proving the following theorem: Let κ be a measurable cardinal. Assume that V_κ has no inner model with a Woodin cardinal. Then there is V–definable inner model K of V_κ which, relativized to V_κ, has he above two properties. This result, which was exposited in [CMI] was an enormous breakthrough, which laid the foundation for all that has been done in inner model theory since then. There remained, however, the pesky problem of doing without the measurable — i.e. constructing K and proving its properties assuming only "ZFC+ there is no inner model with a Woodin". The first step was to construct the model K^C from this assumption. This was almost achieved by Mitchell and Schindler in 2001, except that they needed the additional hypothesis: GCH. Steel then showed that this hypothesis was superfluous. These results were obtained by directly weakening the "background condition" originally used by Steel in constructing K^C. The result of Mitchell and Schindler were published in [UEM]. Independently, Jensen found a construction of K^C using a different background condition called "robustness". This is exposited in [RE]. There reamained the problem of extracting a core model K from K^C. Jensen and Steel finally achieved this result in 2007. It was exposited in [JS].

In the next section we deal with the notion of *extenders*, which is essential to the rest of the book. (We shall, however, deal only with so called "short extenders", whose length is less than or equal to the image of the critical point.)

3.2 Extenders

The *extender* is a generalization of the normal ultrafilter. A normal ultrafilter at κ can be described by a two valued function on $\mathbb{P}(\kappa)$. An extender, on the other hand, is characterized by a map of $\mathbb{P}(\kappa)$ to $\mathbb{P}(\lambda)$, where $\lambda > \kappa$. λ is then called the *length* of the extender. Like a normal ultrafilter an extender F induces a canonical elementary embedding of the universe V into an inner model W. We express this in symbols by: $\pi : V \to_F W$. W is then called the *ultrapower* of V by F and π is called the *canonical embedding* induced by F. The pair $\langle W, \pi\rangle$ is called the *extension* of V by F. We will always have: $\lambda \leq \pi(\kappa)$. However, just as with ultrafilters, we shall also want to apply extenders to transitive models M which may be smaller than V. F might then not be an element of M. Moreover $\mathbb{P}(\kappa)$ might not be a subset of M, in which case F is defined on the smaller set $U = \mathbb{P}(\kappa) \cap M$. Thus we must generalize the notion of extenders, countenancing "suitable" subsets of $\mathbb{P}(\kappa)$ as extender domains. (However, the ultrapower of M by F may not exist.)

We first define:

Definition 3.2.1. S is a *base for* κ iff S is transitive and $\langle S, \in\rangle$ models:

$$\mathsf{ZFC}^- + \kappa \text{ is the largest cardinal.}$$

By a *suitable* subset of $\mathbb{P}(\kappa)$ we mean $\mathbb{P}(\kappa) \cap S$, where S is a base for κ.

We note:

Lemma 3.2.1. *Let S be a base for κ. Then S is uniquely determined by $\mathbb{P}(\kappa) \cap S$.*

Proof: For $a, e \in \mathbb{P}(\kappa) \cap S$ set:

$u(a, e) \simeq$: that transitive u such that
$\langle u, \in\rangle$ is isomorphic to $\langle a, \tilde{e}\rangle$,
where $\tilde{e} = \{\langle \nu, \tau\rangle | \prec \nu, \tau \succ \in e\}$.

Claim $S =$ the union of all $u(a, e)$ such that $a, e \in \mathbb{P}(\kappa) \cap S$ and $u(a, e)$ is defined.

Proof: To prove $(\subset)$, note that if $u \in S$ is transitive, then there exist $\alpha \leq \kappa, f \in S$ such that $f : \alpha \leftrightarrow u$. Hence $u = u(\alpha, e)$ where $e = \{\prec \nu, \tau \succ \mid f(\nu) \in f(\tau)\}$. Conversely, if $u = u(a, e)$ and $a, e \in \mathbb{P}(\kappa) \cap S$, then $u \in S$, since the isomorphism can be constructed in S. QED (Lemma 3.2.1)

Definition 3.2.2. An ordinal λ is called *Gödel closed* iff it is closed under Gödel's pair function $\prec, \succ$ as defined in §2.4. (It follows that λ is closed under Gödel n–tuples $\prec x_1, \ldots, x_n \succ$.)

