where χ is a first order formula. Note that this reduction is *uniform*. Hence if $\gamma < \nu \leq \beta$, $z \in J_\nu$ and $J_\nu \models \chi[z, x, \gamma]$, it follows that $J_{\nu+\omega} \models \psi[x, \gamma]$. This means that $J_\nu \models \neg\chi'[x, \gamma]$ for $\gamma < \nu < \beta$, where $\chi = \chi(v_0, v_1, v_n)$ and $\chi' = \bigvee v_0 \chi$. We know that $\gamma < \rho^n_{J_\beta}$ for all n. Choose n such that χ' is Σ_n. Let $M = J_\beta$, $N : M^{n,p}$ when $p = p_N$. Let $X = h_N(\gamma + 1 \cup \{x\})$ and let $\overline{\pi} : \overline{N} \overset{\sim}{\longleftrightarrow} X$, where $\overline{N}$ is transitive. As before, there are $\overline{M}, \overline{p}, \pi \supset \overline{\pi}$ such that $\overline{M}^n \overline{p} = N$, $\pi : \overline{M} \to_{\Sigma_1} M$, and $\pi(\overline{p}) = p$. Let $\overline{M} = J_{\overline{\beta}}$. Then $J_{\overline{\beta}} \models \chi'(x, \gamma)$. Hence $\overline{\beta} = \beta$ and $\pi = \text{id}$. Hence $N = h_N(\gamma + 1 \cup \{x\})$. Hence $\gamma \geq \rho^{n+1} = \rho_N$.

Contradiction! QED (Lemma 2.5.21)

$M = J^A_\alpha$ is a *constructible extension* of $N = J^A_\beta$ iff $\beta \leq \alpha$ and $A \subset N$.

Or methods have some application to constructible extensions. By a slight modification of the proof of (A) we get:

Lemma 2.5.22. *If $M = J^A_\alpha$ is an acceptable constructible extension of $N = J^A_\beta$, then:*

(a) *If $\rho^n_M \geq \beta$, then M is n-sound.*

(b) *If $\rho^{n+1}_M < \beta \leq \rho^n_M$, and $\overline{M} =: M^{n, p^n_M}$, then $\overline{M} = h_{\overline{M}}(\beta \cup q)$ whenever $q \in P_{\overline{M}}$.*

The proof of (B) then gives us:

Lemma 2.5.23. *If $N = J^A_\beta$ is sound and acceptable, and $A \subset N$, then $M = J^A_{\beta+\omega}$ is acceptable.*

The verifications are left to the reader.

2.6 Σ^*–theory

There is an alternative to the Levy hierarchy of relations on an acceptable structure $M = \langle J^A_\alpha, B \rangle$ which — at first sight — seems more natural. Σ_0, we recall, consists of the relation on M which are Σ_0 definable in the predicates of M. Σ_1 then consists of relations of the form $\bigvee y R(y, \vec{x})$ where R is Σ_0. Call these levels $\Sigma^{(0)}_0$ and $\Sigma^{(0)}_1$. Our next level in the new hierarchy, call it $\Sigma^{(1)}_0$, consists of relations which are "Σ_0 in $\Sigma^{(0)}_1$" — i.e. $\Sigma_0(\langle M, \vec{A} \rangle)$ where $A_1, \ldots, A_n$ are $\Sigma^{(0)}_1$. $\Sigma^{(1)}_1$ then consists of relations of the form $\bigvee y R(y, \vec{x})$

where R is $\Sigma_0^{(1)}$. $\Sigma_0^{(2)}$ then consists of relations which are Σ_0 in $\Sigma_1^{(1)} \ldots$ etc. By a $\underline{\Sigma}_i^{(n)}$ relation we of course mean a relation of the form

$$R(\vec{x}) \leftrightarrow R'(\vec{x}, \vec{p}),$$

where $p_1, \ldots, p_m \in M$ and R' is $\Sigma_i^{(n)}(m)$. It is clear that there is natural class of $\Sigma_i^{(n)}$–formulae such that R is a $\Sigma_i^{(n)}$–relation iff it is defined by a $\Sigma_i^{(n)}$–formula. Thus e.g. we can define the $\Sigma_0^{(1)}$ formula to be the smallest set Σ of formulae such that

- All primitive formulae are in Σ.
- All $\Sigma_1^{(0)}$ formulae are in Σ.
- Σ is closed under the sentential operations $\vee, \rightarrow, \leftrightarrow, \neg$.
- If φ is in Σ, then so are $\bigwedge v \in u\, \varphi$, $\bigvee v \in u\, \varphi$ (where $v \neq u$).

By a $\Sigma_1^{(1)}$ formula we then mean a formula of the form $\bigvee v\varphi$, where φ is $\Sigma_0^{(1)}$.

How does this hierarchy compare with the Levy hierarchy? If no projectum drops, it turns out to be a useful refinement of the Levy hierarchy:
If $\rho_M^n = \alpha$, then $\Sigma_0^{(n)} \subset \Delta_{n+1}$ and $\Sigma_1^{(n)} = \Sigma_{n+1}$. If, however, a projectum drops, it trivializes and becomes useless. Suppose e.g. that $M = J_\alpha$ and $\rho = \rho_M^1 < \alpha$. Then *every* M–definable relation becomes $\underline{\Sigma}_0^{(1)}(M)$. To see this let $R(\vec{x})$ be defined by the formula $\varphi(\vec{v})$, which we may suppose to be in prenex normal form:

$$\varphi(\vec{v}) = Q_1 u_1 \ldots Q_m u_m \varphi'(\vec{v}, \vec{u}),$$

where φ' is quantifier free (hence Σ_0). Then:

$$R(\vec{x}) \leftrightarrow Q_1 y_1 \in M \ldots Q_m y_m \in M R'(\vec{x}, \vec{y})$$

where R' is Σ_0. By soundness we know that there is a $\underline{\Sigma}_1(M)$ partial map f of ρ onto M. But then:

$$R(\vec{x}) \leftrightarrow Q_1 \xi_\xi \in \mathrm{dom}(f) \ldots Q_m \xi_m \in \mathrm{dom}(f) R'(\vec{x}, f(\vec{\xi})).$$

Since f is $\underline{\Sigma}_1$, the relation $R'(\vec{x}, f(\vec{\xi}))$ is $\underline{\Sigma}_1$. But $\mathrm{dom}(f)$ is $\underline{\Sigma}_1$ and $\mathrm{dom}(f) \subset \rho$, hence by induction on m:

$$R(\vec{x}) \leftrightarrow Q_1 \xi_1 \in \rho \ldots Q_m \xi_m \in \rho R''(\vec{x}, \vec{\xi}),$$

where R'' is a sentential combination of $\underline{\Sigma}_1$ relations. Hence R'' is $\underline{\Sigma}_0^{(1)}(M)$ and so is R.

The problem is that, in passing from $\Sigma_1^{(0)}$ to $\Sigma_0^{(1)}$ our variables continued to range over the whole of M, despite the fact that M had grown "soft" with respect to $\underline{\Sigma}_1$ sets. Thus we were able to reduce unbounded quantification over M to quantification bounded by ρ, which lies in the "soft" part of M. in section 2.5 we acknowledged softness by reducing to the part $H = H_\rho^M$ which remained "hard" wrt $\underline{\Sigma}_1$ sets. We then formed a reduct M^p containing just the sets in H. If M is sound, we can choose p such that M^p contains complete information about M. In the general case, however, this may not be possible. It can happen that *every* reduct entails a loss of information. Thus we want to hold on to the original structure M. In passing to $\Sigma_0^{(1)}$, however, we want to restrict our variables to H. We resolve this conundrum by introducing *new* varibles which range only over H. We call these variables of *Type 1*, the old ones being of *Type 0*. Using $u^h, v^h (h = 0, 1)$ as metavariables for variables of Type h, we can then reformulate the definition of $\Sigma_0^{(1)}$ formula, replacing the last clause by:

- If φ is in Σ, then so are $\bigwedge v^i \in u^1\varphi$, $\bigvee v^i \in u^1\varphi$ where $i = 0, 1$ and $v^i \neq u^1$.

A $\Sigma_1^{(1)}$ formula is then a formula of the form $\bigvee v^1\varphi$, where φ is $\Sigma_0^{(1)}$. We call $A \subset M$ a $\underline{\Sigma}_1^{(1)}$ set if it is definable in parameters by a $\Sigma_1^{(1)}$ formula. The *second projectum* ρ^2 is then the least ρ such that $\rho \cap B \notin M$ for some $\underline{\Sigma}_1^{(1)}$ set B. We then introduce type 2 variables $v^2, u^2, \ldots$ ranging over $|J_{\rho^2}^A|$ ($|J_\gamma^A|$ being the set of elements of the structure J_γ^A, where e.g. $M = \langle J_\alpha^A, B\rangle$.) Proceeding in this way, we arrive at a many sorted language with variables of type n for each $n < \omega$. The resulting hierarchy of $\Sigma_h^{(n)}$ formulae ($h = 0, 1$) offers a much finer analysis of M–definabilty than was possible with the Levy hierarchy alone. This analysis is known as Σ^* theory. In this section we shall develop Σ^* theory systematically and *ab ovo*.

Before beginning, however, we address a remark to the reader: Most people react negatively on their first encounter with Σ^* theory. The introduction of a many sorted language seems awkward and cumbersome. It is especially annoying that the variable domains diminish as the types increase. The author confesses to having felt these doubts himself. After developing Σ^*–theory and making its first applications, we spent a couple of months trying vainly to redo the proofs without it. The result was messier proofs and a pronounced loss of perspicuity. It has, in fact, been our consistent experience that Σ^* theory facilitates the fine structural analysis which lies at the heart of inner model theory. We therefore urge the reader to bear with us.

Definition 2.6.1. Let $M = \langle J_\alpha^{\vec{A}}, \vec{B}\rangle$ be acceptable.

The Σ^* *M–language* $\mathbb{L}^* = \mathbb{L}^*_M$ has

- a binary predicate $\dot{\in}$
- unary predicates $\dot{A}_1, \dots, \dot{A}_n, \dot{B}_1, \dots, \dot{B}_m$
- variables $v_i^j (i, j < \omega)$

Definition 2.6.2. By induction on $n < \omega$ we define sets $\Sigma_h^{(n)} (h = 0, 1)$ of formulae

$\Sigma_0^{(n)}$ = the smallest set of formulae such that

- all primitive formulae are in Σ.
- $\Sigma_0^{(m)} \cup \Sigma_1^{(m)} \subset \Sigma$ for $m < n$.
- Σ is closed under sentential operations $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$.
- If φ is in $\Sigma, j \leq n$, and $v^j \neq u^n$, then $\bigwedge v^j \in u^n \varphi$, $\bigvee v^j \in u^n \varphi$ are in Σ.

