Claim 2 M is amenable.

Let $x \in S^A_{\sigma(\nu)}$. Then $\sigma(\overline{B} \cap S^{\overline{A}}_\nu) = B \cap S^A_{\sigma(\nu)}$ and $x \cap B = (B \cap S^A_\nu) \cap x \in U$, since S^A_ν is transitive. QED (Lemma 2.4.18)

Lemma 2.4.19. *Let $\overline{M}, M$ be J–models. Then $\sigma : \overline{M} \to_{\Sigma_0} M$ cofinally iff $\sigma : \overline{M} \to_{\Sigma_0} M$ and σ takes $\mathrm{On}_{\overline{M}}$ to On_M cofinally.*

Proof: $(\to)$ is obvious. We prove $(\leftarrow)$. The proof of $\sigma(S^{\overline{A}}_\nu) = S^A_{\sigma(\nu)}$ goes through as before. Thus if $x \in M$, we have $x \in S^A_\xi$ for some ξ. Let $\xi \leq \sigma(\nu)$. Then $x \in S^A_{\sigma(\nu)} = \sigma(S^{\overline{A}}_\nu)$. QED (Lemma 2.4.19)

2.5 The Σ_1 projectum

2.5.1 Acceptability

We begin by defining a class of J–models which we call *acceptable*. Every J_α is acceptable, and we shall see later that there are many other naturally occurring acceptable structures. Accepability says essentially that if something dramatic happens to β at some later stage ν of the construction, then ν is, in fact, collapsed to β at that stage:

Definition 2.5.1. $J^{\vec{A}}_\alpha$ is *acceptable* iff for all $\beta \leq \nu < \alpha$ in Lm we have:

(a) If $a \subset \beta$ and $a \in J^{\vec{A}}_{\nu+\omega} \setminus J^{\vec{A}}_\nu$, then $\overline{\overline{\nu}} \leq \beta$ in $J^{\vec{A}}_{\nu+\omega}$.

(b) If $x \in J^{\vec{A}}_\beta$ and ψ is a Σ_1 condition such that $J^{\vec{A}}_{\nu+\omega} \models \psi[\beta, x]$ but $J^{\vec{A}}_\nu \not\models \psi[\beta, x]$, then $\overline{\overline{\nu}} \leq \beta$ in $J^{\vec{A}}_{\nu+\omega}$.
A J–model $\langle J^{\vec{A}}_\alpha, \vec{B}\rangle$ is *acceptable* iff $J^{\vec{A}}_\alpha$ is acceptable.

Note. 'Acceptability' referred originally only to property (a). Property (b) was discovered later and was called 'Σ_1 acceptability'.

In the following we shall always suppose M to be acceptable unless otherwise stated. We recall that by Corollary 2.4.8 every $x \in M$ has a cardinal $\overline{\overline{x}} = \overline{\overline{x}}^M$. We call γ a cardinal in M iff $\gamma = \overline{\overline{\gamma}}$ (i.e. no smaller ordinal is mappable onto γ in M).

Lemma 2.5.1. *Let $M = \langle J^A_\alpha, B\rangle$ be acceptable. Let $\gamma > \omega$ be a cardinal in M. Then:*

(a) $\gamma \in \mathrm{Lm}^*$

(b) $J^A_\gamma \prec_{\Sigma_1} J^A_\alpha$

(c) $x \in J^A_\gamma \to M \cap \mathbb{P}(x) \subset J^A_\gamma$.

Proof: We first prove (a). Suppose not. Then $\gamma = \beta + \omega$, where $\beta \in \mathrm{Lm}, \beta \geq \omega$. Then $f \in M$ maps β onto γ where: $f(2i) = i, f(2i+1) = \beta + i, f(\xi) = \xi$ for $\xi \geq \omega$.
Contradiction! QED (a)

If (b) were false, there would be ν such that $\gamma \leq \nu < \alpha$, and for some $x \in J^A_\gamma$ and some Σ_1 formula ψ we have:

$$J^A_{\nu+\omega} \models \psi[x], J^A_\nu \models \neg\psi[x].$$

But then $x \in J^A_\beta$ for some $\beta < \gamma$ in Lm. Hence $\overline{\overline{\gamma}} \leq \overline{\overline{\nu}} \leq \beta$.
Contradiction! QED (b)

To prove (c) suppose not. Then x is not finite. Let $\beta = \overline{\overline{x}}$ in J^A_γ. Then $\beta \geq \omega, \beta \in \mathrm{Lm}$ by (a). Let $f \in J^A_\gamma$ map β onto x. Let $u \subset x$ such that $u \notin J^A_\gamma$. Then $v = f^{-1}{}''u \notin J^A_\gamma$. Let $\nu \geq \gamma$ such that $v \in J^A_{\nu+\omega} \setminus J^A_\nu$. Then $\gamma \leq \overline{\overline{\nu}} \leq \beta$.
Contradiction! QED (Lemma 2.5.1)

Remark We have stated and proven this lemma for M of type $\langle 1, 1\rangle$, since the extension to M of arbitrary type is self evident.

The most general form of GCH says that if $\mathbb{P}(x)$ exists and $\overline{\overline{x}} \geq \omega$, then $\overline{\overline{\mathbb{P}(x)}} = \overline{\overline{x}}^+$ (where α^+ is the least cardinal $> \alpha$).

As a corollary of Lemma 2.5.1 we have:

Corollary 2.5.2. *Let M, γ be as above. Let $a \in M, a \subset J^A_\gamma$. Then:*

(a) $\langle J^A_\gamma, a\rangle$ models the axiom of subsets and GCH.

(b) If γ is a successor cardinal in M, then $\langle J^A_\gamma, a\rangle$ models ZFC^-.

(c) If γ is a limit cardinal in M, then $\langle J^A_\gamma, a\rangle$ models Zermelo set theory.

Proof: (a) follows easily from Lemma 2.5.1 (c). (c) follows from (a) and rud closure of J^A_γ. We prove (b). We know that J^A_γ is rud closed and that the axiom of choice holds in the strong form: $\bigwedge x \bigvee \nu \bigvee f$ f maps ν onto x. We must prove the axiom of collection. Let $R(x, y)$ be $\underline{\Sigma}_\omega(J^A_\gamma)$ and let $u \in J^A_\gamma$ such that $\bigwedge x \in u \bigvee y R(x, y)$.

Claim $\bigvee \nu < \gamma \bigwedge x \in u \bigvee y \in J^A_\nu R(x,y)$. Suppose not.

Let $\gamma = \beta^+$ in M. For each $\nu < \gamma$ there is a partial map $f \in M$ of β onto ν. But then $f \in J^A_\gamma$ since $f \subset \nu \times \beta \in J^A_\gamma$. Set f_ν — the $<_{J^A_\gamma}$ — least such f. For $x \in u$ set:

$$h(x) = \text{ the least } \mu \text{ such that } \bigvee y \in J^A_\mu R(y,x).$$

Then $\sup h''u = \gamma$ by our assumption. Define a partial map k on $u \times \beta$ by: $k(x,\xi) \simeq f_{h(x)}(\xi)$. Then k is onto γ. But $k \in M$, since k is $\underline{\Sigma}_1(J^A_\gamma)$. Clearly $\overline{\overline{u \times \beta}} = \beta$ in M, so $\overline{\overline{\gamma}} \leq \beta < \gamma$ in M.