We now define

Definition 3.2.3. Let S be a base for κ. Let λ be Gödel closed. F is an *extender at κ with length λ, base S and domain* $\mathbb{P}(\kappa) \cap S$ iff the following hold:

- F is a function defined on $\mathbb{P}(\kappa) \cap S$
- There exists a pair $\langle S', \pi\rangle$ such that
  (a) $\pi : S \prec S'$ where S' is transitive
  (b) $\kappa = \text{crit}(\pi), \pi(\kappa) \geq \lambda > \kappa$

(c) Every element of S' has the form $\pi(f)(\alpha)$ where $\alpha < \lambda$ and $f \in S$ is a function defined on κ.

(d) $F(X) = \pi(X) \cap \lambda$ for $X \in \mathbb{P}(\kappa) \cap S$.

Note. If F is an extender at κ, then κ is its critical point in the sense that $F \upharpoonright \kappa = \mathrm{id}$, $F(\kappa)$ is defined, and $\kappa < F(\kappa)$. Thus we set: $\mathrm{crit}(F) =: \kappa$.

Note. (c) can be equivalenly replaced by:

$$\pi : S \prec S' \text{ cofinally.}$$

We leave this to the reader.

Note. $\mathbb{P}(\kappa) \cap S \subset S'$ since $X = \pi(X) \cap \kappa \in S'$. But the proof of Lemma 3.2.1 then shows that $S \subset S'$. (We leave this to the reader.)

Note. As an immediate consequence of this definition we get a form of *Łos Theorem* for the base:

$$\begin{aligned} S' \models &\varphi[\pi(f_1)(\alpha_1), \ldots, (f_n)(\alpha_n)] \leftrightarrow \\ &\prec \vec{\alpha} \succ \in F(\{\langle \vec{\xi} \rangle | S \models \varphi[f_1(\xi_1), \ldots, f_n(\xi_n)]\}) \end{aligned}$$

where $\alpha_1, \ldots, \alpha_n < \lambda$ and $f_i \in S$ is a function defined on κ for $i = 1, \ldots, n$.

Note. $\langle S', \pi \rangle$ is uniquely determined by F since if $\langle \tilde{S}, \tilde{\pi} \rangle$ were a second such pair, we would have:

$$\begin{aligned} \pi(f)(\alpha) \in \pi(g)(\beta) &\leftrightarrow \prec \alpha, \beta \succ \in F(\{\prec \xi, \delta \succ | f(\xi) \in g(\xi)\}) \\ &\leftrightarrow \tilde{\pi}(f)(\alpha) \in \tilde{\pi}(g)(\beta). \end{aligned}$$

Thus there is an isomorphism $i : S' \tilde{\leftrightarrow} \tilde{S}$ defined by $i(\pi(f)(\alpha)) = \tilde{\pi}(f)(\alpha)$. Since $S', \tilde{S}$ are transitive, we conclude that $i = id, S' = \tilde{S}$.

But then we can define:

Definition 3.2.4. Let S, F, S', π be as above. We call $\langle S', \pi \rangle$ the *extension of S by F* (in symbols: $\pi : S \to_F S'$).

Note. It is easily seen that:

- S' is a base for $\pi(\kappa)$
- The embedding $\pi : S \to S'$ is cofinal (since $\pi(f)(\alpha) \in \pi(\mathrm{rng}(f))$).