We then set:

$$\Sigma_1^{(n)} =: \text{ The set of formulae } \bigvee v^n \varphi, \text{ where } \varphi \in \Sigma_0^{(n)}.$$

We also generalize the last part of this definition by setting:

Definition 2.6.3. Let $n < \omega$, $1 \leq h < \omega$. $\Sigma_h^{(n)}$ is the set of formulae

$$\bigvee v_1^n \bigwedge v_2^n \dots Q v_h^n \varphi,$$

where φ is $\Sigma_0^{(n)}$ (and Q is $\bigvee$ if h is odd and $\bigwedge$ if h is even).

We now turn to the interpretation of the formualae in M.

Definition 2.6.4. Let Fml^n be the set of formulae in which only variables of type $\leq n$ occur.

By recursion on n we define:

- The n–th projectum $\rho^n = \rho^n_M$.
- The n–th variable domain $H^n = H^n_M$.

- The satisfaction relation $\models^n$ for formulae in Fml^n.

$\models^n$ is defined by interpreting variables of type i as ranging over H^i for $i \leq n$. We set: $\rho^0 = \alpha$, $H^0 = |M| = |J_\alpha^{\vec{A}}|$, when $M = \langle J_\alpha^{\vec{A}}, \vec{B} \rangle$.

Now let ρ^n, H^n be given (hence $\models^n$ is given). Call a set $D \in H^n$ a $\underline{\Sigma}_1^{(n)}$ *set.* if it is definable from parameters by a $\Sigma_1^{(n)}$ formula φ:

$$Dx \leftrightarrow M \models^n \varphi[x, a_1, \ldots, a_p],$$

where $\varphi = \varphi(v^n, u^{i_1}, \ldots, u^{i_m})$ is $\Sigma_1^{(n)}$. ρ^{n+1} is then the least ρ such that there is a $\underline{\Sigma}_1^{(n)}$ set $D \subset \rho$ with $D \notin M$. We then set:

$$H^{n+1} = |J_\rho^{\vec{A}}|.$$

This then defines $\models^{n+1}$.

It is obvious that $\models^i$ is contained in $\models^j$ for $i \leq j$, so we can define the full Σ^* satisfaction relation for M by:

$$\models = \bigcup_{n<\omega} \models^n .$$

Satisfaction is defined in the usual way. We employ v^i, u^i, ω^i etc. as metavariables for variables of type i. We also employ x^i, y^i, z^i etc. as metavariables for elements of H^i. We call $v_1^{i_1}, \ldots, v_n^{i_n}$ a *good sequence* for the formula φ iff it is a sequence of distinct variables containing all the variables which occur free in φ. If $v_1^{i_1}, \ldots, v_n^{i_n}$ is good we write:

$$\models_M \varphi[v_1^{i_1}, \ldots, v_n^{i_n} / x_1^{i_1}, \ldots, x_n^{i_n}]$$

to mean that φ becomes true if $v_h^{i_n}$ is interpreted by $x_h^{i_n}(h = 1, \ldots, n)$. We shall follow normal usage in suppressing the sequence $v_1^{i_1}, \ldots, v_n^{i_n}$ writing only:

$$\models_M \varphi[x_1^{i_1}, \ldots, x_n^{i_n}].$$

(However, it is often important for our understanding to retain the upper indices $i_1, \ldots, i_n$.) We often write $\varphi = \varphi(v_1^{i_1}, \ldots, v_n^{i_n})$ to indicate that these are the suppressed variables. φ (together with $v_1^{i_1}, \ldots, v_n^{i_n}$) defines a relation:

$$R(x_1^{i_1}, \ldots, x_n^{i_n}) \leftrightarrow \models_M \varphi[x_1^{i_1}, \ldots, x_n^{i_n}].$$

Since we are using a many sorted language, however, we must also employ *many sorted relations*.

The number of argument places of an ordinary one sorted relation is often called its "arity". In the case of a many sorted relation, however, we must know not only the number of argument places, but also the type of each argument place. We refer to this information as its "arity". Thus the arity of the above relation is not n but $\langle i_1, \ldots, i_n\rangle$. An ordinary 1–sorted relation is usually identified with its field. We shall identify a many sorted relation with the pair consisting of its field and its arity:

Definition 2.6.5. A *many sorted relation* R on M is a pair $\langle |R|, r\rangle$ such that for some n:

(a) $|R| \subset M^n$

(b) $r = \langle r_1, \ldots, r_n\rangle$ where $r_i < \omega$

(c) $R(x_1, \ldots, x_n) \to x_i \subset H^{r_i}$ for $i = 1, \ldots, n$.

$|R|$ is called the *field* of R and r is called the *arity* of R.

In practice we adopt a rough and ready notation, writing $R(x_1^{i_1}, \ldots, x_n^{i_n})$ to indicate that R is a many sorted relation of arity $\langle i_1, \ldots, i_n\rangle$.

Note. Let $\mathbb{L} = \mathbb{L}_M$ be the ordinary first order language of M (i.e. it has only variables of type 0.

Since $H^n \in M$ or $H^n = M$ for all $n < \omega$, it follows that every $\mathbb{L}^*$–definable many sorted relation has a field which is $\mathbb{L}$–definable in parameters from M.)

Note. If R is a relation of arity $\langle i_1, \ldots, i_n\rangle$, then its *complement* is $\Gamma \setminus R$, where:

$$\Gamma = \{\langle x_1, \ldots, x_n\rangle | x_h \in H^{i_n} \text{ for } h = 1, \ldots, n\},$$

the arity remaining unchanged.

Definition 2.6.6. $R(x_1^{i_1}, \ldots, x_m^{i_m})$ is a $\Sigma_h^{(n)}(M)$ *relation* iff it is defined by a $\Sigma_h^{(n)}$ formula. R is $\Sigma_h^{(n)}(M)$ *in the parameters* $p_1, \ldots, p_r$ iff $R(\vec{x}) \leftrightarrow R'(\vec{x}, \vec{p})$, where R' is $\Sigma_h^{(n)}(M)$. R is a $\underline{\Sigma}_h^{(n)}(M)$ relation iff it is $\Sigma_h^{(n)}(M)$ in some parameters.

It is easily checked that:

Lemma 2.6.1.
- *If $R(y^n, \vec{x})$ is $\Sigma_1^{(n)}$, so is $\bigvee y^n R(y^n, \vec{x})$*
- *If $R(\vec{x}), P(\vec{x})$ are $\Sigma_1^{(n)}$, then so are $R(\vec{x}) \vee P(\vec{x})$, $R(\vec{x}) \wedge P(\vec{x})$.*

Moreover, if $R(x_0^{i_0}, \ldots, x_{m-1}^{i_{m-1}})$ is $\Sigma_1^{(n)}$, so is any relation $R'(y_0^{j_0}, \ldots, y_{r-1}^{j_{r-1}})$ obtained from R by permutation of arguments, insertion of dummy arguments and fusion of arguments having the same type — i.e.

$$R'(y_0^{j_0}, \ldots, y_{r-1}^{j_{r-1}}) \leftrightarrow R(y_{\sigma(0)}^{j_{\sigma(0)}}, \ldots y_{\sigma(m-1)}^{j_{\sigma(m-1)}})$$

where $\sigma : m \to r$ such that $j_{\sigma(l)} = i_l$ for $l < m$.

Using this we get the analogue of Lemma 2.5.6

Lemma 2.6.2. *Let $M = \langle J_\alpha^A, B\rangle$ be acceptable. Let $\rho = \rho^n$, $H = H^n$. Then*

(a) If $\rho \in M$, then ρ is a cardinal in M. (Hence $H = H_\rho^M$)

(b) If D is $\underset{\sim}{\Sigma}_1^{(n)}(M)$ and $D \subset H$, then $\langle H, D\rangle$ is amenable.

(c) If $u \in H$, there is no $\Sigma_1^{(n)}(M)$ partial map of u onto H.

(d) $\rho \in \mathrm{Lm}^*$ if $n > 0$.

Proof: By induction on n. The induction step is a virtual repetition of the proof of Lemma 2.5.6. QED (Lemma 2.6.2)

Definition 2.6.7. Let $R(x_1^{i_1}, \ldots, x_m^{i_m})$ be a many sorted relation. By an *n–specialization* of R we mean a relation $R'(x_1^{j_1}, \ldots, x_m^{j_m})$ such that

- $j_l \geq i_l$ for $l = 1, \ldots, m$
- $j_l = i_l$ if $l < n$
- If $z_1, \ldots, z_m$ are such that $z_l \in H^{j_l}$ for $l = 1, \ldots, m$, then: $R(\vec{z}) \leftrightarrow R'(\vec{z})$.

Given a formula φ in which all bound quantifiers are of type $\leq n$, we can easily devise a formula φ' which defines a specialization of the relation defined by φ:

Fact Let $\varphi = \varphi(v_1^{i_1}, \ldots, v_m^{i_m})$ be a formula in which all bound variables are of type $\leq n$. Let $u_1^{j_1}, \ldots, u_m^{j_m}$ be a sequence of distinct variables such that $j_l \geq i_l$ and $j_l = i_l$ if $i_l < n (l = 1, \ldots, m)$. Suppose that $\varphi' = \varphi'(\vec{u})$ is obtained by replacing each free occurence of $v_l^{i_l}$ by a free occurence of $u_l^{j_l}$ for $l = 1, \ldots, m$. Then for all $x_1, \ldots, x_m$ such that $x_l \in H^{j_l}$ for $l = 1, \ldots, m$ we have:

$$\models_M \varphi(\vec{v})[\vec{x}] \leftrightarrow \models_M \varphi'(\vec{u})[\vec{x}].$$

The proof is by induction on φ. We leave it to the reader. Using this, we get:

Lemma 2.6.3. *Let $R(x_1^{i_1}, \ldots, x_m^{i_m})$ be $\Sigma_l^{(n)}$. Then every n–specialization of R is $\Sigma_l^{(n)}$.*

Proof: $R'(x_1^{i_1},\ldots,x_m^{i_m})$ be an n–spezialization. Let R be defined by $\varphi(v_1^{i_1},\ldots,v_m^{i_m})$. Suppose $(u_1^{j_1},\ldots,v_m^{j_m})$ is a sequence of distinct variables which are new — i.e. none of them occur free or bound in φ. Let φ' be obtained by replacing every free occurence of $v_l^{i_l}$ by $u_l^{j_l}(l=1,\ldots,m)$. Then $\varphi'(u_1^{j_1},\ldots,v_m^{j_m})$ defines R' by the above fact. QED (Lemma 2.6.3)

Corollary 2.6.4. *Let R be $\Sigma_1^{(n)}$ in the parameter p. Then every n–spezialization of R is $\Sigma_1^{(n)}$ in p.*

Lemma 2.6.5. *Let $R'(x_1^{j_1},\ldots,x_m^{j_m})$ be $\Sigma_1^{(n)}$. Then R' is an n–specialization of a $\Sigma_1^{(n)}$ relation $R(x_1^{i_1},\ldots,x_m^{i_m})$ such that $i_l \leq n$ for $l=1,\ldots,m$.*