Contradiction! QED (Corollary 2.5.2)

Corollary 2.5.3. *Let M, γ be as above. Then*

$$J^A_\gamma = H^M_\gamma =: \bigcup \{u \in M | u \textit{ is transitive } \wedge \overline{\overline{u}} < \gamma \textit{ in } M\}.$$

Proof: Let $u \in M$ be transitive and $\overline{\overline{u}} < \gamma$ in M. It suffices to show that $u \in J^A_\gamma$. Let $\nu = \overline{\overline{u}} < \gamma$ in M. Let $f \in M$ map ν onto u. Set:

$$r = \{\langle \xi, \delta \rangle \in \nu^2 | f(\xi) \in f(\delta)\}.$$

Then $r \in J^A_\gamma$ by Lemma 2.5.1 (c), since $\nu^2 \in J^A_\gamma$. Let $\beta = \overline{\overline{\nu}}^+ =$ the least cardinal $> \nu$ in M. then J^A_β models ZFC^- and $r, \nu \in J^A_\beta$. But then $f \in J^A_\beta \subset J^A_\gamma$, since f is defined by recursion on $r : f(x) = f''r''\{x\}$ for $x \in \nu$. Hence $u = \mathrm{rng}(f) \in J^A_\gamma$. QED (Corollary 2.5.3)

Lemma 2.5.4. *If $\pi : \overline{M} \to_{\Sigma_1} M$ and M is acceptable, then so is $\overline{M}$.*

Proof: $\overline{M}$ is a J–model by §4. Let e.g. $M = J^A_\alpha, \overline{M} = J^{\overline{A}}_{\overline{\alpha}}$. Then $\overline{M}$ has a counterexample — i.e. there are $\overline{\nu} < \overline{\alpha}, \overline{\beta} < \overline{\nu}, \overline{a}$ such that $\mathrm{card}(\overline{\nu}) > \overline{\beta}$ in $J_{\overline{\nu}+\omega}$ and either $\overline{a} \subset \overline{\beta}$ and $\overline{a} \in J^{\overline{\mathbb{A}}}_{\overline{\nu}+\omega} \setminus J^{\overline{A}}_{\overline{\nu}}$ or else $\overline{a} \in J^{\overline{A}}_{\overline{\beta}}, J^{\overline{A}}_{\overline{\nu}+\omega} \models \psi[\overline{a}, \overline{\beta}]$ and $J^{\overline{A}}_{\overline{\nu}} \models \neg\psi[\overline{a}, \overline{\beta}]$, where ψ is Σ_1. But then letting $\pi(\overline{\beta}, \overline{\nu}, \overline{a}) = \beta, \nu, a$ it follows easily that β, ν, a is a counterexample in M.

Contradiction! QED (Lemma 2.5.4)

Lemma 2.5.5. *If $\pi : \overline{M} \to_{\Sigma_0} M$ cofinally and $\overline{M}$ is acceptable, then so is M.*

Proof: M is a J–model by §4. Let $M = J^A_\alpha, \overline{M} = J^{\overline{A}}_{\overline{\alpha}}$.

Case 1 $\overline{\alpha} = \omega$.

Then $\overline{M} = M = J^A_\omega, \pi = \mathrm{id}$.

Case 2 $\overline{\alpha} \in \mathrm{Lm}^*$.

Then "$\overline{M}$ is acceptable" is a $\Pi_1(\overline{M})$ condition. But then $\alpha \in \mathrm{Lm}^*$ and M must satisfy the same Π_1 condition.

Case 3 $\overline{a} = \overline{\beta} + \omega, \overline{\beta} \in \mathrm{Lm}$.

Then $\alpha = \beta + \omega, \beta \in \mathrm{Lm}$ and $\beta = \pi(\overline{\beta})$. Then $J^A_\beta = \pi(J^{\overline{A}}_{\overline{\beta}})$ is acceptable, so there can be no counterexample $\langle \delta, \nu, a\rangle \in J^A_\beta$.

We show that there can be no counterexample of the form $\langle \delta, \beta, a\rangle$. Let $\overline{\gamma} = \mathrm{card}(\overline{\beta})$ in $\overline{M}$. The statement $\mathrm{card}(\overline{\beta}) \leq \overline{\gamma}$ is $\Sigma_1(M)$. Hence $\mathrm{card}(\beta) \leq \gamma = \pi(\overline{\gamma})$ in M. Hence there is no counterexample $\langle \delta, \beta, a\rangle$ with $\delta \geq \gamma$. But since $\overline{M}$ is acceptable and $\overline{\gamma} \leq \overline{\beta}$ is a cardinal in $\overline{M}$, the following Π_1 statements hold in $\overline{M}$ by Lemma 2.5.1

$$\bigwedge \delta < \overline{\gamma} \bigwedge a \subset \delta a \in J^{\overline{A}}_{\overline{\gamma}}$$

$$\bigwedge \delta < \overline{\gamma} \bigwedge x \in J^{\overline{A}}_{\delta} (\bigvee y R(x, \delta) \to \bigvee y \in J^{\overline{A}}_{\overline{\gamma}})$$

where R is $\Sigma_0(\overline{M})$.

But then the corresponding statements hold in M. Hence $\langle \delta, \beta, a\rangle$ cannot be a counterexample for $\delta < \gamma$. QED (Lemma 2.5.5)

2.5.2 The projectum

We now come to a central concept of fine structure theory.

Definition 2.5.2. Let M be acceptable. The Σ_1*–projectum* of M (in symbols ρ_M) is the least $\rho \leq \mathrm{On}_M$, such that there is a $\underline{\Sigma}_1(M)$ set $a \subset \rho$ with $a \notin M$.

Lemma 2.5.6. *Let $M = \langle J^A_\alpha, B\rangle, \rho = \rho_M$. Then*

(a) If $\rho \in M$, then ρ is cardinal in M.

(b) If D is $\underline{\Sigma}_1(M)$ and $D \subset J^A_\rho$, then $\langle J^A_\rho, D\rangle$ is amenable.

(c) If $u \in J^A_\rho$, there is no $\underline{\Sigma}_1(M)$ partial map of u onto J^A_ρ.

(d) $\rho \in \mathrm{Lim}^*$

Proof:

(a) Suppose not. Then there are $f \in M$, $\gamma < \rho$ such that f maps γ onto ρ. Let $a \subset \rho$ be $\underline{\Sigma}_1(M)$ such that $a \notin M$. Set $\tilde{a} = f^{-1}{}''a$. Then $\tilde{a}$ is $\Sigma_1(M)$

and $\tilde{a} \subset \gamma$. Hence $\tilde{a} \in M$. But then $a = f''\tilde{a} \in M$ by rud closure.
Contradiction! QED (a)

(b) Suppose not. Let $u \in J^A_\rho$ such that $D \cap u \notin J^A_\rho$. We first note:

Claim $D \cap u \notin M$.