Note. The concept of extender was first introduced by Bill Mitchell. He regarded it as a sequence of ultrafilters (or *measures*) $\langle F_\alpha | \alpha < \lambda \rangle$, where $F_\alpha = \{X | \alpha \in F(X)\}$. For this reason he called it a *hypermeasure*. We shall retain this name and call $\langle F_\alpha | \alpha < \lambda \rangle$ the *hypermeasure representation of F*. We can recover F by: $F(X) = \{\alpha | X \in F_\alpha\}$.

Definition 3.2.5. We call an extender F on κ with base S and extension $\langle S', \pi\rangle$ *full* iff $\pi(\kappa)$ is the length of F.

In later sections we shall work almost entirely with full extenders. We leave it to the reader to show that if S is a ZFC^- model with largest cardinal κ and $\pi : S \prec S'$ cofinally. Then $\pi \restriction \mathbb{P}(\kappa)$ is a full extender with base S and extension $\langle S', \pi\rangle$.

Lemma 3.2.2. *Let F be an extender with base S and extension $\langle S', \pi\rangle$. Then:*

(a) $\langle S', \pi\rangle$ is amenable

(b) If F is full, then $\langle S', F\rangle$ is amenable.

(c) If φ is Σ_0, then $\{\langle\vec{x}\rangle : \langle S, \pi\rangle \models \varphi[\vec{x}]\}$ is uniformly $\Sigma_1(\langle S, F\rangle)$ in $x_1, \ldots, x_n$.

Proof: (b) follows from (a), since then:

$$F \cap u = \{\langle Y, X\rangle \in \pi \cap u | X \subset \kappa \wedge Y \subset \lambda\}.$$

We prove (a). Since π takes S to S' cofinally, it suffices to show: $\pi \cap \pi(u) \in S'$ for $u \in S$. We can assume w.l.o.g. that u is transitive and non empty. If $\langle \pi(X), X\rangle \in \pi \cap \pi(u)$, then $\pi(X) \in \pi(u)$ by transitivity, hence $X \in u$. Thus $\pi \cap \pi(u) = (\pi \restriction u) \cap \pi(u)$ and it suffices to show:

Claim $\pi \restriction u \in S'$.
Let $f = \langle f(i) | i < \kappa\rangle$ enumerate u. Then $\pi \restriction u = \{\langle \pi(f)(i), f(i)\rangle | i < \kappa\}$.

This proves (a). We now prove (c). It suffices to show:
Claim. ($\nu \neq \varnothing$ is transitive and $y = \pi \restriction \nu$) is uniformly $\Sigma_1(\langle S, F\rangle)$ in ν, y, since then $\langle S, \pi\rangle \models \varphi[\vec{x}]$ is expressed by:

$$\bigvee w \bigvee u (u, w \text{ are transitive } \wedge \vec{x} \in u \wedge \pi \restriction u \subset w \wedge \langle w, \pi \restriction u\rangle) \models \varphi[\vec{x}]$$

We prove the Claim. Let $u \neq \varnothing$ be transitive. Then:

$$y = \pi \restriction u \iff \bigvee f (f : k \longrightarrow u \wedge y = \{\langle \pi(f)(i), f(i)\rangle : i < \kappa\}.)$$

$\{\kappa\}, \{\pi(\kappa)\}$ are uniformly $\Sigma_1(\langle S, F\rangle)$, since

$$\langle \pi(\kappa), \kappa\rangle = \text{ the unique } \prec \beta, \alpha \succ \in F.$$

Hence it suffices to show that $\{\pi(f)\}$ is uniformly $\Sigma_1(\langle S, F\rangle)$ in f. Let:

$$X = \{\prec j, i \succ \in \kappa : f(i) \in f(j)\}.$$

Then f is the unique function g such that

$$\mathrm{dom}(g) = \kappa \wedge g(j) = \{g(i) :\prec j, i \succ \in X\} \text{ for } i < \kappa.$$