Proof: Let R' be defined by $\varphi'(u_1^{j_1},\ldots,v_m^{j_m})$, when φ' is $\Sigma_1^{(n)}$. Let $v_1^{i_n},\ldots,v_m^{i_m}$ be a sequence of distinct new variables, where $i_l=\min(n,j_l)$ for $l=1,\ldots,m$. Replace each free occurence of $u_l^{j_l}$ by $v_l^{i_l}$ for $l=1,\ldots,m$ to get $\varphi(u_1^{i_1},\ldots,v_m^{i_m})$. Let R be defined by φ. Then R' is a specialization of R by the above fact. QED (Lemma 2.6.5)

Corollary 2.6.6. *Let $R'(x_1^{j_1},\ldots,x_m^{j_m})$ be $\Sigma_1^{(n)}$ in p. Then R' is a specialization of a relation $R(x_1^{i_1},\ldots,x_m^{i_m})$ which is $\Sigma_1^{(n)}$ in p with $i_l \leq n$ for $l=1,\ldots,m$.*

Every $\Sigma_1^{(m)}$ formula can appear as a "primitive" component of a $\Sigma_0^{(m+1)}$ formula. We utilize this fact in proving:

Lemma 2.6.7. *Let $n=m+1$. Let $Q_j(z_{j,1}^n,\ldots,z_{j,p_j}^n,x_1^{i_1},\ldots,x^{i_p})$ be $\Sigma_1^{(m)}(j=1,\ldots,r)$.*
Set: $Q_{j,\vec{x}} =: \{\langle \vec{z}_j^n\rangle | Q_j(\vec{z}_j^n,\vec{x})\}$.
Set: $H_{\vec{x}} =: \langle H^n, Q_{1,\vec{x}},\ldots,Q_{r,\vec{x}}\rangle$.
Let $\varphi=\varphi(v_1,\ldots,v_q)$ be Σ_l in the language of $H_{\vec{x}}$. Then

$$\{\langle \vec{x}^n,\vec{x}\rangle | H_{\vec{x}} \models \varphi[\vec{x}^n]\} \text{ is } \Sigma_l^{(n)}.$$

Proof: We first prove it for $l=0$, showing by induction on φ that the conclusion holds for any sequence $v_1,\ldots,v_l$ of variables which is good for φ.

We describe some typical cases of the induction.

Case 1 φ is primitive.

Let e.g. $\varphi=\dot{Q}_j(v_{h_1},\ldots,v_{h_{p_i}})$, where $\dot{Q}_j$ is the predicate for $Q_{j\vec{x}}$. Then $H_{\vec{x}} \models \varphi[\vec{x}^n]$ is equivalent to: $Q_j(x_{h_1}^n,\ldots,x_{h_{p_j}}^n,\vec{x})$, which is $\Sigma_1^{(m)}$ (hence $\Sigma_0^{(n)}$). QED (Case 1)

Case 2 φ arises from a sentential operation.

Let e.g. $\varphi = (\varphi_0 \wedge \varphi_1)$. Then $H_{\vec{x}} \models \varphi[\vec{x}^n]$ is equivalent to:

$$H_{\vec{x}} \models \varphi_0[\vec{x}^n] \wedge H_{\vec{x}} \models \varphi_1[\vec{x}^n]$$

which, by the induction hypothesis is $\Sigma_0^{(n)}$. QED (Case 2)

Case 3 φ arises from a quantification.

Let e.g. $\varphi = \bigwedge w \in v_i \Psi$. By bound relettering we can assume *w.l.o.g.* that w is not among $v_1, \ldots, v_p$. We apply the induction hypothesis to $\Psi(w, v_1, \ldots, v_p)$. Then $H_{\vec{x}} \models \varphi[\vec{x}^n]$ is equivalent to:

$$\bigwedge z \in x_i^n H_{\vec{x}} \models \Psi[w, \vec{x}^n]$$

which is $\Sigma_0^{(n)}$ by the induction hypothesis. QED (Case 3)

This proves the case $l = 0$. We then prove it for $l > 0$ by induction on l, essentially repeating the proof in case 3. QED (Lemma 2.6.7)

Note. It is clear from the proof that the set $\{\langle \vec{x}^n, \vec{x}\rangle | H_{\vec{x}} \models \varphi[\vec{x}^n]\}$ is *uniformly* $\Sigma_l^{(n)}$ — i.e. its defining formula χ depends only on φ and the defining formula Ψ_i for $Q_i (i = 1, \ldots, p)$. In fact, the proof implicitly describes an algorithm for the function $\varphi, \Psi_1, \ldots, \Psi_p \mapsto \chi$.

We can invert the argument of Lemma 2.6.7 to get a weak converse:

Lemma 2.6.8. *Let $n = m+1$. Let $R(\vec{x}^n, x_1^{i_1}, \ldots, x_g^{i_g})$ be $\Sigma_l^{(n)}$ where $i_l \leq m$ for $l = 1, \ldots, g$. Then there are $\Sigma_1^{(n)}$ relations $Q_i(\vec{z}_i^n, \vec{x})(i = 1, \ldots, p)$ and a Σ_l formula φ such that*

$$R(\vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n],$$

where $H_{\vec{x}}$ is defined as above.

Note. This is weaker, since we now require $i_l \leq m$.

Proof: We first prove it for $l = 0$. By induction on χ we prove:

Claim Let χ be $\Sigma_0^{(n)}$. Let $\vec{v}^n, v_1^{i_1}, \ldots, v_q^{i_q}$ be good for χ, where $i_1, \ldots, i_q \leq m$. Let $\chi(\vec{v}^n, \vec{v})$ define the relation $R(\vec{x}^n, \vec{x})$. Then the conclusion of Lemma 2.6.8 holds for this R (with $l = 0$).

Case 1 χ is $\Sigma_1^{(m)}$.

Let $\chi(\vec{x}^n, \vec{x})$ define $Q(\vec{x}^n, \vec{x})$. Then $R(\vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \dot{Q}\vec{v}^n[\vec{x}^n]$.

QED (Case 1)

Case 2 χ arises from a sentential operation.

Let e.g. $\chi = (\Psi \wedge \Psi')$. Appliyng the induction hypothesis we get $Q_i(\vec{x}_i^n, \vec{x})(i = 1, \ldots, p)$ and φ such that

$$M \models \Psi[\vec{x}^n, \vec{x}] \leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n]$$

where $H_{\vec{x}} = \langle H^n, Q_{1\vec{x}}, \ldots, Q_{p\vec{x}} \rangle$. Similarly we get $Q_i'(\vec{y}_i^n, \vec{x})(i = 1, \ldots, q')$ and φ'

$$M \models \Psi'[\vec{x}^n, \vec{x}] \leftrightarrow H'_{\vec{x}} \models \varphi'[\vec{x}^n].$$

Let $\dot{Q}_i$ be the predicate for $Q_{i\vec{x}}$ in the language of $H_{\vec{x}}$. Let $\dot{Q}_i'$ be the predicate for $Q'_{i\vec{x}}$ in the language of $H'_{\vec{x}}$. Assume *w.l.o.q.* that $\dot{Q}_i \neq \dot{Q}_j'$ for all i, j. Putting the two languages together we get a language for

$$H^*_{\vec{x}} = \langle H^n, \vec{Q}_{\vec{x}}, \vec{Q}'_{\vec{x}} \rangle.$$

Clearly:

$$M \models (\chi \wedge \chi')[\vec{x}^n, \vec{x}] \leftrightarrow H^*_{\vec{x}} \models (\varphi \wedge \varphi')[\vec{x}^n].$$

QED (Case 2)

Case 3 χ arises from the application of a bounded quantifier.

Let e.g. $\chi = \bigwedge w^n \in v_j^n \chi'$. By bound relettering we can assume *w.l.o.g.* that w^n is not among $\vec{v}^n$. Then $w^n \vec{v}^n, \vec{v}$ is a good sequence for χ' and by the induction hypothesis we have for $\chi' = \chi'(w^n, \vec{v}^n, \vec{v})$:

$$M \models \chi'[z^n, \vec{x}^n, x] \leftrightarrow H_{\vec{x}} \models \varphi[z^n, \vec{x}^n, \vec{x}].$$

But then:

$$\begin{aligned} M \models \chi[\vec{x}^n, \vec{x}] \quad & \leftrightarrow \bigwedge z^n \in x_j^n M \models \chi'[z^n, \vec{x}^n, \vec{x}] \\ & \leftrightarrow \bigwedge z^n \in x_j^n H_{\vec{x}} \models \varphi[z^n, \vec{x}^n] \\ & \leftrightarrow H_{\vec{x}} \models \bigwedge w \in v_j \varphi[\vec{x}^n]. \end{aligned}$$

QED (Lemma 2.6.8)

Note. Our proof again establishes uniformity. In fact, if χ is the $\Sigma_l^{(n)}$–definition of R, the proof implicitely describes an algorithm for the function

$$\chi \mapsto \varphi, \Psi_1, \ldots, \Psi_p$$

where Ψ_i is a $\Sigma_1^{(m)}$ definition of Q_i.

Remark. Lemma 2.6.7 and 2.6.8 taken together give an inductive definition of "$\Sigma_l^{(n)}$ relation" which avoids the many sorted language. It would, however, be difficult to work directly from this definition.

By a function of *arity* $\langle i_1, \ldots, i_n\rangle$ *to* H^j we mean a relation $F(y^j, x^{i_1}, \ldots, x^{i_n})$ such that for all $x^{i_1}, \ldots, x^{i_n}$ there is at most one such y^j. If this y exists, we denote it by $F(x^{i_1}, \ldots, x^{i_n})$. Of particular interest are the $\Sigma_1^{(i)}$ functions to H^i.

Lemma 2.6.9. *$R(y^n, \vec{x})$ be a $\Sigma_1^{(n)}$ relation. Then R has a $\Sigma_1^{(n)}$ uniformizing function $F(\vec{x})$.*

Proof: We can assume *w.l.o.g* that the arguments of R are all of type $\leq n$. (Otherwise let R be a specialization of R', where the arguments of R' are of type $\leq n$. Let F' uniformize R'. Then the appropriate specialization F of F' uniformizes R.)

Case 1 $n = 0$.

Set:

$$F(\vec{x}) \simeq: y \text{ where } \langle z, y\rangle \text{ is } <_M \text{–least such that } R'(z, y, \vec{x}).$$

By section 2.3 we know that $u_M(x)$ is Σ_1, where $u_M(x) = \{y | y <_M x\}$. Thus for sufficient r we have:

$$\begin{gathered} y = F(\vec{x}) \leftrightarrow \bigvee z(R'(z, y, \vec{x}) \wedge \\ \wedge w \in u_M(\langle z, y\rangle) \bigwedge z', y' \in C_r(w) \\ (w = \langle z', y'\rangle \to \neg R(z', y', \vec{x})), \end{gathered}$$

which is uniformly $\Sigma_1(M)$.

Case 2 $n > 0$. Let $n = m + 1$.