If $\rho = \alpha$ this is trivial, so let $\rho < \alpha$. Then ρ is a cardinal by (a) and by Lemma 2.5.1 we know that $\mathbb{P}(u) \cap M \subset J^A_\rho$. QED (Claim)

By Corollary 2.5.2 there is $f \in J^A_\rho$ mapping a $\nu < \rho$ onto u. Then $d = f^{-1u}(D \cap u)$ is $\underline{\Sigma}_1(M)$ and $d \subset \nu < \rho$. Hence $d \in M$. Hence $D \cap u = f''d \in M$ by rud closure. QED (b)

(c) Suppose not. Let f ba a counterexample. Set $a = \{x \in u | x \in \mathrm{dom}(f) \wedge x \notin f(x)\}$. Then a is $\underline{\Sigma}_1(M)$, $a \subset u \in M$. Hence $a \in J^A_\rho$ by (b). Let $a = f(x)$. Then $x \in f(x) \leftrightarrow x \notin f(x)$.
Contradiction! QED (c)

(d) If not, then $\rho = \beta + \omega$ where $\beta \in \mathrm{Lim}$. But then there is a $\underline{\Sigma}_1(M)$ partial map of β onto ρ, violating (c). QED (Lemma 2.5.6)

Remark We have again stated and proven the theorem for the special case $M = \langle J^A_\alpha, B\rangle$, since the general case is then obvious. We shall continue this practice for the rest of the book. A *good parameter* is a $p \in M$ which witnesses that $\rho = \rho_M$ is the projectum — i.e. there is $B \subset M$ which is $\Sigma_1(M)$ in p with $B \cap H^M_\rho \notin M$. But by §3 any $p \in M$ has the form $p = f(a)$ where f is a $\Sigma_1(M)$ function and a is a finite set of ordinals. Hence a is good if p is. For technical reasons we shall restrict ourselves to good parameters which are finite sets of ordinals:

Definition 2.5.3. $P = P_M =:$ The set of $p \in [\mathrm{On}_M]^{<\omega}$ which are good parameters.

Lemma 2.5.7. *If $p \in P$, then $p \setminus \rho_M \in P$.*

Proof: It suffices to show that if $\nu = \min(p)$ and $\nu < \rho$, then $p' = p \setminus (\nu + 1) \in P$. Let B be $\Sigma_1(M)$ in p such that $B \cap H^M_\rho \notin M$. Let $B(x) \leftrightarrow B'(x, p)$ where B' is $\Sigma_1(M)$.

Set:
$$B^*(x) \leftrightarrow: \bigvee z \bigvee \nu (x = \langle z, \nu\rangle \wedge B'(z, p' \cup \{\nu\})).$$
Then $B^* \cap H_\rho \notin M$, since otherwise
$$B \cap H_\rho = \{x | \langle x, \nu\rangle \in B^* \cap H_\rho\} \in M.$$

Contradiction! QED (Lemma 2.5.7)

For any $p \in [\mathrm{On}_M]^{<\omega}$ we define the *standard code* T^p *determined by* p as:

Definition 2.5.4.

$$T^p = T^p_M =: \{\langle i, x\rangle | \models_M \varphi_i[x, p]\} \cap H^M_{\rho_M}\}$$

where $\langle \varphi_i | i < \omega\rangle$ is a fixed recursive enumeration of the Σ_1–fomulae.

Lemma 2.5.8. *$p \in P \leftrightarrow T^p \notin M$.*

Proof:

($\leftarrow$) $T^p = T \cap H^M_p$ for a T which is $\Sigma_1(M)$ in p.

($\rightarrow$) Let B be $\Sigma_1(M)$ in p such that $B \cap H^M_p \notin M$. Then for some i:

$$B(x) \leftrightarrow \langle i, x\rangle \in T^p$$

for $x \in H^M_p$. Hence $T^p \notin M$. QED (Lemma 2.5.8)

A parameter p is *very good* if every element of M is Σ_1 definable from parameters in $\rho_M \cup \{p\}$. R is the set of very good parameters lying in $[\mathrm{On}_M]^{<\omega}$.

Definition 2.5.5. $R = R_M =:$ the set of $r \in [\mathrm{On}_M]^{<\omega}$ such that $M = h_M(\rho_M \cup \{r\})$.

Note. This is the same as saying $M = h_M(\rho_M \cup r)$, since

$$h(\rho \cup r) = h"(\omega \times [\rho \cup r]^{<\omega}).$$

But $\rho \cup r = \rho \cup (r \setminus \rho)$. Hence:

Lemma 2.5.9. *If $r \in R$, then $r \setminus \rho \in R$. We also note:*

Lemma 2.5.10. *$R \subset P$.*

Proof: Let $r \in R$. We must find $B \subset M$ such that B is $\Sigma_1(M)$ in r and $B \cap H^M_\rho \notin M$. Set:

$$B = \{\langle i, x\rangle | \bigvee y\, y = h(i, \langle x, r\rangle) \wedge \langle i, x\rangle \notin y\}.$$

If $b = B \cap H^M_\rho \in M$, then $b = h(i, \langle x, r\rangle)$ for some i. Then $\langle i, x\rangle \in b \leftrightarrow \langle i, x\rangle \notin b$.
Contradiction! QED (Lemma 2.5.10)

However, R can be empty.

Lemma 2.5.11. *There is a function h^r uniformly $\Sigma_1(M)$ in r such that whenever $r \in R_M$, then $M = h^{r\prime\prime}\rho_M$.*

Proof: Let $x \in M$. Since $x \in h(\rho \cup \{r\})$ there is an f which is $\Sigma_1(M)$ in r such that $x = f(\xi_1, \dots, \xi_n)$. But ρ is closed under Gödel pairs, so $x = f'(\prec \xi_1, \dots, \xi_n \succ)$, where

$$x = f'(\xi) \leftrightarrow \bigvee \xi_1, \dots, \xi_n(\xi = \prec \vec{\xi} \succ \wedge x = f(\vec{\xi})).$$

f' is $\Sigma_1(M)$ in r. Hence $x = h(i, \langle\langle\vec{\xi}\rangle, r\rangle)$ for some $i < \omega$. Set

$$x = h^r(\delta) \leftrightarrow \bigvee \xi \bigvee i < \omega(\delta = \langle i, \xi\rangle \wedge x = h(i, \langle \xi, r\rangle)).$$

Then $x = h^r(\langle i, \langle\vec{\xi}\rangle\rangle)$. QED (Lemma 2.5.11)

Lemma 2.5.11 explains why we called T^p a *code*: If $r \in R$, then T^r gives complete information about M. Thus the relation $\in' = \{\langle x, \tau\rangle | h^r(\nu) \in h^r(\tau)\}$ is rud in T^r, since $\nu \in' \tau \leftrightarrow \langle i, \langle \nu, \tau\rangle\rangle \in T^r$ for some $i < \omega$. Similarly, if $M = \langle J_\alpha^{\vec{A}}, \vec{B}\rangle$, then $A_i' = \{\nu | h^r(\nu) \in A_i\}$ and $B_j' = \{\nu | h^r(\nu) \in B_i\}$ are rud in T^r (as is, indeed, R' whenever R is a relation which is $\Sigma_1(M)$ in p). Note, too, that if $B \subset H_\rho^M$ is $\underline{\Sigma}_1(M)$, then B is rud in T^r. However, if $p \in P^1 \setminus R^1$, then T^p does not completely code M.