Since $F(X) = \pi(X)$ we conclude that $\pi(f)$ is the unique function g such that

$$\mathrm{dom}(g) = \pi(\kappa) \wedge g(j) = \{g(i) :\prec j, i \succ \in F(X)\} \text{ for } i < \pi(\kappa).$$

The conclusion is immediate. QED (Lemma 3.2.2)

Definition 3.2.6. Let F be an extender at κ with base S, length λ, and extension $\langle S', \pi\rangle$. The *expansion* of F is the function F^* on $\bigcup_{n<\omega} \mathbb{P}(\kappa^n) \cap S$ defined by:

$$F^*(X) = \pi(X) \cap \lambda^n \text{ for } X \in \mathbb{P}(\kappa^n) \cap S.$$

We also expand the hypermeasure by setting:

$$F^*_{\alpha_1,\ldots,\alpha_n} = \{X | \langle\vec{\alpha}\rangle \in F^*(X)\}$$

for $\alpha_1, \ldots, \alpha_n < \lambda$. By an abuse of notation we shall usually not distinguish between F and F^*, writing $F(X)$ for $F^*(X)$ and $F_{\vec{\alpha}}$ for $F^*_{\vec{\alpha}}$.

Using this notation we get another version of Łos Lemma:

$$\begin{aligned} S' \models \quad & \varphi[\pi(f_1)(\vec{\alpha}), \ldots, \pi(f_n)(\vec{\alpha})] \leftrightarrow \\ & \{\langle\vec{\xi}\rangle | S \models \varphi[f_1(\vec{\xi}), \ldots, f_n(\vec{\xi})]\} \in F_{\vec{\alpha}} \end{aligned}$$

for $\alpha_1, \ldots, \alpha_m < \lambda$ and $f_i \in M$ a function with domain k^m for $i = 1, \ldots, n$.

Note. Most authors permit extenders to have length which are not Gödel closed. We chose not to for a very technical reason: If λ is not Gödel closed, the expanded extender F^* is not necessarily determined by $F = F^* \upharpoonright \mathbb{P}(\kappa)$.

Hence if we drop the requirement of Gödel completeness, we must work with expanded extenders from the beginning. We shall, in fact, have little reason to consider extenders whose length is not Gödel closed, but for the sake of completeness we give the general definition:

Definition 3.2.7. Let S be a base for κ. Let $\lambda > \kappa$. F is an *expanded extender at κ with base S, length* λ, and *extension* $\langle S', \pi\rangle$ iff the following hold:

- F is a function defined on $\bigcup_{n<\omega} \mathbb{P}(\kappa^n) \cap S$
- $\pi : S \prec S'$ where S' is transitive
- $\kappa = \text{crit}(\pi), \pi(\kappa) \geq \lambda$
- Every element of S' has the form $\pi(f)(\alpha_1, \dots, \alpha_n)$ where $\alpha_1, \dots, \alpha_n < \lambda$ and $f \in S$ is a function defined on κ^n
- $F(X) = \pi(X) \cap \kappa^n$ for $X \in \mathbb{P}(\kappa^n) \cap S$.

This makes sense for any $\lambda > \kappa$. If, indeed, λ is Gödel closed and F is an extender of length λ as defined previously, then F^* is the unique expanded extender with $F = F^* \restriction \mathbb{P}(\kappa)$.

Definition 3.2.8. Let F be an extender at κ of length λ with base S and extension $\langle S', \pi \rangle$. $X \subset \lambda$ is a set of *generators* for F iff every $\beta < \lambda$ has the form $\beta = \pi(f)(\vec{\alpha})$ where $\alpha_1, \dots, \alpha_n \in X$ and $f \in S$.