Rearranging the arguments of R if necessary, we can assume that R has the form $R(y^n, \vec{x}^n, \vec{x})$, where the $\vec{x}$ are of type $\leq m$. Then there are $Q_i(\vec{z}_i^n, \vec{x}^n, \vec{x})(i = 1, \ldots, p)$ such that Q_i is $\Sigma_1^{(m)}$ and

$$R(y^n, \vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi[y^n, \vec{x}^n],$$

where φ is Σ_1 and

$$H_{\vec{x}} = \langle H^n, Q_{1\vec{x}}, \ldots, Q_{n\vec{x}}\rangle.$$

If e.g. $M = \langle J^A, B\rangle$, we can assume *w.l.o.g.* that $Q_1(z^n, \vec{x}) \leftrightarrow A(z^n)$. Then $<_{H\vec{x}}, u_{H\vec{x}}$ are uniformly $\Sigma_1(H_{\vec{x}})$ and by the argument of Case 1 there is a Σ_1 formula φ' such that F uniformies R where

$$y = F(\vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi'[\vec{x}^n, \vec{x}].$$

QED (2.6.9)

Note. The proof shows that $F(\vec{x})$ is *uniformly* $\Sigma_1^{(n)}$ — i.e. its $\Sigma_1^{(n)}$ definition depends only on the $\Sigma_1^{(n)}$ definition of $R(y^n, \vec{x})$, regardless of M.

Note. It is clear from the proof that the $\Sigma_1^{(n)}$ definition of F is *functionally absolute* — i.e. it defines a function over every acceptable M of the same type. Thus:

Corollary 2.6.10. *Every $\Sigma_1^{(n)}$ function $F(\vec{x})$ to H^n has a functionally absolute $\Sigma_1^{(n)}$ definition.*

Note. The $\Sigma_1^{(n)}$ functions are closed under permutation of arguments, insertion of dummy arguments, and fusion of arguments of same type. Thus if $F(x_1^{i_1}, \dots x_n^{i_n})$ is $\Sigma_1^{(n)}$, so is $F'(y_1^{j_1}, \dots, y_m^{j_m})$ where

$$F'(y_1^{j_1}, \dots, y_m^{j_m}) \simeq F(y_{\sigma(1)}^{j_{\sigma(1)}}, \dots, y_{\sigma(n)}^{j_{\sigma(n)}})$$

and $\sigma : n \to m$ such that $j_{\sigma(l)} = i_l$ for $l < n$.

If $R(x_1^{j_1}, \dots, x_p^{j_p})$ is a relation and $F_i(\vec{z})$ is a function to H^{j_i} for $i = 1, \dots, n$, we sometimes use the abbreviation:

$$R(\vec{F}(\vec{z})) \leftrightarrow: \bigvee x_1^{j_1}, \dots x_p^{j_p} (\bigwedge_{i=1}^{p} x_i^{j_i} = F_i(\vec{z}) \wedge R(\vec{x})).$$

Note that $R(\vec{F}(\vec{z}))$ is then false if some $F_i(\vec{z})$ does not exist. $\Sigma_1^{(n)}$ relations are not, in general, closed under substitution of $\Sigma_1^{(n)}$ functions, but we do get:

Lemma 2.6.11. *Let $R(x_1^{j_1}, \dots, x_p^{j_p})$ be $\Sigma_1^{(n)}$ such that $j_i \leq n$ for $i = 1, \dots, p$. Let $F_i(\vec{z})$ be a $\Sigma_1^{(j_i)}$ map to H^{j_i} for $i = 1, \dots, p$. Then $R(\vec{F}(\vec{z}))$ is $\Sigma_1^{(n)}$ (uniformly in the $\Sigma_1^{(n)}$ definitions of $R, F_1, \dots, F_p$)*

Before proving Lemma 2.6.11 we show that it has the following corollary:

Corollary 2.6.12. *Let $R(\vec{x}, y_1^{j_1}, \dots, y_p^{j_p})$ be $\Sigma_1^{(n)}$ where $j_i \leq n$ for $i = 1, \dots, p$. Let $F_i(\vec{z})$ be a $\Sigma_1^{(j_i)}$ map to H^{j_i} for $i = 1, \dots, p$. Then $R(\vec{x}, \vec{F}(\vec{z}))$ is (uniformly) $\Sigma_1^{(n)}$.*

Proof: We can assume *w.l.o.g.* that each of $\vec{x}$ has type $\leq n$, since otherwise R is a specialization of an R' with this property. But then $R(\vec{x}, \vec{F}(z))$ is a specialization of $R'(\vec{x}, \vec{F}(z))$. Let $\vec{x} = x_1^{h_1}, \dots, x_q^{h_q}$ with $h_i \leq n$ for $i = 1, \dots, q$. For $i = 1, \dots, p$ set:

$$F'(\vec{x}, \vec{z}) \simeq F(\vec{z}).$$

For $i = 1, \ldots, q$ set:

$$G_h(\vec{x}, \vec{z}) \simeq x_i^{h_i}.$$

By Lemma 2.6.11, $R(\vec{G}(\vec{x}, \vec{z}), F'(\vec{x}, \vec{z}))$ is $\Sigma_1^{(n)}$. But

$$R(\vec{G}(\vec{x}, \vec{z}), F'(\vec{x}, \vec{z})) \leftrightarrow R(\vec{x}, \vec{F}(\vec{z})).$$

QED (Corollary 2.6.12)

We now prove Lemma 2.6.11 by induction on n.

Case 1 $n = 0$.

The conclusion is immediate by the definition of $R(\vec{F}(\vec{z}))$:

$$R(\vec{F}(\vec{z})) \leftrightarrow \bigvee x_1^0 \ldots x_p^0 (\bigwedge_{i=1}^{p} x_1^0 = F_i(\vec{z}) \wedge R(\vec{x})).$$

Case 2 $n = m + 1$.

Then Lemma 2.6.11 holds at m and it is clear from the above proof that Corollary 2.6.12 does, too.

Rearranging the arguments of R if necessary, we can bring R into the form:

$$R(\vec{x}^n, x_1^{l_1}, \ldots, x_q^{l_q}) \text{ where } l_i \leq m \text{ for } i = 1, \ldots, q.$$

We first show:

Claim $R(\vec{x}^n, \vec{F}(\vec{z}))$ is $\Sigma_1^{(n)}$.

Proof: Let $Q_i(\vec{z}_i^n, \vec{x})$ be $\Sigma_1^{(m)}(i = 1, \ldots, r)$ such that

$$R(x^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n]$$

where φ is Σ_1 and:

$$H_{\vec{x}} = \langle H^n, Q_{1,\vec{x}}, \ldots, Q_{r,\vec{x}} \rangle.$$

Set:

$$\begin{aligned} \overline{Q}_i(\vec{z}_i^n, \vec{z}) \quad &\leftrightarrow: Q_i(z_i^n, F(\vec{z})) \\ &\leftrightarrow \bigvee \vec{x} (\textstyle\bigwedge_{i=1}^{q} x_i^{l_i} = F_i(\vec{z}) \wedge R(\vec{x})) \\ &\overline{H}_{\vec{z}} =: \langle H^n, \overline{Q}_{1,\vec{z}}, \ldots, \overline{Q}_{r,\vec{z}} \rangle. \end{aligned}$$

If $x_i^{l_i} = F_i(\vec{z})$ for $i = 1, \ldots, q$, then $\overline{Q}_i(\vec{z}_i^n, \vec{z}) \leftrightarrow Q_i(\vec{z}^n, \vec{x})$ and $\overline{H}_{\vec{z}} = H_{\vec{x}}$. Hence:

$$\begin{aligned} \overline{H}_{\vec{z}} \models \varphi[\vec{x}^n] \quad &\leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n] \\ &\leftrightarrow R(\vec{x}^n, \vec{x}) \\ &\leftrightarrow R(\vec{x}^n, \vec{F}(\vec{z})). \end{aligned}$$

If, on the other hand, $F_i(\vec{z})$ does not exist for some i, then $R(\vec{x}^n, \vec{F}(\vec{z}))$ is false. Hence:

$$\begin{aligned} R(\vec{x}^n, \vec{F}(\vec{z})) \quad &\leftrightarrow (\bigwedge_{i=1}^{q} \bigvee x_i^{l_i}(x_i^{l_i} = F_i(\vec{z})) \\ &\wedge \overline{H}_{\vec{z}} \models \varphi[\vec{x}^n]). \end{aligned}$$

But $\bigwedge\limits_{i=1}^{q} \bigvee x_i^{l_i}(x_i^{l_i} = F_i(\vec{z}))$ is $\Sigma_0^{(n)}$, so the result follows by applying Lemma 2.6.7 to φ. QED (Claim)

But then, setting: $R'(\vec{x}^n, \vec{z}) \leftrightarrow R(\vec{x}^n, F(\vec{z}))$, we have:

$$R(\vec{F}(\vec{x})) \leftrightarrow \vee\vec{x}^n(\bigwedge_{i=1}^{q} x_i^n = F_i(\vec{z}) \wedge R'(\vec{x}^n, \vec{z})).$$

QED (Lemma 2.6.11)

Note that if, in the last claim, we took $R(\vec{x}^n, x_1^{l_1}, \ldots, x_q^{l_q})$ as being $\Sigma_0^{(n)}$ instead of $\Sigma_1^{(n)}$, then in the proof of the claim we could take φ as being Σ_0 instead of Σ_1. But then the application of Lemma 2.6.7 to $\overline{H}_{\vec{z}} \models \varphi[\vec{x}^n]$ yields a $\Sigma_0^{(n)}$ formula. Then we have, in effect, also proven:

Corollary 2.6.13. *Let $R(\vec{x}^n, y_1^{l_1}, \ldots, y_q^{l_q})$ be $\Sigma_0^{(n)}$ where $l_1, \ldots, l_r < n$. Let $F_i(\vec{z})$ be a $\Sigma_1^{(l_i)}$ map to H^{l_i} for $i = 1, \ldots, r$. Then $R(x^n, \vec{F}(\vec{z}))$ is (uniformly) $\Sigma_0^{(n)}$.*

As corollaries of Lemma 2.6.11 we then get:

Corollary 2.6.14. *Let $G(x_1^{j_1}, \ldots, x_p^{j_p})$ be a $\Sigma_1^{(n)}$ map to H^n, where $j_1, \ldots, j_p \leq n$. Let $F_i(\vec{z})$ be a $\Sigma_1^{(n)}$ map to H^{j_i} for $i = 1, \ldots, p$. Then $H(\vec{z}) \simeq G(\vec{F}(\vec{z}))$ is uniformly $\Sigma_1^{(n)}$.*

Proof:

$$y = H(\vec{z}) \leftrightarrow \bigvee \vec{x}(\bigwedge_{i=1}^{p} x_i^{j_i} = F_i(\vec{z}) \wedge y = G(\vec{x})).$$

QED (Corollary 2.6.14)

Corollary 2.6.15. *Let $R(x_1^{j_1}, \ldots, x_p^{j_p})$ be $\Sigma_1^{(n)}$ where $j_i \leq n$ for $i = 1, \ldots, p$. There is a $\Sigma_1^{(n)}$ relation $R'(z_1^0, \ldots, z_p^0)$ with the same field*

Proof: Set:

$$R'(\vec{z}) \leftrightarrow: \bigvee \vec{x}(\bigwedge_{i=1}^{p} x_i^{j_i} = z_i^0 \wedge R(\vec{x})).$$

QED (Corollary 2.6.15)

Thus in theory we can always get by with relations that have only arguments of type 0. (Lest one make too much of this, however, we remark that the defining formula of R' will still have bounded many sorted variables.)