Definition 2.5.6. Let $p \in [\mathrm{On}_M]^{<\omega}$. Let $M = \langle J_\alpha^{\vec{A}}, \vec{B}\rangle$.

The *reduct of M by p* is defined to be

$$M^p =: \langle J_{\rho_M}^{\vec{A}}, T_M^p\rangle.$$

Thus M^p is an acceptable model which — if $p \in R_M$ — incorporates complete information about M.

The *downward extension of embeddings lemma* says:

Lemma 2.5.12. *Let $\pi : N \to_{\Sigma_0} M^p$ where N is a J–model and $p \in [\mathrm{On}_M]^{<\omega}$.*

(a) There are unique $\overline{M}, \overline{p}$ such that $\overline{M}$ is acceptable, $\overline{p} \in R_{\overline{M}}, N = \overline{M}^{\overline{p}}$.

(b) There is a unique $\tilde{\pi} \supset \pi$ such that $\tilde{\pi} : \overline{M} \to_{\Sigma_0} M$ and $\pi(\overline{p}) = p$.

(c) $\tilde{\pi} : \overline{M} \to_{\Sigma_1} M$.

Proof: We first prove the existence claim. We then prove the uniqueness claimed in (a) and (b).

Let e.g. $M = \langle J^A_\alpha, B\rangle, M^p = \langle J^A_\rho, T\rangle, N = \langle J^{\overline{A}}_{\overline{\rho}}, \overline{T}\rangle$. Set: $\tilde{\rho} = \sup \pi''\overline{\rho}$, $\tilde{M} = M^p|\tilde{\rho} = \langle J^A_{\tilde{\rho}}, \tilde{T}\rangle$ where $\tilde{T} = T \cap J^A_{\tilde{\rho}}$. Set $X = \text{rng}(\pi)$, $Y = h_M(X \cup \{p\})$. Then $\tilde{\pi} : N \to_{\Sigma_0} \tilde{M}$ cofinally by §4.

(1) $Y \cap \tilde{M} = X$
Proof: Let $y \in Y \cap \tilde{M}$. Since X is closed under ordered pairs, we have $y = f(x,p)$ where $x \in X$ and f is $\Sigma_1(M)$. Then

$$\begin{aligned} y = f(x,p) \quad &\leftrightarrow \models_M \varphi_i[\langle y, x\rangle, p] \\ &\leftrightarrow \langle i, \langle y, x\rangle\rangle \in \tilde{T}. \end{aligned}$$

Since $X \prec_{\Sigma_1} \tilde{M}$, there is $y \in X$ such that $\langle i, \langle y, x\rangle\rangle \in \tilde{T}$. Hence $y = f(x, \rho) \in X$. QED (1)

Now let $\tilde{\pi} : \overline{M} \overset{\sim}{\leftrightarrow} Y$, where $\overline{M}$ is transitive. Clearly $p \in Y$, so let $\tilde{\pi}(\overline{p}) = p$. Then:

(2) $\tilde{\pi} : \overline{M} \to_{\Sigma_1} M,\ \tilde{\pi} \upharpoonright N = \pi,\ \tilde{\pi}(\overline{p}) = p.$
But then:

(3) $\overline{M} = h_{\overline{M}}(N \cup \{\overline{p}\})$.
Proof: Let $y \in \overline{M}$. Then $\tilde{\pi}(y) \in Y = h_M''(\omega x (X x \{p\}))$, since X is closed under ordered pairs. Hence $\tilde{\pi}(y) = h_M(i, \langle \pi(x), p\rangle)$ for an $x \in \overline{M}$. Hence $y = h_{\overline{M}}(i, \langle x, \overline{p})$. QED (3)

(4) $\overline{\rho} \geq \rho_{\overline{M}}$.
Proof: It suffices to find a $\Sigma_1(\overline{M})$ set b such that $b \subset N$ and $b \notin \overline{M}$. Set

$$\begin{aligned} b = \{\langle i, x\rangle \in \omega \times N | \bigvee y \quad &(y = h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \\ &\wedge \langle i, x\rangle \notin y)\} \end{aligned}$$

If $b \in \overline{M}$, then $b = h_{\overline{M}}(i, \langle x, \overline{p}\rangle)$ for some $x \in N$. Hence

$$\langle i, x\rangle \in b \leftrightarrow \langle i, x\rangle \notin b.$$

Contradiction! QED (4)

(5) $\overline{T} = \{\langle i, x\rangle \in \omega \times N | \models_{\overline{M}} \varphi_i[i, \langle x, p\rangle]\}$.
Proof: $\overline{T} \subset \omega \times N$, since $\tilde{T} \subset \omega \times \tilde{M}$. But for $\langle i, x\rangle \in \omega \times N$ we have:

$$\begin{aligned} \langle i, x\rangle \in \overline{T} \quad &\leftrightarrow \langle i, \pi(x)\rangle \in \tilde{T} \\ &\leftrightarrow M \models \varphi_i[\langle (x), p\rangle] \\ &\leftrightarrow \overline{M} \models \varphi_i[\langle x, p\rangle] \text{ by (2)} \end{aligned}$$

QED (5)

(6) $\overline{\rho} = \rho_{\overline{M}}$.
Proof: By (4) we need only prove $\overline{\rho} \le \rho_{\overline{M}}$. It suffices to show that if $b \subset N$ is $\underline{\Sigma}_1(\overline{M})$, then $\langle J^{\overline{A}}_{\overline{\rho}}, b\rangle$ is amenable. By (3) b is $\Sigma_1(\overline{M})$ in $x, \overline{p}$ where $x \in \overline{N}$.
Hence
$$\begin{aligned} b &= \{z | \overline{M} \models \varphi_i[\langle z, x\rangle, \overline{p}]\} = \\ &= \{z | \langle i, z, x\rangle \in \overline{T}\} \end{aligned}$$
Hence b is rud in $\overline{T}$ where $N = \langle J^{\overline{A}}_{\overline{\rho}}, \overline{T}\rangle$ is amenable. QED (6)

But then $\overline{M} = h_{\overline{M}}(\overline{\rho} \cup \{\overline{p}\})$ by (3) and the fact that $h_{J^{\overline{A}}_{\overline{\rho}}}(\overline{\rho}) = J^{\overline{A}}_{\overline{\rho}}$. Hence

(7) $\overline{p} \in R_{\overline{M}}$.
By (6) we then conclude:

(8) $N = \overline{M}^{\overline{p}}$.
This proves the existence assertions. We now prove the uniqueness assertion of (a). Let $\hat{M}^{\hat{p}} = N$ where $\hat{p} \in R_{\hat{M}}$.
We claim: $\hat{M} = \overline{M}$, $\hat{p} = \overline{p}$.