If X is a set of generators, then every $x \in S'$ will have the form $\pi(f)(\vec{\alpha})$ where $\alpha_1, \dots, \alpha_n \in X$ and $f \in S$. Thus only the generators are relevant. In some cases $\{\kappa\}$ will be a set of generators. (This will happen for instance if λ is the first admissible above κ or if $\lambda = \kappa + 1$ and F is the expanded extender.) This means that every element of S' has the form $\pi(f)(\kappa)$ and that:

$$S' \models \varphi[\pi(\vec{f})(\kappa)] \leftrightarrow \{\xi | S \models \varphi[\vec{f}(\xi)]\} \in F_\kappa.$$

Thus, in this case, S' is the ultrapower of S by the normal ultrafilter F_κ.

In §2.7 we used a "term model" construction to analyze the conditions under which the liftup of a given embedding exists. This construction emulated the well known construction of the ultrapower by a normal ultrafilter. We could use a similar construction to determine wheter a given F is, in fact, an extender with base S — i.e. whether the extension $\langle S', \pi \rangle$ by F exists. However, the only existence theorem for extenders which we shall actually need is:

Lemma 3.2.3. *Let S be a base for κ. Let $\pi^* : S \prec S^*$ such that $\kappa = \text{crit}(\pi^*)$ and $\kappa < \lambda \leq \pi^*(\kappa)$ where λ is Gödel closed. Set*

$$F(X) =: \pi^*(X) \cap \lambda \textit{ for } X \in \mathbb{P}(\kappa) \cap S.$$

Then

(a) F is an extender of length λ.

(b) *Let $\langle S', \pi \rangle$ be the extension by F. Then there is a unique $\sigma : S' \prec S^*$ such that $\sigma\pi = \pi^*$ and $\pi \upharpoonright \lambda = \mathrm{id}$.*

Proof: We first prove (a). Let Z be the set of $\pi^*(f)(\alpha)$ such that $\alpha < \lambda$ and $f \in S$ is a function on κ.

(1) $Z \prec S^*$

Proof: Let $S^* \models \bigvee v\varphi[\vec{x}]$ where $x_1, \ldots, x_n \in Z$. We must show:

Claim $Vy \in ZS^* \models \varphi[y, \vec{x}]$.
We know that there are functions $f_i \in S$ and $\alpha_i < X$ such that $x_i = \pi^*(f_i)(\alpha_i)$ for $i = 1, \ldots, n$. By replacement there is a $g \in S$ such that $\mathrm{dom}(g) = \kappa$ and in S:

$$\begin{aligned} \textstyle\bigwedge_{\xi_1 \ldots \xi_n} < \kappa \quad & (\textstyle\bigvee y\varphi(y, f_1(\xi_1), \ldots, f_n(\xi_n)) \rightarrow \\ & \varphi(g(\prec \xi_1, \ldots, \xi_n \succ, f_1(\xi_1), \ldots, f_n(\xi_n)))). \end{aligned}$$

But then the corresponding statement holds of $\pi^*(\kappa), \pi^*(g), \pi^*(f_1), \ldots, \pi^*(f_n)$ in S^*. Hence, setting $\beta = \prec \alpha_1, \ldots, \alpha_n \succ$ we have:

$$S^* \models \varphi[\pi^*(g)(\beta), \pi^*(f_1)(\alpha_1), \ldots, \pi^*(f_n)(\alpha_n)].$$

QED (1)

Now let $\sigma : S' \stackrel{\sim}{\leftrightarrow} Z$ where S' is transitive. Set: $\pi = \sigma^{-1}\pi^*$. Then $S \prec S'$. $\sigma : S' \prec S^*$, and $\sigma(\pi(f)(\alpha)) = \pi^*(f)(\alpha)$ for $\alpha < \lambda$. It follows easily that F is an extender and $\langle S', \pi \rangle$ is the extension by F.

