=0}^{3} G_i''U^3 \cup \{A \cap S_{\beta+m}^A\}$.

Set:

$$g_i = \{\langle F_i(f'(\xi), f'(\zeta)), \langle i, \langle \xi, \zeta\rangle\rangle\rangle | \xi, \zeta < \beta\}$$
for $i = 0, \ldots, 8$
$$g_{8+i+1} = \{\langle G_i | f'(\xi), f'(\zeta), f'(\mu)), \langle 8 + i + 1, \langle \xi, \zeta, \mu\rangle\rangle | \xi, \zeta, \mu < \beta\}$$
for $i = 0, \ldots, 3$
$$g_{13} = \{\langle A \cap S_{\beta+m}^A \langle 13, \emptyset\rangle\rangle\}$$

Then $g = \bigcup_{i=0}^{13} g_i \in J_\alpha^A$ is a partial map of J_β^A onto $S_{\beta+n}^A$ and $gh \in J_\alpha^A$ is a partial map of β onto S_β^A. QED (Corollary 2.4.8)

Define the *cardinal* of x in M by:

Definition 2.4.4. $\overline{\overline{x}} = \overline{\overline{x}}^M$ =: the least γ such that some $f \in M$ maps γ onto x.

Note. this is a non standard definition of cardinal numbers. If M is e.g. *pr* closed, we get that there is $f \in M$ bijecting $\overline{\overline{x}}$ onto x.

Definition 2.4.5. Let $X \subset M$. $h(X) = h_M(X)$ =: The set of all $y \in M$ such that $y = f(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in X$ and f is a $\Sigma_1(M)$ function

Since $\Sigma_1(M)$ functions are closed under composition, it follows easily that $Y = h(X)$ is closed under $\Sigma_1(M)$ functions.

By Corollary 2.4.2 we then have:

Lemma 2.4.9. *Let $Y = h(X)$. Then $M|Y \prec_{\Sigma_1} M$ where*

$$M|Y =: \langle Y, A_1 \cap Y, \ldots, A_n \cap Y, B_1 \cap Y, \ldots, B_m \cap Y\rangle.$$

Note. We shall often ignore the distinction between Y and $M|Y$, writing simply: $Y \prec_{\Sigma_1} M$.

If f is a $\Sigma_1(M)$ function, there is $i < \omega$ such that $h(i, \langle \vec{x} \rangle) \simeq f(\vec{x})$. Hence:

Corollary 2.4.10. $h(X) = \bigcup_{n<\omega} h''(\omega \times X^n)$.

There are many cases in which $h(X) = h''(\omega \times X)$, for instance:

Corollary 2.4.11. $h(\{x\}) = h''(\omega \times \{x\})$.

Gödels pair function on ordinals is defined by:

Definition 2.4.6. $\prec \gamma, \delta \succ =: p^{-1}(\prec \gamma, \delta \succ)$, where p is the function defined in the proof of Lemma 2.4.6.

We can then define *Gödel n–tuples* by iterating the pair function:

Definition 2.4.7. $\prec \gamma \succ =: \gamma; \prec \gamma_1, \ldots, \gamma_n \succ =: \prec \gamma_1, \prec \gamma_2, \ldots, \gamma_n \succ \succ$ $(n \geq 2)$.

Hence any X which is closed under Gödel pairs is closed under the tuple–function. Imitating the proof of Lemma 2.4.7 we get:

Corollary 2.4.12. *If $Y \subset \mathrm{On}_M$ is closed under Gödel pairs, then:*

(a) $h(Y) = h''(\omega \times Y)$

(b) $h(Y \cup \{p\}) = h''(\omega \times (Y \times \{p\}))$ *for* $p \in M$.

Proof: We display the proof of (b). Let $y \in h(Y \cup \{p\})$. Then $y = f(\gamma_1, \ldots, \gamma_n, p)$, where $\gamma_1, \ldots, \gamma_n \in Y$ and f is $\Sigma_1(M)$.

Hence $y = f^*(\langle \delta, p \rangle)$ where $\delta = \prec \gamma_1, \ldots, \gamma_n \succ$ and

$$y = f^*(z) \leftrightarrow \bigvee \gamma_1, \ldots, \gamma_n \bigvee p(z = \langle \prec \gamma_1, \ldots, \gamma_n \succ, p \rangle \wedge \\ \wedge y = f(\vec{\gamma}, p)).$$

Hence $y = h(i, \langle \delta, p \rangle)$ for some i. QED (Corollary 2.4.12)

Similarly we of course get:

Corollary 2.4.13. *If $Y \subset M$ is closed under ordered pairs, then:*

(a) $h(Y) = h''(\omega \times Y)$

(b) $h(Y \cup \{p\}) = h''(\omega \times (Y \times \{p\})$ *for* $p \in M$.

By Lemma 2.4.5 we easily get:

Corollary 2.4.14. *Let $Y \subset \mathrm{On}_M$. Then $h(Y) = h''(\omega \times \mathbb{P}_\omega(Y))$.*

In fact:

Corollary 2.4.15. *Let $A \subset \mathbb{P}_\omega(\mathrm{On}_M)$ be directed (i.e. $a, b \in A \to \bigvee c \in A\ a, b \subset c$). Let $Y = \bigcup A$. Then $h(Y) = h''(\omega \times A)$.*

By the condensation lemma we get:

Lemma 2.4.16. *Let $\pi : \overline{M} \to_{\Sigma_1} M$ where M is a J–model and $\overline{M}$ is transitive. Then $\overline{M}$ is a J–model.*

Proof: $\overline{M}$ is amenable by Σ_1 preservation. But then it is a J–model by the condensation lemma. QED (Lemma 2.4.16)

We can get a theorem in the other direction as well. We first define:

Definition 2.4.8. Let $\overline{M}, M$ be transitive structures. $\sigma : \overline{M} \to M$ *cofinally* iff σ is a structural embedding of $\overline{M}$ into M and $M = \bigcup \sigma''\overline{M}$.