Since the Skolem function is uniformly Σ_1 there is a $j < \omega$ such that
$$\begin{aligned} & h_{\hat{M}}(i, \langle x, \hat{p}\rangle) \in h_{\hat{M}}(i, \langle y, \hat{p}\rangle) \leftrightarrow \\ & \quad \leftrightarrow \hat{M} \models \varphi_j[\langle x, y\rangle, p] \leftrightarrow \langle j, \langle x, y\rangle\rangle \in \overline{T} \\ & \quad \leftrightarrow h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \in h_{\overline{M}}(i, \langle y, \overline{p}\rangle) \end{aligned}$$
Similarly:
$$\begin{aligned} h_{\hat{M}}(i, \langle x, \hat{p}\rangle) \in \hat{A} \leftrightarrow h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \in \overline{A} \\ h_{\hat{M}}(i, \langle x, \hat{p}\rangle) \in \hat{B} \leftrightarrow h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \in \overline{B} \end{aligned}$$
where $\hat{M} = \langle J^{\hat{A}}_{\hat{\alpha}}, \hat{B}\rangle$, $\overline{M} = \langle J^{\overline{A}}_{\overline{\alpha}}, \overline{B}\rangle$. Then there is an isomorphism $\sigma : \hat{M} \overset{\sim}{\leftrightarrow} \overline{M}$ defined by $\sigma(h_{\hat{M}}(i, \langle x, \hat{p}\rangle) \simeq h_{\overline{M}}(i, \langle x, \overline{p}\rangle)$ for $x \in N$. Clearly $\sigma(\hat{p}) = \overline{p}$. Hence $\sigma = \mathrm{id}, \hat{M}, \overline{M}, \hat{p} = \overline{p}$, since $\overline{M}, \hat{M}$ are transitive.

We now prove (b). Let $\hat{\pi} \supset \pi$ such that $\hat{\pi} : \overline{M} \to_{\Sigma_0} M$ and $\hat{\pi}(\overline{p}) = p$. If $x \in N$ and $h_{\overline{M}}(i, \langle x, \overline{p}\rangle)$ is defined, it follows that:
$$\hat{\pi}(h_{\overline{M}}(i, \langle x, \overline{p}\rangle) = h_M(i, \langle \pi(x), p\rangle) = \tilde{\pi}(h_M(i, \langle x, \overline{p}\rangle)).$$
Hence $\hat{\pi} = \pi$. QED (Lemma 2.5.12)

If we make the further assumption that $p \in R_M$ we get a stronger result:

Lemma 2.5.13. *Let $M, N, \overline{M}, \pi, \overline{\pi}, p, \overline{p}$ be as above where $p \in R_M$ and $\pi : N \to_{\Sigma_l} M^p$ for an $l < \omega$. Then $\tilde{\pi} : \overline{M} \to_{\Sigma_{l+1}} M$.*

Proof: For $l = 0$ it is proven, so let $l \geq 1$ and let it hold at l. Let R be $\Sigma_{l+1}(M)$ if l is even and $\Pi_{l+1}(M)$ if l is odd. Let $\overline{R}$ have the same definition over $\overline{M}$. It suffices to show:

$$\overline{R}(\vec{x}) \leftrightarrow R(\tilde{\pi}(\vec{x})) \text{ for } x_1, \ldots, x_n \in \overline{M}.$$

But:

$$R(\vec{x}) \leftrightarrow Q_1 y_1 \in M \ldots Q_l y_l \in M R'(\vec{y}, \vec{x})$$

and

$$\overline{R}(\vec{x}) \leftrightarrow Q_1 y_1 \in \overline{M} \ldots Q_l y_l \in \overline{M}\overline{R}'(\vec{y}, \vec{x})$$

where $Q_1 \ldots Q_l$ is a string of alternating quantifiers, R' is $\Sigma_1(M)$, and $\overline{R}'$ is $\Sigma_1(\overline{M})$ by the same definition. Set

$$D =: \{\langle i, x\rangle \in \omega \times J_\rho^A | h_M(i, \langle x, p\rangle) \text{ is defined}\}$$
$$\overline{D} =: \{\langle i, x\rangle \in \omega \times J_{\overline{\rho}}^{\overline{A}} | h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \text{ is defined}\}.$$

Then D is $\Sigma_1(M)$ in p and $\overline{D}$ is $\Sigma_1(\overline{M})$ in $\overline{p}$ by the same definition. Then D is rud in T_M^p and $\overline{D}$ is rud in $T_{\overline{M}}^{\overline{p}}$ by the same definition, since for some $j < \omega$ we have:

$$\langle i, x\rangle \in D \leftrightarrow \langle j, x\rangle \in T_M^p,\ x \in \overline{D} \leftrightarrow \langle j, x\rangle \in T_{\overline{M}}^{\overline{p}}.$$

Define k on D

$$k(\langle i, x\rangle) = h_M(i, \langle x, p\rangle);\ \overline{k}(\langle i, x\rangle) = h_{\overline{M}}(i, \langle x, \overline{p}\rangle).$$

Set:

$$P(\vec{w}, \vec{z}) \leftrightarrow (\vec{w}, \vec{z} \in D \wedge R'(k(\vec{w}), k(\vec{z}))$$
$$\overline{P}(\vec{w}, \vec{z}) \leftrightarrow (\vec{w}, \vec{z} \in \overline{D} \wedge \overline{R}'(\overline{k}(\vec{w}), \overline{k}(\vec{z}))$$

Then: as before, P is rud in T_M^p and $\overline{D}$ is rud in $T_{\overline{M}}^{\overline{p}}$ by the same definition. Now let $x_i = k(z_i)$ for $i = 1, \ldots, n$. Then $\tilde{\pi}(x_i) = k(\pi(z_i))$. But since π is Σ_l–preserving, we have:

$$\begin{aligned} \overline{R}(\vec{x}) \quad &\leftrightarrow Q_1 w_1 \in \overline{D} \ldots Q_l w_l \in \overline{D}\ \overline{P}(\vec{w}, \vec{z}) \\ &\leftrightarrow Q_1 w_1 \in D \ldots Q_l w_l \in D P(\vec{w}, \pi(\vec{z})) \\ &\leftrightarrow R(\tilde{\pi}(\vec{x})) \end{aligned}$$

QED (Lemma 2.5.13)

2.5.3 Soundness and iterated projecta

The reduct of an acceptable structure is itself acceptable, so we can take its reduct etc., yielding a sequence of reducts and nonincreasing projecta $\langle\rho_M^n|n<\omega\rangle$. this is the classical method of doing fine structure theory, which was used to analyse the constructible hierarchy, yielding such results as the $\Box$ principles and the covering lemma. In this section we expound the basic elements of this classical theory. As we shall see, however, it only works well when our acceptable structures have a property called *soundness*. In this book we shall often have to deal with unsound structures, and will, therefore, take recourse to a further elaboration of fine structure theory, which is developed in §2.6.