This proves (a). It also proves the existence part of (b), since $\sigma \upharpoonright \lambda = \mathrm{id}$ and $\sigma\pi = \pi^*$. But if σ' also has the properties, then $\sigma'(\pi(f)(\alpha)) = \pi^*(f)(\alpha) = \sigma(\pi(f)(\alpha))$. Then $\sigma' = \sigma$ and σ is unique. QED (Lemma 3.2.3)

Definition 3.2.9. Let F be an extender at κ with extension $\langle S', \pi \rangle$. Let $\kappa < \lambda \leq \pi(\kappa)$ where λ is Gödel closed. $F|\lambda$ is the function F' defined by: $\mathrm{dom}(F') = \mathrm{dom}(F)$ and

$$F'(X) = \pi(X) \cap \lambda \text{ for } X \in \mathrm{dom}(F).$$

It follows immediately from Lemma 3.2.3 that $F|\lambda$ is an extender at κ with length λ.

The main use of an extender F with base S is to embed a larger model M with $\mathbb{P}(\kappa) \cap M = \mathbb{P}(\kappa) \cap S \in M$ into another transitive model M', which we then call the *ultrapower of M by F*. Ther is a wide class of models to which F can be so applied, but we shall confine ourselves to J–models.

Definition 3.2.10. Let M be a J–model. F is an *extender at* κ *on* M iff F is an extender with base S and $\mathbb{P}(\kappa) \cap M = \mathbb{P}(\kappa) \cap S \in M$, where κ is the largest cardinal in S. (In other words $S = H_\tau^M \in M$ where $\tau = \kappa^+$.)

Making use of the notion of liftups developed in §2.7.1 we define:

Definition 3.2.11. Let F be an extender at κ on M. Let $H = H_\tau^M$ be the base of F and let $\langle H', \pi' \rangle$ be the extension of H by F. We call $\langle N, \pi \rangle$ the *extension of* M *by* F (in symbols $\pi : M \to_F N$) iff $\langle N, \pi \rangle$ is the liftup of $\langle M, \pi' \rangle$.

We then call N the *ultrapower of* M *by* F. We call π the *canonical embedding* given by F.

Note. that π is Σ_0 preserving but not necessarily elementary.

Lemma 3.2.4. *Let F be an extender at κ on M of length λ. Let $\langle N, \pi \rangle$ be the extension of M by F. Then every element of N has the form $\pi(f)(\alpha)$ where $\alpha < \lambda$ and $f \in M$ is a function with domain κ.*

Proof: Let $H = H_\tau^M$ and let $\langle H', \pi' \rangle$ be the extension of H by F, where $\tau = \kappa^{+M}$. Each $x \in N$ has the form $x = \pi(f)(z)$, where $f \in M$ is a function, $\mathrm{dom}(f) \in H$ and $z \in \pi(\mathrm{dom}(f))$. But then $z = \pi(g)(\alpha)$ where $\alpha < \lambda, g \in H$ and $\mathrm{dom}(g) = \kappa$. We may assume w.l.o.g. that $\mathrm{rng}(g) \subset \mathrm{dom}(f)$. (Otherwise redefine g slightly.) Thus $x = \pi(f \circ g)(\alpha)$. QED (Lemma 3.2.4)

Using the expanded extenders we then get Łos Theorem in the form:

Lemma 3.2.5. *Let M, F, λ, N, π be as above. Let $\alpha_1, \ldots, \alpha_n < \lambda$ and let $f_i \in M$ be such that $f_i : \kappa^m \to M$ for $i = 1, \ldots, n$. Let φ be Σ_0. Then*

$$N \models \varphi[\pi(\vec{f}(\vec{\alpha})] \leftrightarrow \{\langle \vec{\xi} \rangle | M \models \varphi[\vec{f}(\vec{\xi})]\} \in F_{\vec{\alpha}}.$$

Proof: As in §2.7.1 we set:

$$\begin{aligned} \Gamma^0 = \quad & \Gamma^0(\tau, M) = \text{ the set of } f \in M \text{ such that} \\ & f \text{ is a function and } \mathrm{dom}(f) \in H_\tau^M. \end{aligned}$$