Generalizing this, we see that if R is a relation with arguments of type $\leq n$, then the property of being $\Sigma_1^{(n)}$ depends only on the field of R. Let us define:

Definition 2.6.8. $R'(z_1^{j_1}, \ldots, z_r^{j_r})$ is a *reindexing* of the relation $R(x_1^{i_1}, \ldots, x_r^{i_r})$ iff both relations have the same field i.e.

$$R'(\vec{y}) \leftrightarrow R(\vec{y}) \text{ for } y_1, \ldots, y_r \in M.$$

Then:

Corollary 2.6.16. *Let $R(x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_1^{(n)}$ where $i_1, \ldots, i_r \leq n$. Let $R'(z_1^{j_1}, \ldots, z_r^{j_r})$ be a reindexing of R, where $j_1, \ldots, j_r \leq n$. Then R' is $\Sigma_1^{(n)}$.*

Proof:

$$\begin{aligned} R'(\vec{z}) & \leftrightarrow R(F_1(z_1), \ldots, F_r(z_r)) \\ & \leftrightarrow \vee \vec{x} (\textstyle\bigvee_{l=1}^{r} x_l^{i_l} = z_l^{j_l} \wedge R(\vec{x})) \end{aligned}$$

where

$$x^{i_l} = F_l(z^{j_l}) \leftrightarrow: x^{i_l} = z^{j_l}.$$

QED (Corollary 2.6.16)

We now consider the relationship between Σ^* theory and the theory developed in §2.5. $\Sigma_1^{(0)}$ is of course the same as Σ_1 and ρ_1 is the same as the Σ_1 projectum ρ which we defined in §2.5.2. In §2.5.2 we also defined the set P of good parameters and the set R of very good parameters. We then defined the reduct M of M_P for any $p \in [\mathrm{On}_M]^{<\omega}$. We now generalize these notions to $\Sigma_1^{(n)}$. We have already defined the $\Sigma_1^{(n)}$ projectum ρ^n. In analogy with the above we now define the sets P^n, R^n of $\Sigma_1^{(n)}$*–good* parameters. We also define the $\Sigma_1^{(n)}$ *reduct* M^{np} of M by $p \in [\mathrm{On}_M]^{<\omega}$.

Under the special assumption of soundness, these will turn out to be the same as the concepts defined in §2.5.3.

Definition 2.6.9. Let $M = \langle J_\alpha^A, B \rangle$ be acceptable. We define sets $M^n_{x^{n-1},\ldots,x^0}$ and predicates $T^n(x^n, \ldots, x^0)$ as follows:

$$\begin{aligned} & M^0 =: M, T^0 =: B \text{ (i.e. } M^n_{\vec{x}} = M \text{ for } n = 0) \\ & M^{n+1}_{\vec{x}} =: \langle J^A_{\rho^{n+1}}, T^{n+1}_{\vec{x}} \rangle \text{ for } \vec{x} = x^n, \ldots, x^0 \\ & \quad T^{n+1}(x^{n+1}, \vec{x}) \leftrightarrow \bigvee z^{n+1} \bigvee i < \omega (x^{n+1} = \langle i, z^{n+1} \rangle \\ & \qquad \wedge M^n_{x^{n-1},\ldots,x^0} \models \varphi_i[z^{n+1}, x^n]) \end{aligned}$$

(where $\langle\varphi_i | i < \omega\rangle$ is our fixed canonical enumeration of Σ_1 formulae.)

(Then $T^{n+1}(\langle i, x^{n+1}\rangle, x^n, \ldots, x^0) \leftrightarrow M^n_{x^{n-1},\ldots,x^0} \models \varphi_i[x^{n+1}, x^n]$).

Clearly T^{n+1} is uniformly $\Sigma_1^{(n)}(M)$.

Lemma 2.6.17.

(a) T^{n+1} *is* $\Sigma_1^{(n)}$

(b) Let φ *be* Σ_j*. Then* $\{\langle\vec{x}^{n+1}, \vec{x}\rangle | M^{n+1}_{\vec{x}} \models \varphi[\vec{x}^{n+1}]\}$ *is* $\Sigma_j^{(n+1)}$.

Proof: We first note that $M^{n+1}_{\vec{x}}$ can be written as $H_{\vec{x}} = \langle H^{n+1}, A^{n+1}_{\vec{x}}, T^{n+1}_{\vec{x}}\rangle$, where $A^{n+1}(x^{n+1}, \vec{x}) \leftrightarrow: A(x^{n+1})$. Hence by Lemma 2.6.7:

(1) If (a) holds at n, so does (b). But (a) then follows by induction on n:

Case 1 $n = 0$ is trivial since $\Vdash^{\Sigma_1}_N$ is $\Sigma_1(N)$ for all rud closed N.

Case 2 $n = m + 1$. Then $T^{(n+1)}$ is $\Sigma_1^{(n)}$ by (1) applied to m.

QED (Lemma 2.6.17)

We now prove *a* converse to Lemma 2.6.17.

Lemma 2.6.18. *(a) Let* $R(x^{n+1}, \ldots, x^0)$ *be* $\Sigma_1^{(n)}$*. Then there is* $i < \omega$ *such that*

$$R(x^{n+1}, \vec{x}) \leftrightarrow T^{n+1}(\langle i, x^{n+1}\rangle, \vec{x}).$$

(b) Let $R(\vec{x}^{n+1}, \ldots, x^0)$ *be* $\Sigma_1^{(n+1)}$*. Then there is a* Σ_1 *formula* φ *such that*

$$R(\vec{x}^{n+1}, \vec{x}) \leftrightarrow M^{n+1}_{\vec{x}} \models \varphi[\vec{x}^{n+1}].$$

Proof:

(1) Let (a) hold at n. Then so does (b).

Proof: We know that

$$R(\vec{x}^{n+1}, \vec{x}) \leftrightarrow \bigvee z^{n+1} P(z^{n+1}, x^{n+1}, \vec{x})$$

for a $\Sigma_0^{(n+1)}$ formula P. Hence it suffices to show:

Claim Let $P(\vec{x}^{n+1}, \vec{x})$ be $\Sigma_0^{(n+1)}$. Then there is a Σ_1 formula φ such that

$$P(\vec{x}^{n+1}, \vec{x}) \leftrightarrow M_{\vec{x}}^{n+1} \models \varphi[\vec{x}^{n+1}].$$

Proof: We know that there are $Q_i(\vec{z}_i^{n+1}, \vec{x})(i = 1, \ldots, p)$ such that Q_i is $\Sigma_1^{(n)}$ and

(2) $P(\vec{x}^{n+1}, \vec{x}) \leftrightarrow H_{\vec{x}}^{n+1} \models \Psi[\vec{x}^{n+1}]$ where Ψ is Σ_0 and

$$H_{\vec{x}}^n = \langle H^{n+1}, \vec{Q}_{\vec{x}} \rangle.$$

Applying (a) to the relation:

$$\bigvee u^{n+1}(u^{n+1} = \langle \vec{z}_i^{n+1} \rangle \wedge Q_i(\vec{z}_i^{n+1}, \vec{x}))$$

we see that for each i there is $j_i < \omega$ such that

$$Q_i(\vec{z}_i^{n+1}, \vec{x}) \leftrightarrow \langle j_i, \langle \vec{z}^{n+1} \rangle \rangle \in T_{vecx}^{n+1}.$$

Thus $Q_i, \vec{x}$ is uniformly rud in $T_{\vec{x}}^{n+1}$ for $i = 1, \ldots, p$. $P_{\vec{x}}$ is the restriction of a relation rud in $Q_{i,\vec{x}}(i = 1, \ldots, p)$ to H^{n+1}, by (2). By §2 Corollary 2.2.8 it follows that $P_{\vec{x}}$ is the restriction of a relation rud in $T_{\vec{x}}^{n+1}$ to H^{n+1} uniformly. Since $M_{\vec{x}}^{n+1} = \langle J_{\rho n+1}^A, T_{\vec{x}}^{n+1} \rangle$ is rud closed, it follows by §2 Corollary 2.2.8 that:

$$P(\vec{x}^{n+1}, \vec{x}) \leftrightarrow M_{\vec{x}}^{n+1} \models \varphi[\vec{x}^{n+1}]$$

for a Σ_1 formula φ. QED (1)

Given (1) we can now prove (a) by induction on n.

Case 1 $n = 0$.

Since $\Sigma_1 = \Sigma_1^{(0)}$, there is φ_i such that

$$\begin{aligned} R(x^1, x^0) &\leftrightarrow M \models \varphi_i[x^1, x^0] \\ &\leftrightarrow T^1(\langle i, x^1 \rangle, x^0). \end{aligned}$$

Case 2 $n = m + 1$.

Let $R(x^{n+1}, \ldots, x^0)$ be $\Sigma_1^{(n)}$. By the induction hypothesis and (1) we know that (b) holds at n. Hence:

$$\begin{gathered} R(x^{n+1}, x^{m+1}, x^m, \ldots, x^0) \leftrightarrow \\ \leftrightarrow M_{x^m,\ldots,x^0}^n \models \varphi_i[x^{n+1}, x^{m+1}] \end{gathered}$$

for some i. But then

$$R(x^{n+1}, \ldots, x^0) \leftrightarrow T^{n+1}(\langle i, x^{n+1} \rangle, x^{m+1}, \ldots, x^0).$$

QED (Lemma 2.6.18)

Note. The reductions in (a) and (b) are both uniform. We have in fact implicitly defined algorithms which in case (a) takes us from the $\Sigma_1^{(n)}$ definition of R to the integer i, and in case (b) takes us from the $\Sigma_1^{(n+1)}$ definition of R to the Σ_1 formula φ.

We now generalize the definition of *reduct* given in §2.5.2 as follows:

Definition 2.6.10. Let $a \in [\mathrm{On}_M]^{<\omega}$. $M^{0,a} =: M$; $M^{n+1,a} =: M^{n+1}_{a^{(0)},\ldots,a^{(n)}}$ where $a^{(i)} = a \cap \rho^i_M$.

Thus $M^{n+1,a} = \langle J^A_{\rho^{n+1}}, T^{n+1,a}\rangle$ where $T^{n+1,a} =: T^{n+1}_{a^{(0)},\ldots,a^{(n)}}$.

Thus by Lemma 2.6.18

Corollary 2.6.19. *Set* $a^{(i)} = a \cap \rho^i$ *for* $a \in [\mathrm{On}_M]^{<\omega}$.