It is easily seen that:

Lemma 2.5.14. *Let $p \in R_M$. Let B be $\underline{\Sigma}_1(M)$. Then $B \cap J_\rho^A$ is* rud *in parameters over M^p.*

Proof: Let B be Σ_1 in r, where $r = h_M(i, \langle v, p\rangle)$ and $\nu < \rho$. Then B is Σ_1 in ν, p. Let:

$$B(x) \leftrightarrow M \models \varphi_i[\langle x, \nu\rangle, p]$$

where $\langle\varphi_i|i<\omega\rangle$ is our canonical enumeration of Σ_1 formulae. Then:

$$x \in B \leftrightarrow \langle i, \langle x, \nu\rangle\rangle \in T^p$$

QED(Lemma 2.5.14)

It follows easily that:

Corollary 2.5.15. *Let $p, q \in R_M$. Let $D \subset J_\rho^A$. Then D is $\underline{\Sigma}_1(M^p)$ iff it is $\underline{\Sigma}_1(M^q)$.*

Assuming that $R_M \neq \emptyset$, there is then a uniquely defined *second projectum* defined by:

Definition 2.5.7. $\rho_M^2 \simeq: \rho_{M^p}$ for $p \in R_M$.

We can then define:

$$\begin{aligned} R_M^2 =: \quad & \text{The set of } a \in [\mathrm{On}_M]^{<w} \text{ such that} \\ & a \in R_M \text{ and } a \cap \rho \in R_{M^{(a\setminus\rho)}}. \end{aligned}$$

If $R_M^2 \neq \emptyset$ we can define the *second reduct:*

$$M^{2,a} =: (M^a)^{a\cap\rho} \text{ for } a \in R_M^2.$$

But then we can define the *third projectum:*

$$\rho^3 = \rho_{M^{2,a}} \text{ for } a \in R_M^2.$$

Carrying this on, we get R_M^n, $M^{n,a}$ for $a \in R_M^n$ and ρ^{n+1}, as long as $R_M^n \neq \emptyset$. We shall call M *weakly n–sound* if $R_M^n \neq \emptyset$.

The formal definitions are as follows:

Definition 2.5.8. Let $M = \langle J_\alpha^A, B\rangle$ be acceptable.

By induction on n we define:

- The set R_M^n of *very good n–parameters.*
- If $R_M^n \neq \emptyset$, we define the $n+1$*st projectum* ρ_M^{n+1}.
- For all $a \in R_M^n$ the *n–th reduct* $M^{n,a}$.

We inductively verify:

* If $D \subset J_{\rho^n}^A$ and $a, b \in R^n$, then D is $\underline{\Sigma}_1(M^{n,a})$ iff it is $\underline{\Sigma}_1(M^{n,b})$.

Case 1 $n = 0$. Then $R^0 =: [\mathrm{On}_M]^{<\omega}, \rho^0 = \mathrm{On}_M, M^{0,a} = M$.

Case 2 $n = m+1$. If $R^m = \emptyset$, then $R^n = \emptyset$ and ρ^n is undefined. Now let $R^m \neq \emptyset$. Since (*) holds at m, we can define

- $\rho^n =: \rho_{M^{m,a}}$ whenever $a \in R^m$.
- $R^n =:$ the set of $a \in [\alpha]^{<\omega}$ such that $a \in R^m$ and $a \cap \rho^m \in R_{M^{m,a}}$.
- $M^{n,a} =: (M^{m,a})^{a\cap\rho^m}$ for $a \in R^n$.

Note. It follows inductively that $a \setminus \rho^n \in R^n$ whenever $a \in R^n$.

We now verify (*). It suffices to prove the direction ($\rightarrow$). We first note that $M^{n,a}$ has the form $\langle J_{\rho n}^A, T\rangle$, where T is the restriction of a $\underline{\Sigma}_1(M^{m,a})$ set T' to $J_{\rho n}^A$. But then T' is $\underline{\Sigma}_1(M^{m,b})$ by the induction hypothesis. Hence T is rudimentary in parameters over $M^{n,b} = (M^{m,b})^{b\cap\rho^n}$ by Lemma 2.5.14.

Hence, if $D \subset J_{\rho n}^A$ is $\underline{\Sigma}_1(M^{n,a})$, it is also $\underline{\Sigma}_1(M^{n,b})$. QED

This concludes the definition and the verification of (*). Note that $R_M^1 = R_M$, $\rho^1 = \rho_M^1$, and $M^{1,a} = M^a$ for $a \in R_M$.

We say that M is *weakly n–sound* iff $R^n_M \neq \emptyset$. It is *weakly sound* iff it is weakly n–sound for $n < \omega$. A stronger notion is that of *full soundness*:

Definition 2.5.9. M is *n–sound* (or *fully n– sound*) iff it is weakly n–sound and for all $i < n$ we have: If $a \in R^i$, then $P_{M^{i,a}} = R_{M^{i,a}}$.

Thus $R_M = P_M$, $R_{M^{1,a}} = P_{M^{1,a}}$ for $a \in P_M$ etc. If M is n–sound we write P^i_M for $R^i_M (i \leq n)$, since then: $a \in P^{i+1} \leftrightarrow (a \smallsetminus \rho^i \in P^i \wedge a \cap \rho^i \in R_{M^{i, a\cap\rho^i}}$ for $i < n$).

There is an alternative, but equivalent, definition of soundness in terms of *standard parameters*. in order to formulate this we first define:

Definition 2.5.10. Let $a, b \in [\mathrm{On}]^{<\omega}$.

$$a <_* b \leftrightarrow= \bigvee \mu (a \setminus \mu = b \setminus \mu \wedge \mu \in b \setminus a).$$

Lemma 2.5.16. *$<_*$ is a well ordering of* $[\mathrm{On}]^{<\omega}$.

Proof: It suffices to show that every non empty $A \subset [\mathrm{On}]^{<\omega}$ has a unique $<_*$–minimal element. Suppose not. We derive a contradiction by defining an infinite descending chain of ordinals $\langle \mu_i | i < \omega \rangle$ with the properties:

- $\{\mu_0, \ldots, \mu_n\} \leq_* b$ for all $b \in A$.
- There is $b \in A$ such that $b \setminus \mu_n = \{\mu_0, \ldots, \mu_n\}$.