Then $f_i \in \Gamma^0, \mathrm{dom}(f_i) = \kappa^m$. By Łos Theorem for liftups we get:

$$N \models \varphi[\pi(\vec{f})(\vec{\alpha})] \leftrightarrow \langle \vec{\alpha} \rangle \in \pi(e) \cap \lambda^m = F(e)$$

where

$$e = \{\langle \vec{\xi} \rangle | M \models \varphi[\vec{f}(\vec{\xi})]\}.$$

QED (Lemma 3.2.5)

The following lemma is often useful:

Lemma 3.2.6. *Let F, κ, M, π be as above. Let τ be regular in M such that $\tau \neq \kappa$. Then $\pi(\tau) = \sup \pi''\tau$.*

Proof: If $\tau < \kappa$ this is trivial. Now let $\tau > \kappa$. Let $\xi = \pi(f)(\alpha) < \pi(\tau)$, where $\alpha < \lambda$. Set $\beta = \sup f''\kappa$. Then $\beta < \tau$ by regularity. Hence:

$$\xi = \pi(f)(\alpha) \leq \sup \pi(f)''\pi(\kappa) = \pi(\beta) < \pi(\tau).$$

QED (Lemma 3.2.6)

3.2.1 Extendability

Definition 3.2.12. Let F be an extender at κ on M. M is *extendible* by F iff the extension $\langle N, \pi \rangle$ of M by F exists.

Note. This requires that N be a transitive model.

$\langle N, \pi \rangle$, if it exists, is the liftup of $\langle M, \pi' \rangle$ where $H = H_\tau^M, \tau = \kappa^{+M}$ and $\langle H', \pi' \rangle$ is the extension of its base H by F. In §2.7.1 we formed a term model $\mathbb{D}$ in order to investigate when this liftup exists. The points of $\mathbb{D}$ consisted of pairs $\langle f, z \rangle$ where

$$f \in \Gamma^0(\tau, M) := \text{ the set of functions } f \in M \text{ such that } \operatorname{dom}(f) \in H.$$

The equality and set membership relation were defined by

$$\langle f, z \rangle \simeq \langle g, w \rangle \leftrightarrow: \langle z, w \rangle \in \pi'(\{\langle x, y \rangle | f(x) = g(y)\})$$
$$\langle f, z \rangle \tilde{\in} \langle g, w \rangle \leftrightarrow: \langle z, w \rangle \in \pi'(\{\langle x, y \rangle | f(x) = g(y)\})$$

Now set:

Definition 3.2.13. $\Gamma^0_* = \Gamma^0_*(\kappa, M) =: \{f \in \Gamma^0 | \operatorname{dom}(f) = \kappa\}$.

Set $\mathbb{D}^* = \mathbb{D}^*(\kappa, M) =:$ the restriction of $\mathbb{D}$ to terms $\langle t, \alpha \rangle$ such that $t \in \Gamma^0_*$ and $\alpha < \lambda$. The proof of Lemma 3.2.4 implicitly contains a barely disguised proof that:

$$\bigwedge x \in \mathbb{D} \bigvee y \in \mathbb{D}^* x \simeq y.$$

The set membership relation of $\mathbb{D}^*$ is:

$$\langle f, \alpha \rangle \in^* \langle g, \beta \rangle \leftrightarrow \prec \alpha, \beta \succ \in \pi'(\{\xi, \zeta\} | f(\xi) \in g(\zeta)\}).$$

In §2.7.1. we used the term model to show that the liftup $\langle N, \pi \rangle$ exists if and only if $\tilde{\in}$ is well founded. In this case $\mathbb{D}^*$ contains all the points of interest, so we may conclude:

Lemma 3.2.7. *M is extendible iff $\in^*$ is well founded.*

Note. In the future, when dealing with extenders, we shall often fail to distinguish notationally between $\Gamma^0_*, \mathbb{D}^*, \in^*$ and $\Gamma^0, \mathbb{D}, \tilde{\in}$.