(a) If $D \subset H^{n+1}$ is $\Sigma_1^{(n)}$ in $a^{(0)}, \ldots, a^{(n)}$, there is (uniformly) an $i < \omega$ such that

$$D(x^{n+1}) \leftrightarrow \langle i, x^{n+1}\rangle \in T^{n+1,a}$$

(b) If $D(\vec{x}^{n+1})$ is $\Sigma_1^{(n+1)}$ in $a^{(0)}, \ldots, a^{(n)}$ there is (uniformly) a Σ_1 formula φ such that $D(\vec{x}^{n+1}) \leftrightarrow M^{n+1,a} \models \varphi[\vec{x}^{n+1}]$.

Note. Being $\Sigma_1^{(n)}$ in a is the same as being $\Sigma_1^{(n)}$ in $a^{(0)}, \ldots, a^{(n)}$, but I do not see how this is uniformly so. To see that a $\Sigma_1^{(n)}$ relation R in $a^{(0)}, \ldots, a^{(n)}$ is $\Sigma_1^{(n)}$ in a we note that for each n there is k such that $y = a \cap \rho^n \leftrightarrow \bigvee f$ (f is the monotone enumeration of a and $y = f''k$), which is Σ_1 in a. However, k cannot be inferred from the $\Sigma_1^{(n)}$ definition of R, so the reduction is not uniform.

We can generalize the good parameter sets P, R of §2.5.2 as follows:

Definition 2.6.11. $P^0_M =: [\mathrm{On}]^{<\omega}$.

$P^{n+1}_M =:$ the set of $a \in P^n_M$ such that there is D which is $\Sigma_1^{(n)}(M)$ in a with $D \cap H^n_M \notin M$.

(Thus we obviously have $P^1 = P$.)

Similarly:

Definition 2.6.12. $R^0_M =: P^0_M$.

R_M^{n+1} =: The set of $a \in R_M^n$ such that

$$M^{n,a} = h_{M^{n,a}}(\rho^{n+1} \cup (a \cap \rho^n)).$$

Comparing these definitions with those in §2.5.6 it is apparent that R_M^n has the same meaning and that, whenever $a \in R_M^n$, then $M^{n,a}$ is the same structure.

By a virtual repetition of the proof of Lemma 2.5.8 we get:

Lemma 2.6.20. *$a \in P^n \leftrightarrow T^{na} \notin M$.*

We also note the following fact:

Lemma 2.6.21. *Let $a \in R^n$. Let D be $\underline{\Sigma}_1^{(n)}$. Then D is $\Sigma_1^{(n)}$ in parameters from $\rho^{n+1} \cup \{a^{(0)}, \ldots, a^{(n)}\}$, where $a^{(i)} =: a \cap \rho^i$. (Hence D is $\Sigma_1^{(n)}(M)$ in parameters from $\rho^{n+1} \cup \{a\}$.)*

Proof: We use induction on n. Let it hold below n. Then:

$$D(\vec{x}) \leftrightarrow D'(\vec{x}; a^{(0)}, \ldots, a^{(n-1)}, \vec{\xi}),$$

where $\xi_1, \ldots, \xi_r < \rho^n$. (If $n = 0$ the sequence $a^{(0)}, \ldots, a^{(n-1)}$ is vacuous and $\rho^n = \mathrm{On}_M$.)

Let $\xi_i = h_{M^{n+1}}(j_i, \langle \mu_i, a^{(n)} \rangle)$, where $\mu_1, \ldots, \mu_r < \rho^{n+1}$. The functions:

$$F_i(x) \simeq h_{M^{na}}(j_i, \langle x, a^{(n)} \rangle)$$

are $\Sigma_1^{(n)}$ to H^n in the parameters $a^{(0)}, \ldots, a^{(n)}$. But $D(\vec{x})$ then has the form:

$$D'(\vec{x}, a^{(0)}, \ldots, a^{(n-1)}, F_1(\mu_1), \ldots, F_r(\mu_r)),$$

which is $\Sigma_1^{(n)}$ in $a^{(0)}, \ldots, a^{(n)}, \mu_1, \ldots, \mu_k$ by Corollary 2.6.12.

QED (Lemma 2.6.21)

Definition 2.6.13. π is a $\Sigma_h^{(n)}$ *preserving map of* $\overline{M}$ to M (in symbols $\pi : \overline{M} \to_{\Sigma_h^{(n)}} M$) iff the following hold:

- $\overline{M}, M$ are acceptable structures of the same type.
- $\pi'' H_{\overline{M}}^i \subset H_M^i$ for $i \leq n$.
- Let $\varphi = \varphi(v_1^{j_1}, \ldots, v_m^{j_m})$ be a $\Sigma_h^{(n)}$ formula with a good sequence $\vec{v}$ of variables such that $j_1, \ldots, j_m \leq n$. Let $x_i \in H_{\overline{M}}^{j_i}$ for $i = 1, \ldots, m$. Then:
$$\overline{M} \models \varphi[\vec{x}] \leftrightarrow M \models \varphi[\pi(\vec{x})].$$

π is then a structure preserving injection. If it is $\Sigma_h^{(n)}$–preserving, it is $\Sigma_1^{(m)}$–preserving for $m < n$ and $\Sigma_i^{(n)}$–preserving for $i < h$. If $h \geq 1$ then $\pi^{-1}{}''H_M^n \subset H_{\overline{M}}^n$, as can be seen using:

$$x \in H_M^n \leftrightarrow M \models \bigvee u^n u^n = v^0[x].$$

We say that π is *strictly* $\Sigma_h^{(n)}$ *preserving* (in symbols $\pi : \overline{M} \to_{\Sigma_h^{(n)}} M$ strictly) iff it is $\Sigma_h^{(n)}$ preserving and $\pi^{-1}{}''H_M^n \subset H_{\overline{M}}^n$. (Only if $h = 0$ can the embedding fail to be strict.)

We say that π is Σ^* *preserving* ($\pi : \overline{M} \to_{\Sigma^*} M$) iff it is $\Sigma_1^{(n)}$ preserving for all $n < \omega$. We call π $\Sigma_\omega^{(n)}$ *preserving* iff it is $\Sigma_h^{(n)}$ preserving for all $h < \omega$.

Good functions

Let $n < \omega$. Consider the class $\mathbb{F}$ of all $\Sigma_1^{(n)}$ functions $F(x^{i_1}, \dots, x^{i_m})$ to H^j, where $j, i_1, \dots, i_m \leq n$. This class is not necessarily closed under composition. If, however, $\mathbb{G}^0$ is the class of $\Sigma_1^{(j)}$ functions $G(z^{i_1}, \dots, z^{i_m})$ to H^j where $j, i_1, \dots, i_m \leq n$, then $\mathbb{G}^0 \subset \mathbb{F}$ and, as we have seen, elements of $\mathbb{G}^0$ can be composed into elements of $\mathbb{F}$ — i.e. if $F(z^{i_1}, \dots, z^{i_m})$ is in $\mathbb{F}$ and $G_l(\vec{x})$ is in $\mathbb{G}^0$ for $l = 1, \dots, m$, then $F(\vec{G}(\vec{x}))$ lies in $\mathbb{F}$. The class $\mathbb{G}$ of *good* $\Sigma_1^{(n)}$ functions is the result of closing $\mathbb{G}^0$ under composition. The elements of $\mathbb{G}$ are all $\Sigma_1^{(n)}$ functions and $\mathbb{G}$ is closed under composition. The precise definition is:

Definition 2.6.14. Fix acceptable M. We define sets $\mathbb{G}^k = \mathbb{G}_n^k$ of $\Sigma_1^{(n)}$ functions by:

$\mathbb{G}^0 =$ The set of partial $\Sigma_1^{(i)}$ maps $F(x_1^{j_1}, \dots, x_m^{j_m})$ to H^i, where $i \leq n$ and $j_1, \dots, j_m \leq n$.

$\mathbb{G}^{k+1} =$ The set of $H(\vec{x}) \simeq G(\vec{F}(\vec{x}))$, such that $G(y^{j_1}, \dots, y_m^{j_m})$ is in G^k and $F_l \in \mathbb{G}^0$ is a map to j_l for $l = 1, \dots, m$.

It follows easily that $\mathbb{G}^k \subset \mathbb{G}_{k+1}^k$ (since $G(\vec{y}) \simeq G(\vec{h}(\vec{y}))$ where $h(y_1^{j_1}, \dots, y_m^{j_m}) = y_i^{j_i}$ for $i = 1, \dots, m$). $\mathbb{G} = \mathbb{G}_n =: \bigcup_k \mathbb{G}^k$ is then the set of all *good* $\Sigma_1^{(n)}$ *functions* $\mathbb{G}^* = \bigcup_n \mathbb{G}_n$ is the set of all *good* Σ^* *functions.* All good $\Sigma_1^{(n)}$ functions have a functionally absolute $\Sigma_1^{(n)}$ definition. Moreover, the good $\Sigma_1^{(n)}$ functions are closed under permutation of arguments, insertion of dummy arguments, and fusion of arguments of same type (i.e. if $F(x_0^{i_1},\ldots,x_{m-1}^{j_p})$ is good, then so is $F'(\vec{y}) \simeq F(y_{\sigma(1)}^{j_{\sigma(1)}},\ldots,y_{\sigma(m)}^{j_{\sigma(m)}})$ where $\sigma : m \to p$ such that $j_{\sigma(l)} = i_l$ for $l < m$.

To see this, one proves by a simple induction on k that:

Lemma 2.6.22. *Each $\mathbb{G}_n^k$ has the above properties.*

The proof is quite straightforward. We then get:

Lemma 2.6.23. *The good $\Sigma_1^{(n)}$ functions are closed under composition: Let $G(y_1^{j_1},\ldots,y_m^{j_m})$ be good and let $F_l(\vec{x})$ be a good function to H^{j_l} for $l = 1,\ldots,m$. Then the function $G(\vec{F}(\vec{x}))$ is good.*

Proof: By induction in $k < \omega$ we prove:

Claim The above holds for $F_l \in \mathbb{G}^k (l = 1,\ldots,m)$.

Case 1 $k = 0$.
This is trivial by the definition of "good function".

Case 2 $k = h + 1$.
Let:

$$F_l(\vec{x}) \simeq H_l(F_{l,1}(\vec{x}),\ldots,F_{l,p_l}(\vec{x}))$$

for $l = 1,\ldots,m$, where $H_l(z_{l,1},\ldots,z_{l,p_l})$ is in $\mathbb{G}^h$ and $F_{l,i} \in G^0$ is a map to $H^{j_{l,i}}$ for $l = 1,\ldots,m, i = 1,\ldots,p_l$.

Let $\langle\langle l_\xi, i_\xi\rangle | \xi = 1,\ldots,p\rangle$ enumerate

$$\{\langle l,i\rangle | l = 1,\ldots,m;\ i = 1,\ldots,p_l\}.$$

Define $\sigma_l : \{1,\ldots,p_l\} \to \{1,\ldots,p\}$ by:

$$\sigma_l(i) = \text{ that } \xi \text{ such that } \langle l,i\rangle = \langle l_\xi, i_\xi\rangle.$$

Set:

$$H'_l(z_1,\ldots,z_p) \simeq H_l(z_{\sigma_l(1)},\ldots,z_{\sigma_l(p_l)})$$

for $l = 1,\ldots,m$. $F'_\xi = F_{l_\xi,i_\xi}$ for $\xi = 1,\ldots,p$.