$\emptyset \notin A$, since otherwise $\emptyset$ would be the unique minimal element, so set: $\mu_0 = \min\{\max(b) | b \in A\}$. Given μ_n we know that $\{\mu_0, \ldots, \mu_n\} \notin A$, since it would otherwise be the $<_*$–minimal element. Set:

$$\mu_{n+1} = \min\{\max(b \cap \mu_n) | b \in A \cap b \setminus \mu_n = \{\mu_0, \ldots, \mu_n\}\}.$$

QED (Lemma 2.5.16)

Definition 2.5.11. The *first standard parameter* p_M is defined by:

$$p_M =: \text{ The } <_*\text{–least element of } P_M.$$

Lemma 2.5.17. $P_M = R_M$ *iff* $p_M \in R_M$.

Proof: $(\rightarrow)$ is trivial. We prove $(\leftarrow)$. Suppose not. Then there is $r \in P \setminus R$. Hence $p <_* r$, where $p = p_M$. Hence in M the statement:

(1) $\bigvee q <_* r\ r = h(i, \langle \nu, q\rangle)$

holds for some $i < \omega$, $\nu < p_M$. Form M^r and let $\overline{M}, \overline{r}, \pi$ be such that $\overline{M}^{\overline{r}} = M^r$, $\overline{r} \in R_{\overline{M}}$, $\pi : \overline{M} \to_{\Sigma_1} M$, and $\pi(\overline{r}) = r$. The statement (1) then holds of $\overline{r}$ in $\overline{M}$.

Let $\overline{q} \in \overline{M}$, $\overline{r} = h_{\overline{M}}(i, \overline{q})$ where $\overline{q} <_* \overline{r}$. Set $q = \pi(\overline{q})$. Then $r = h(i, q)$ in M, where $q <_* r$. Hence $q \in P_M$. But then $q \in R_M$ by the minimality of r. This impossible however, since

$$q \in \pi''\overline{M} = h_M(\rho_M \cup r) \neq M.$$

Contradiction! QED (Lemma 2.5.17)

Definition 2.5.12. The n–th *standard parameter* p^n_M is defined by induction on n as follows:

Case 1 $n = 0$. $p^0 = \emptyset$.

Case 2 $n = m + 1$. If $p^m \in R^m$
$p^n = p^m \cup p_{M^{m,p^m}}$

Note. that we always have: $p^n \cap \rho^{n+1} = \emptyset$ by $<_*$–minimality and Lemma 2.5.7.

If $p^m \notin R^m$, then p^n is undefined. By Lemma 2.5.17 it follows easily that:

Corollary 2.5.18. *M is n–sound iff p^n_M is defined and $p^n_M \in R^n_M$.*

This is the definition of soundness usually found in the literature.

Note. That the sequences of projecta ρ^n will stabilize at some n, since it is monotony non increasing. If it stabilizes at n, we have $R^{n+h} = R^n$ and $P^{n+h} = P^n$ for $h < \omega$.

By iterated application of Lemma 2.5.13 we get:

Lemma 2.5.19. *Let $a \in R^n_M$ and let $\overline{\pi} : N \to_{\Sigma_l} M^{na}$. Then there are $\overline{M}, \overline{a}$ and $\pi \supset \overline{\pi}$ such that $\overline{M}^{n\overline{a}} = M^{na}$, $\overline{a} \in R^n_{\overline{M}}$, $\pi : \overline{M} \to_{\Sigma_{n+l+1}} M$ and $\pi(\overline{a}) = a$.*

We also have:

Lemma 2.5.20. *Let $a \in R^n_M$. There is an M–definable partial map of ρ^n onto M which is M–definable in the parameter a.*

Proof: By induction on n. The case $n = 0$ is trivial. Now let $n = m + 1$. Let f be a partial map of ρ^m onto M which is definable in $a \setminus \rho^m$. Let $N = M^{m,a\setminus\rho^n}$, $b = a \cap \rho^m$. Then $N = h_N(\rho^n \cup \{b\}) = h_N{}''(w \times (\rho^n \times \{b\}))$. Set:

$$g(\prec i, \nu \succ) \simeq: h_N(i, \langle \nu, b \rangle) \text{ for } \nu < \rho^n.$$

Then $N = g''\rho^n$. Hence $M = fg''\rho^n$, where fg is M–definable in a. QED

We have now developend the "classical" fine structure theory which was used to analyze L. Its applicability to L is given by:

Lemma 2.5.21. *Every J_α is acceptable and sound.*

Unfortunately, in this book we shall sometimes have to deal with acceptable structures which are not sound and can even fail to be weakly 1–sound. This means that the structure is not coded by any of its reducts. How can we deal with it? It can be claimed that the totality of reducts contains full information about the structure, but this totality is a very unwieldy object. In §2.6 we shall develop methods to "tame the wilderness".

We now turn to the proof of Lemma 2.5.21:

We first show:

(A) If J_α is acceptable, then it is sound.

Proof: By induction on n we show that J_α is n–sound. The case $n = 0$ is trivial. Now let $n = m + 1$. Let $p = p^m_M$. Let $q = p_{M^{m,p}}$ = The $<_*$–least $q \in P_{M^{m,p}}$.

Claim $q \in R_{M^{m,p}}$.

Suppose not. Let $X = h_{M^{m,p}}(\rho^n \cup q)$. Let $\overline{\pi} : N \xleftrightarrow{\sim} X$, where N is transitive. Then $\overline{\pi} : N \to_{\Sigma_1} M^{np}$ and there are $\overline{M}, \overline{p}, \pi \supset \overline{\pi}$ such that $\overline{M}^{m\overline{p}} = M^{mp}$, $\overline{p} \in R^m_{\overline{M}}$, $\pi : \overline{M} \to_{\Sigma_n} M$, and $\pi(\overline{p}) = p$. Then $\overline{M} = J_{\overline{\alpha}}$ for some $\overline{\alpha} \leq \alpha$ by the condensation lemma for L.

Let A be $\Sigma_1(M^{mp})$ in q such that $A \cap \rho^n_M \notin M^{m,p}$ Then $A \cap \rho^n_M \notin M$. Let $\overline{A}$ be $\Sigma_1(N)$ in $\overline{q} = \pi^{-1}(q)$ by the same definition. Then $A \cap \rho^n = \overline{A} \cap \rho^n$ is $J_{\overline{\alpha}}$ definable in $\overline{q}$. Hence $\overline{\alpha} = \alpha$, $\overline{M} = M$, since otherwise $A \cap \rho^n \in M$. But then $\pi = id$ and $N = \overline{M}^{m\overline{p}} = M^m$. But by definition: $N = h_{M^{m,p}}(\rho^n \cup q)$. Hence $q \in R_{M^{np}}$. QED

By induction on α we then prove:

(B) J_α is acceptable.

Proof: The case $\alpha = \omega$ is trivial. The case $\alpha \in \mathrm{Lim}^*$ is also trivial. There remains the case $\alpha = \beta + \omega$, where β is a limit ordinal. By the induction hypothesis J_β is acceptable, hence sound.