Then:

Lemma 2.4.17. *If $\sigma : \overline{M} \to_{\Sigma_0} M$ cofinally. Then σ is Σ_1 preserving.*

Proof: Let $R(y, \vec{x})$ be $\Sigma_0(M)$ and let $\overline{R}(y, \vec{x})$ be $\Sigma_0(\overline{M})$ by the same definition. We claim:

$$\bigvee y R(y, \sigma(\vec{x})) \to \bigvee y \overline{R}(y, \vec{x})$$

for $x_1, \ldots, x_n \in \overline{M}$. To see this, let $R(y, \sigma(\vec{x}))$. Then $y \in \sigma(u)$ for a $u \in \overline{M}$. Hence $\bigvee y \in \sigma(u) R(y, \sigma(\vec{x}))$, which is a Σ_0 statement about $\sigma(u), \sigma(\vec{x})$. Hence $\bigvee y \in u \overline{R}(y, \vec{x})$. QED (Lemma 2.4.17)

Lemma 2.4.18. *Let $\sigma : \overline{M} \to_{\Sigma_0} M$ cofinally, where $\overline{M}$ is a J–model. Then M is a J–model.*

Proof: Let e.g. $\overline{M} = \langle J_{\overline{\alpha}}^{\overline{A}} \rangle, M = \langle U, A, \overline{B} \rangle$.

Claim 1 $U = J_\alpha^A$ where $\alpha = \mathrm{On}_M$.

> **Proof:** $y = S^{\overline{A}} \restriction \nu$ is a Σ_0 condition, so $\sigma(S^{\overline{A}} \restriction \nu) = S^A \restriction \sigma(\nu)$. But σ takes $\overline{\alpha}$ cofinally to α, so if $\xi < \alpha, \xi < \sigma(\nu)$, then $S_\xi^A(S^A \restriction \sigma(\nu))(\xi) \in U$. Hence $J_\alpha^A \subset U$. To see $U \subset J_\alpha^A$, let $x \in U$. Then $x \in \sigma(u)$ where $u \in J_{\overline{\alpha}}^{\overline{A}}$. Hence $u \subset S_\nu^{\overline{A}}$ and $x \in \sigma(S_\nu^{\overline{A}}) = S_{\sigma(\nu)}^A \subset J_\alpha^A$. QED (Claim 1)

Claim 2 M is amenable.

Let $x \in S^A_{\sigma(\nu)}$. Then $\sigma(\overline{B} \cap S^{\overline{A}}_\nu) = B \cap S^A_{\sigma(\nu)}$ and $x \cap B = (B \cap S^A_\nu) \cap x \in U$, since S^A_ν is transitive. QED (Lemma 2.4.18)

Lemma 2.4.19. *Let $\overline{M}, M$ be J–models. Then $\sigma : \overline{M} \to_{\Sigma_0} M$ cofinally iff $\sigma : \overline{M} \to_{\Sigma_0} M$ and σ takes $\mathrm{On}_{\overline{M}}$ to On_M cofinally.*

Proof: $(\to)$ is obvious. We prove $(\leftarrow)$. The proof of $\sigma(S^{\overline{A}}_\nu) = S^A_{\sigma(\nu)}$ goes through as before. Thus if $x \in M$, we have $x \in S^A_\xi$ for some ξ. Let $\xi \leq \sigma(\nu)$. Then $x \in S^A_{\sigma(\nu)} = \sigma(S^{\overline{A}}_\nu)$. QED (Lemma 2.4.19)

2.5 The Σ_1 projectum

2.5.1 Acceptability

We begin by defining a class of J–models which we call *acceptable*. Every J_α is acceptable, and we shall see later that there are many other naturally occurring acceptable structures. Accepability says essentially that if something dramatic happens to β at some later stage ν of the construction, then ν is, in fact, collapsed to β at that stage:

Definition 2.5.1. $J^{\vec{A}}_\alpha$ is *acceptable* iff for all $\beta \leq \nu < \alpha$ in Lm we have:

(a) If $a \subset \beta$ and $a \in J^{\vec{A}}_{\nu+\omega} \setminus J^{\vec{A}}_\nu$, then $\overline{\overline{\nu}} \leq \beta$ in $J^{\vec{A}}_{\nu+\omega}$.

(b) If $x \in J^{\vec{A}}_\beta$ and ψ is a Σ_1 condition such that $J^{\vec{A}}_{\nu+\omega} \models \psi[\beta, x]$ but $J^{\vec{A}}_\nu \not\models \psi[\beta, x]$, then $\overline{\overline{\nu}} \leq \beta$ in $J^{\vec{A}}_{\nu+\omega}$.
A J–model $\langle J^{\vec{A}}_\alpha, \vec{B} \rangle$ is *acceptable* iff $J^{\vec{A}}_\alpha$ is acceptable.

Note. 'Acceptability' referred originally only to property (a). Property (b) was discovered later and was called 'Σ_1 acceptability'.

In the following we shall always suppose M to be acceptable unless otherwise stated. We recall that by Corollary 2.4.8 every $x \in M$ has a cardinal $\overline{\overline{x}} = \overline{\overline{x}}^M$. We call γ a cardinal in M iff $\gamma = \overline{\overline{\gamma}}$ (i.e. no smaller ordinal is mappable onto γ in M).

Lemma 2.5.1. *Let $M = \langle J^A_\alpha, B \rangle$ be acceptable. Let $\gamma > \omega$ be a cardinal in M. Then:*