Clearly we have:

$$F_l(\vec{x}) = H'_l(F'_1(\vec{x}),\ldots,F'_p(\vec{x}))$$

where $H'_l \in \mathbb{G}^h$ for $l = 1,\ldots,m$. Set:

$$G'(z_1,\ldots,z_p| \simeq G(H_1(\vec{z}),\ldots,H_m(\vec{z})).$$

Then G' is a good $\Sigma_1^{(n)}$ function by the induction hypothesis. But:

$$G(\vec{F}(\vec{x})) \simeq G'(F_1'(\vec{x}), \ldots, F_p'(\vec{x})).$$

The conclusion then follows by Case 1, since $F_i' \in \mathbb{G}^0$ for $i = 1, \ldots, p$.

QED (Lemma 2.6.23)

An entirely similar proof yields:

Lemma 2.6.24. *Let $R(x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_1^{(n)}$ where $i_1, \ldots, i_r \le n$. Let $F_l(\vec{z})$ be a good $\Sigma_1^{(n)}$ map to $H^{i_l}(L = 1, \ldots, m)$. Then $R(\vec{F}(\vec{z}))$ is $\Sigma_1^{(n)}$.*

Recall that $R(\vec{F}(\vec{z}))$ means:

$$\bigvee y_1, \ldots, y_r (\bigwedge_{l=1}^{r} y_l = F_l(\vec{z}) \wedge R(\vec{y})).)$$

Applying Corollary 2.6.13 we also get:

Lemma 2.6.25. *Let $n = m + 1$. Let $R(\vec{x}^n, x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_0^{(n)}$ where $i_1, \ldots, i_r \le m$. Let $F_l(\vec{z})$ be a good $\Sigma_1^{(n)}$ map to H^{i_l} for $l = 1, \ldots, r$. Then $R(\vec{x}^n, \vec{F}(\vec{z}))$ is $\Sigma_0^{(n)}$.*

By a *reindexing* of a function $G(x_1^{i_1}, \ldots, x_r^{i_r})$ we mean any function G' which is a reindexing of G as a relation. (In other words G, G' have the same field, i.e.

$$G(\vec{x}) \simeq G'(\vec{x}) \text{ for all } x_1, \ldots, x_r \in M.)$$

Then:

Corollary 2.6.26. *Let $G(x_1^{i_1}, \ldots, x_r^{i_r})$ be a good $\Sigma_1^{(m)}$ map to H^i. Let $G'(y_1^{j_1}, \ldots, y_r^{j_r})$ be a map to H^j, where $j, j_1, \ldots, j_r \le n$. If G' is a reindexing of G, then G' is a good $\Sigma_1^{(m)}$ function.*

Proof: $G'(y) \simeq F(G(F_1(y_1^{j_1}), \ldots, F(y_r^{j_r})))$ where F is defined by $x^i = y^i$ and F_l is defined by $x_l^{i_l} = y_l^{j_l}$. (Then e.g.

$$F(y) = \begin{cases} y \text{ if } y \in H_M^{\min\{i,j\}}, \\ \text{undefined if not.} \end{cases}$$

where F is a map to i with arity j.)
But $F, F_1 \ldots, F_r$ are $\Sigma_1^{(n)}$ good. QED (Corollary 2.6.26)

The statement made earlier that every good $\Sigma_1^{(n)}$ function has a functionally absolute $\Sigma_1^{(n)}$ definition can be improved. We define:

Definition 2.6.15. φ is a *good* $\Sigma_1^{(n)}$ *definition* iff φ is a $\Sigma_1^{(n)}$ formula which defines a good $\Sigma_1^{(n)}$ function over any acceptable M of the given type.

Lemma 2.6.27. *Every good $\Sigma_1^{(n)}$ function has a good $\Sigma_1^{(n)}$ definition.*

Proof: By induction on k we show that it is true for all elements of $\mathbb{G}^k$. If $F \in \mathbb{G}^0$, then F is a $\Sigma_1^{(i)}$ map to H^i for an $i \leq n$. Hence any functionally absolute $\Sigma_1^{(i)}$ definition will do. Now let $F \in \mathbb{G}^{k+1}$. Then $F(\vec{x}) \simeq G(H_1(\vec{x}), \ldots, H_p(\vec{x}))$ where $G \in \mathbb{G}^k$ and $H_i \in \mathbb{G}^0$ for $i = 1, \ldots, p$. Then G has a good definition φ and every H_i has a good definition Ψ_i. By the uniformity expressed in Corollary 2.6.14 there is a $\Sigma_1^{(n)}$ formula χ such that, given *any* acceptable M of the given type, if φ defines G' and Ψ_i defines $H_i'(i = 1, \ldots, p)$, then χ defines $F'(\vec{x}) \simeq G'(\vec{H}'(\vec{x}))$. Thus χ is a good $\Sigma_1^{(n)}$ definition of F. QED (Lemma 2.6.27)

Definition 2.6.16. Let $a \in [\mathrm{On}_M]^{<\omega}$. We define partial maps h_a from $\omega \times H^n$ to H^n by:

$$h_a^n(i, x) \simeq: h_{M^{n,a}}(i, \langle x, a^{(n)} \rangle).$$

Then h_a^n is uniformly $\Sigma_1^{(n)}$ in $a^{(n)}, \ldots, a^{(0)}$. We then define maps $\tilde{h}_a^n$ from $\omega \times H^n$ to H^0 by:

$$\begin{aligned}
&\tilde{h}_a^0(i, x) \simeq h_a^o(i, x) \\
&\tilde{h}_a^{n+1}(i, x) \simeq \tilde{h}_a^n((i)_0, h_a^{n+1}((i)_1, x)).
\end{aligned}$$

Then $\tilde{h}_a^n$ is a good $\Sigma_1^{(n)}$ function uniformly in $a^{(n)}, \ldots, a^{(0)}$.

Clearly, if $a \in R^{n+1}$, then

$$h_a^{n\prime\prime}(\omega \times \rho^{n+1}) = H^n.$$

Hence:

Lemma 2.6.28. *If $a \in R^{n+1}$, then $\tilde{h}_a^{n\prime\prime}(\omega \times \rho^{n+1}) = M$.*

Corollary 2.6.29. *If $R^n \neq \emptyset$, then $\underline{\Sigma}_l \subset \underline{\Sigma}_l^{(n)}$ for $l \geq 1$.*

Proof: Trivial for $n = 0$, since $\Sigma_l^{(0)} = \Sigma_l$. Now let $n = m + 1$. Set: $D = H^n \cap \mathrm{dom}(h_a^n)$, where $a \in R^n$. Then D is $\underline{\Sigma}_1^{(n)}$ by Lemma 2.6.24, since:

$$\begin{aligned}
x^n \in D \quad &\leftrightarrow h_a^n(x^n) = h_a^n(x^n) \\
&\leftrightarrow \bigvee z^0 (z^0 = h_a^n(x^n) \wedge z^0 = z^0).
\end{aligned}$$

Let $R(\vec{x})$ be $\Sigma_l(M)$. Let

$$R(\vec{x}) \leftrightarrow Q_1 z_1 \dots Q z_l P(\vec{z}, \vec{x})$$

where P is Σ_0. Set:

$$P'(\vec{u}^n, \vec{x}) \leftrightarrow: P(\vec{h}^n(\vec{u}^n), \vec{x}).$$

Then P' is $\Sigma_1^{(n)}$ in a. But for $u_1^n, \dots, u_l^n \in D$, $\neg P'(\vec{u}^n, \vec{x})$ can also be written as a $\Sigma_1^{(n)}$ formula. Hence

$$R(\vec{x}) \leftrightarrow Q u_1^n \in D \dots Q u_l^n \in D P'(\vec{u}^n, \vec{x})$$

is $\Sigma_l^{(n)}$ in a. QED (Corollary 2.6.29)

We have seen that every $\underline{\Sigma}_\omega^{(n)}$ relation is $\underline{\Sigma}_\omega$. Hence:

Corollary 2.6.30. *Let $R^n \neq \emptyset$. Then $\underline{\Sigma}_\omega^{(n)} = \underline{\Sigma}_\omega$.*

An obvious corollary of Lemma 2.6.28 is:

Corollary 2.6.31. *Let $a \in R_M^n$. Then every element of M has the form $F(\xi, a^{(0)}, \dots, a^{(n)})$ where F is a good $\Sigma_1^{(n)}$ function and $\xi < \rho^n$.*

Using this we now prove a downward extension of embeddings lemma which strengthens and generalizes Lemma 2.5.12

Lemma 2.6.32. *Let $n = m + 1$. Let $a \in [\mathrm{On}_M]^{<\omega}$ and let $N = M^{na}$. Let $\overline{\pi} : \overline{N} \to_{\Sigma_j} N$, where $\overline{N}$ is a J–model. Then:*

(a) There are unique $\overline{M}, \overline{a}$ such that $\overline{a} \in R_{\overline{M}}^n$ and $\overline{M}^{n\overline{a}} = \overline{N}$.

(b) There is a unique $\pi \supset \overline{\pi}$ such that $\pi : \overline{M} \to_{\Sigma_0^{(m)}} M$ strictly and $\pi(\overline{a}) = a$.

(c) $\pi : \overline{M} \to_{\Sigma_j^{(n)}} M$.

Proof: We first prove existence, then uniqueness. The existence assertion in (a) follows by:

Claim 1 There are $\overline{M}, \overline{a}, \hat{\pi} \supset \overline{\pi}$ such that $\overline{M}^{na} = \overline{N}$, $a \in R_{\overline{M}}^n$, $\hat{\pi} : \overline{M} \to_{\Sigma_1} M$, $\hat{\pi}(\overline{a}) = a$.

Proof: We proceed by induction on m. For $m = 0$ this immediate by Lemma 2.5.12. Now let $m = h + 1$. We first apply Lemma 2.5.12 to M^{ma}. It is clear from our definition that $\rho_{M^{m,a}} \geq \rho_M^n$. Set $N' = (M^{m,a})^{a \cap \rho_M^m}$. Then $N' = \langle J_{\rho'}^A, T' \rangle$, where $\rho' = \rho_{M^{ma}}$. But it is clear from our definition that $T^{na} = T' \cap J_{\rho_M^n}^A$. Hence:

(1) $\overline{\pi} : \overline{N} \to_{\Sigma_0} N'$.
By Lemma 2.5.12 there are then $\tilde{M}, \tilde{a}, \tilde{\pi} \supset \overline{\pi}$ such that $\tilde{M}^{\tilde{a}} = N'$, $\tilde{a} \in R_{\tilde{M}}$, $\tilde{\pi} : \tilde{M} \to_{\Sigma_1} M^{m,a}$ and $\tilde{\pi}(\tilde{a}) = a \cap \rho_M^m = a^{(m)}$.
(**Note:** Throughout this proof we use the notation:

$$a^{(i)} =: a \cap \rho^i \text{ for } i = 0, \dots, m.)$$

By the induction hypothesis there are then $\overline{M}, \overline{a}, \hat{\pi} \supset \tilde{\pi}$ such that $\overline{M}^{m\overline{a}} = \tilde{M}$, $\hat{\pi} : \overline{M} \to_{\Sigma_1} M$, and $\hat{\pi}(\overline{a}) = a$.
We observe that:

(2) $\tilde{a} = \overline{a} \cap \rho_{\overline{M}}^m$.