We first verify (a) in the definition of acceptability. Since J_β is acceptable, it suffices to show that if $\gamma \leq \beta$ and $a \in J_\alpha \setminus J_\beta$ with $a \subset \gamma$, then:

Claim $\overline{\overline{\beta}} \leq \gamma$ in J_α.

Suppose not. Since $\mathbb{P}(J_\beta) \cap J_\alpha = \mathrm{Def}(J_\beta)$, we show that a is J_β–definable in a parameter r. We may assume w.l.o.g. that $r \in [\beta]^{<\omega}$. We may also assume that a is $\Sigma_n(J_\beta)$ in r for sufficiently large n. There is then, no partial map $f \in \mathrm{Def}(J_\beta)$ mapping γ onto β. Hence, by Lemma 2.5.20 we have $\gamma < \rho^n = \rho^n_{J_\beta}$ for all $n < \omega$.

Pick n big enough that a is $\Sigma_n(J_\beta)$ in r. Set: $p = p^n \cup r$ (where $p^n = p^n_{J_\beta}$). Then $p \in R^n$. Let $M = J_\beta$, $N = M^{np}$. Let $X = h_N(\gamma \cup q)$ where $q = p \cap \rho^n$. Let $\overline{\pi} : \overline{N} \overset{\sim}{\longleftrightarrow} X$, where $\overline{N}$ is transitive. Then $\overline{\pi} : \overline{N} \to_{\Sigma_1} N$ and hence there are $\overline{M}$, $\overline{p}$, $\pi \supset \overline{\pi}$ such that $\overline{M}^{n,\overline{p}} = \overline{N}$, $\overline{p} \in R^n_{\overline{M}}$, $\pi : \overline{M} \to_{\Sigma_{n+1}} M$, $\pi(\overline{p}) = p^n$. Hence $\overline{M} = J_{\overline{\beta}}$ for $\overline{\beta} \leq \beta$. Moreover, a is $\Sigma_n(\overline{M})$ in $\overline{p}$. Hence $\overline{\beta} = \beta$, since otherwise $a \in \mathrm{Def}(J_{\overline{\beta}}) \subset J_\beta$. But then $\pi = id$, $\overline{N} = N = h_N(\gamma \cup q)$. Hence $\gamma \geq \rho_N = \rho^{n+1}_M$.

Contradiction! QED (Claim)

This proves (a). We now prove (b) in the definition of "acceptable". Most of the proof will be a straightforward imitation of the proof of (a). Assume $J_\alpha \models \psi[x, \gamma]$, but $J_\beta \not\models \psi[x, \gamma]$, where $x \in J_\gamma$, $\gamma \leq \beta$ and ψ is Σ_1. As before we claim:

Claim $\overline{\overline{\beta}} \leq \gamma$ in J_α.

Suppose not. Then $\gamma < \beta$. Let $\psi = \bigvee y\varphi$ where φ is Σ_0. Let $J_\alpha \models \varphi(y, x, \gamma)$. Then $y = f(z, x, \gamma, J_\beta)$ where f is rud and $z \in J_\beta$. But

$$J_\alpha \models \varphi[f(z, x, \gamma, J_\beta), x, \beta]$$

reduces to:

$$J_\alpha \models \varphi'[z, x, \gamma, J_\beta]$$

where φ' is Σ_0. But then

$$J_\beta \cup \{J_\beta\} \models \varphi'[z, x, \gamma, J_\beta).$$

As we have seen in §2.3, this reduces to:

$$J_\beta \models \chi[z, x, \gamma]$$

where χ is a first order formula. Note that this reduction is *uniform*. Hence if $\gamma < \nu \leq \beta$, $z \in J_\nu$ and $J_\nu \models \chi[z, x, \gamma]$, it follows that $J_{\nu+\omega} \models \psi[x, \gamma]$. This means that $J_\nu \models \neg\chi'[x, \gamma]$ for $\gamma < \nu < \beta$, where $\chi = \chi(v_0, v_1, v_n)$ and $\chi' = \bigvee v_0 \chi$. We know that $\gamma < \rho^n_{J_\beta}$ for all n. Choose n such that χ' is Σ_n. Let $M = J_\beta$, $N : M^{n,p}$ when $p = p_N$. Let $X = h_N(\gamma + 1 \cup \{x\})$ and let $\overline{\pi} : \overline{N} \overset{\sim}{\longleftrightarrow} X$, where $\overline{N}$ is transitive. As before, there are $\overline{M}, \overline{p}, \pi \supset \overline{\pi}$ such that $\overline{M}^n\overline{p} = N$, $\pi : \overline{M} \to_{\Sigma_1} M$, and $\pi(\overline{p}) = p$. Let $\overline{M} = J_{\overline{\beta}}$. Then $J_{\overline{\beta}} \models \chi'(x, \gamma)$. Hence $\overline{\beta} = \beta$ and $\pi = \mathrm{id}$. Hence $N = h_N(\gamma + 1 \cup \{x\})$. Hence $\gamma \geq \rho^{n+1} = \rho_N$.

Contradiction! QED (Lemma 2.5.21)

$M = J^A_\alpha$ is a *constructible extension* of $N = J^A_\beta$ iff $\beta \leq \alpha$ and $A \subset N$.

Or methods have some application to constructible extensions. By a slight modification of the proof of (A) we get:

Lemma 2.5.22. *If $M = J^A_\alpha$ is an acceptable constructible extension of $N = J^A_\beta$, then:*

(a) *If $\rho^n_M \geq \beta$, then M is n-sound.*

(b) *If $\rho^{n+1}_M < \beta \leq \rho^n_M$, and $\overline{M} =: M^{n,p^n_M}$, then $\overline{M} = h_{\overline{M}}(\beta \cup q)$ whenever $q \in P_{\overline{M}}$.*

The proof of (B) then gives us:

Lemma 2.5.23. *If $N = J^A_\beta$ is sound and acceptable, and $A \subset N$, then $M = J^A_{\beta+\omega}$ is acceptable.*

The verifications are left to the reader.

2.6 Σ^*–theory

There is an alternative to the Levy hierarchy of relations on an acceptable structure $M = \langle J^A_\alpha, B\rangle$ which — at first sight — seems more natural. Σ_0, we recall, consists of the relation on M which are Σ_0 definable in the predicates of M. Σ_1 then consists of relations of the form $\bigvee y R(y, \vec{x})$ where R is Σ_0. Call these levels $\Sigma^{(0)}_0$ and $\Sigma^{(0)}_1$. Our next level in the new hierarchy, call it $\Sigma^{(1)}_0$, consists of relations which are "Σ_0 in $\Sigma^{(0)}_1$" — i.e. $\Sigma_0(\langle M, \vec{A}\rangle)$ where $A_1, \ldots, A_n$ are $\Sigma^{(0)}_1$. $\Sigma^{(1)}_1$ then consists of relations of the form $\bigvee y R(y, \vec{x})$