Proof:

($\subset$) Let $\tilde{\rho} =: \rho_{\overline{M}}^m = \mathrm{On} \cap \tilde{M}$. Then $\tilde{a} \subset \tilde{\rho}$. But $\hat{\pi}(\tilde{a}) = \tilde{\pi}(\tilde{a}) = a \cap \rho_M^m \subset a = \hat{\pi}(\overline{a})$. Hence $\tilde{a} \subset a$.

($\supset$) $\hat{\pi}(\overline{a} \cap \tilde{\rho}) = \hat{\pi}''(\overline{a} \cap \tilde{\rho}) \subset \rho_M^m \cap a = \hat{\pi}(\tilde{a})$, since $\hat{\pi}''\tilde{\rho} \subset \rho_M^m$. Hence $\overline{a} \cap \tilde{\rho} = \tilde{a}$. QED (2)

Since $\tilde{a} \in R_{\overline{M}}^{m\overline{a}}$ we conclude that $a \in R_{\overline{M}}^n$ and $\overline{N} = (M^{m\overline{a}})^{a \cap \tilde{\rho}} = \overline{M}^{n,\overline{a}}$. QED (Claim 1)

We now turn to the existence assertion in (b).

Claim 2 Let $\overline{M}^{\overline{a}} = N$ and $\overline{a} \in R_{\overline{M}}^n$. There is $\pi \supset \overline{\pi}$ such that $\pi : \overline{M} \to_{\Sigma_1^{(m)}} M$ and $\pi(\overline{a}) = a$.

Proof: Let $x_1, \dots, x_n \in \overline{M}$ with $x_i = \overline{F}_i(z_i) (i = 1, \dots, r)$, where $\overline{F}_i$ is a $\Sigma_1^{(m)}(\overline{M})$ good function in the parameters $\overline{a}^{(0)}, \dots, \overline{a}^{(n)}$ and $z_i \in \overline{N}$. Let F_i have the same $\Sigma_1^{(m)}(M)$–good definition in $a^{(0)}, \dots, a^{(m)}$. Let $\overline{R}(u_1, \dots, u_r)$ be a $\Sigma_1^{(n)}(\overline{M})$ relation and let R be $\Sigma_1^{(n)}(M)$ by the same definition.

Then $\overline{R}(\overline{F}_1(z_1), \dots, \overline{F}_r(z_r))$ is $\Sigma_1^{(m)}(\overline{M})$ in $\overline{a}^{(0)}, \dots, \overline{a}^{(m)}$ and $R(F_1(z_1), \dots, F_r(z_r))$ is $\Sigma_1^{(m)}(M)$ in $a^{(0)}, \dots, a^{(m)}$ by the same definition. Hence there is $i < \omega$ such that

$$\overline{R}(\overline{F}(\vec{z}) \leftrightarrow \langle i, \langle \vec{z} \rangle \rangle \in \overline{T}$$
$$R(F(\vec{z})) \leftrightarrow \langle i, \langle \vec{z} \rangle \rangle \in T$$

where $\overline{N} = \langle J_{\overline{\rho}}^{\overline{A}}, \overline{T} \rangle, N = \langle J_\rho^A, T \rangle$. Thus $\overline{R}(\overline{F}(\vec{z}))$ is rud in $\overline{N}$ and $R(F(\vec{z}))$ is rud in N by the same rud definition. But $\overline{\pi} : \overline{N} \to_{\Sigma_0} N$.

Hence:

$$\overline{R}(\overline{F}_1(z_i), \dots, \overline{F}_r(z_r)) \leftrightarrow R(F_1(\overline{\pi}(z_1)), \dots, F_r(\overline{\pi}(z_r))).$$

Thus there is $\pi : \overline{M} \to_{\Sigma_1^{(n)}} M$ defined by $\pi(\overline{F}(\xi)) =: F(\overline{\pi}(\xi))$ whenever $\xi \in \mathrm{On} \cap \overline{N}$, $\overline{F}$ is $\Sigma_1^{(m)}(\overline{M})$– good in $\overline{a}^{(0)}, \dots, \overline{a}^{(m)}$ and F is $\Sigma_1^{(m)}(M)$–good in $a^{(0)}, \dots, a^{(m)}$ by the same definition. But then

$$\pi(z) = \pi(\mathrm{id}(z)) = \overline{\pi}(z) \text{ for } z \in \overline{N}.$$

Hence $\pi \supset \overline{\pi}$. But clearly

$$\begin{aligned} \pi(\overline{a}) &= \pi(\overline{a}^{(0)} \cup \ldots \cup \overline{a}^{(m)}) \\ &= a^{(0)} \cup \ldots \cup a^{(m)} = a. \end{aligned}$$

QED (Claim 2)

We now verify (c):

Claim 3 Let $\overline{M}, \overline{a}, \pi$ be as in Claim 2. Then $\pi : \overline{M} \to_{\Sigma_j^{(n)}} M$.

Proof: We first note that π, being $\Sigma_1^{(n)}$–preserving, is *strictly* so — i.e. $\rho^i_{\overline{M}} = \pi^{-1}{''}\rho^i_M$ for $i = 0, \dots, m$. It follows easily that:

$$\pi(\overline{a}^{(i)}) = \pi''\overline{a}^{(i)} = a^{(i)} \text{ for } i = 0, \dots, m.$$

We now proceed the cases.

Case 1 $j = 0$.

It suffices to show that if φ is $\Sigma_1^{(n)}$ and $x_1, \dots, x_r \in \overline{N}$, then

$$\overline{M} \models \varphi[x_1, \dots, x_r] \to M \models \varphi[\pi(x_1), \dots, \pi(x_r)].$$

Let $x_1, \dots, x_r \in \overline{M}$. Then $x_i = \overline{F}_i(z_i) (i = 1, \dots, r)$ where $z_i \in \overline{N}$ and $\overline{F}_i$ is $\Sigma_1^{(m)}(\overline{M})$–good in $\overline{a}^{(0)}, \dots, \overline{a}^{(m)}$. Let F_i be $\Sigma_1^{(m)}(M)$–good in $a^{(0)}, \dots, a^{(m)}$ by the same good definition.

By Corollary 2.6.19, we know that $\overline{M} \models \varphi[\overline{F}_1(z_1), \dots, \overline{F}_r(z_r)]$ is equivalent to

$$\overline{N} \models \Psi[z_1, \dots, z_r]$$

for a certain Σ_1 formula Ψ. The same reduction on the M side shows that $M \models \varphi[F_1(z_1), \dots, F_r(z_r)]$ is equivalent to: $N \models \Psi[z_1, \dots, z_r]$ for $z_1, \dots, z_r \in N$, where Ψ is the same formula.

Since π is Σ_0–preserving we then get:

$$\begin{aligned} \overline{M} \models \varphi[\vec{x}] &\leftrightarrow \overline{M} \models \varphi[\overline{F}(\vec{z})] \\ &\leftrightarrow \overline{N} \models \Psi[\vec{z}] \\ &\to N \models \Psi[\pi(\vec{z})] \\ &\leftrightarrow M \models \varphi[F(\pi(\vec{z}))] \\ &\leftrightarrow M \models \varphi[\pi(\vec{x})]. \end{aligned}$$

QED (Case 1)

Case 2 $j > 0$.

This is entirely similar. Let φ be $\Sigma_j^{(n)}$. By Corollary 2.6.19 it follows easily that there is a Σ_j formula Ψ such that: $\overline{M} \models \varphi[\overline{F}_1(z_1), \ldots, \overline{F}_r(z_r)]$ is equivalent to:

$$\overline{N} \models \Psi[z_1, \ldots, z_r].$$

Since the corresponding reduction holds on the M–side, we get

$$\overline{M} \models \varphi[\vec{x}] \leftrightarrow M \models \varphi[\pi(\vec{x})],$$

since $\pi(x_i) = \pi(\overline{F}_i(z_i)) = F_i(\overline{\pi}(z_i))$. QED (Claim 3)

This proves existence. We now prove uniqueness.

Claim 4 The uniqueness assertion of (a) holds.

Proof: Let $\hat{M}, \hat{a}$ be such that $\hat{M}^{n,\hat{a}} = \overline{N}$ and $\hat{a} \in R_{\hat{M}}^N$.
Claim $\hat{M} = \overline{M}, \hat{a} = \overline{a}$.

Proof: By a virtual repetition of the proof in Claim 2 there is a $\pi : \hat{M} \to_{\Sigma_1^{(m)}} \overline{M}$ defined by:

(3) $\pi(\hat{F}(z)) = \overline{F}(z)$ whenever $z \in \overline{N}$, $\hat{F}$ is a good $\Sigma_1^{(m)}(\hat{M})$ function in $\hat{a}^{(0)}, \ldots, \hat{a}^{(m)}$ and $\overline{F}$ is the $\Sigma_1^{(m)}(\overline{M})$ function in $\overline{a}^{(0)}, \ldots, \overline{a}^{(m)}$ with the same good definition.

But π is then onto. Hence π is an isomorphism of $\hat{M}$ with $\overline{M}$. Since $\hat{M}, \overline{M}$ are transitive, we conclude that $\overline{M} = \hat{M}, \overline{a} = \hat{a}$.

QED (Claim 4)

Finally we prove the uniqueness assertion of (b):

Claim 5 Let $\pi' : \overline{M} \to_{\Sigma_0^{(m)}} M$ strictly, such that $\pi'(\overline{a}) = a$. Then $\pi' = \pi$.

Proof: By strictness we can again conclude that $\pi'(\overline{a}^{(i)}) = a^{(i)}$ for $i = 0, \ldots, m$. Let $x \in \overline{M}$, $x = \overline{F}(z)$, where $z \in \overline{N}$ and $\overline{F}$ is a $\Sigma_1^{(m)}(\overline{M})$ good function in the parameters $\overline{a}^{(0)}, \ldots, \overline{a}^{(m)}$. Let F be $\Sigma_1^{(m)}(M)$ in $a^{(0)}, \ldots, a^{(m)}$ by the same good definition.
The statement: $x = \overline{F}(z)$ is $\Sigma_2^{(m)}(\overline{M})$ in $\overline{a}^{(0)}, \ldots, \overline{a}^{(m)}$. Since π' is $\Sigma_0^{(m)}$–preserving, the corresponding statement must hold in M — i.e. $\pi'(x) = F(\overline{\pi}(z)) = \pi(x)$.

QED (Lemma 2.6.32)