Using this principle we develop a further criterion of extendability. We define:

Definition 3.2.14. Let $\overline{F}$ be an extender on $\overline{M}$ at $\overline{\kappa}$ of length $\overline{\lambda}$. Let F be an extender on M at κ of length λ.

$$\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \to \langle M, F\rangle$$

means:

(a) $\pi : \overline{M} \to_{\Sigma_0} M$ and $\pi(\overline{\kappa}) = \kappa$

(b) $g : \overline{\lambda} \to \lambda$

(c) Let $\overline{X} \subset \overline{\kappa}$, $\pi(\overline{X}) = X$, $\alpha_1, \ldots, \alpha_n < \overline{\lambda}$. Let $\beta_i = g(\alpha_i)$ for $i = 1, \ldots, n$. Then

$$\prec \vec{\alpha} \succ \in \overline{F}(\overline{X}) \leftrightarrow \prec \vec{\beta} \succ \in F(X).$$

Lemma 3.2.8. *Let $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \to \langle M, F\rangle$, where M is extendible by F. Then $\overline{M}$ is extendible by $\overline{F}$. Moreover, if $\langle N, \sigma\rangle$, $\langle \overline{N}, \overline{\sigma}\rangle$ are the extensions of M, N respectively, then there is a unique π' such that*

$$\pi' : \overline{N} \to_{\Sigma_0} N,\ \pi'\overline{\sigma} = \sigma\pi,\ \textit{and}\ \pi' \restriction \overline{\lambda} = g.$$

π' is defined by:

$$\pi'(\overline{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha))$$

for $f \in \Gamma^0$ and $\alpha < \overline{\lambda}$.

Proof: We first show that $\overline{M}$ is extendible by $\overline{F}$. Let $\sigma : M \to_F N$. The relation $\tilde{\in}$ on the term model $\overline{\mathbb{D}} = \mathbb{D}(\overline{\kappa}, \overline{M})$ is well founded, since:

$$\begin{aligned}
\langle f, \alpha\rangle \tilde{\in} \langle h, \beta\rangle \quad & \leftrightarrow \prec \alpha, \beta \succ \in \overline{F}(\{\prec \xi, \zeta \succ \mid f(\xi) \in h(\zeta)\}) \\
& \leftrightarrow \prec g(\alpha), g(\beta) \succ \in F(\{\prec \xi, \zeta \succ \mid \pi(f)(\xi) \in \pi(h)(\zeta)\}) \\
& \leftrightarrow \sigma\pi(f)(g(\alpha)) \in \sigma\pi(h)(g(\beta))
\end{aligned}$$

Now let $\overline{\sigma} : \overline{M} \to \overline{N}$. Let φ be a Σ_0 formula.

Then:

$$\begin{aligned}
\overline{N} \quad & \models \varphi[\overline{\sigma}(f_1)(\alpha_1), \ldots, \overline{\sigma}(f_n)(\alpha_n)] \\
& \leftrightarrow \langle \vec{\alpha}\rangle \in \overline{F}(\{\langle \vec{\xi}\rangle | \overline{M} \models \varphi[\vec{f}(\vec{\xi})]\}) \\
& \leftrightarrow \langle g(\vec{\alpha})\rangle \in F(\{\vec{\xi} | M \models \varphi[\pi(\vec{f})(\vec{\xi})]\}) \\
& \leftrightarrow N \models \varphi[\sigma\pi(f_1)(g(\alpha_1)), \ldots, \sigma\pi(f_n)(g(\alpha_n))].
\end{aligned}$$

Hence there is $\pi' : \overline{N} \to_{\Sigma_0} N$ defined by:

$$\pi'(\overline{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha)).$$

But any π' fulfilling the above conditions will satisfy this definition.

QED (Lemma 3.2.8)

3.2.2 Fine Structural Extensions

These lemmas show that N is the ultrapower of M in the usual sense. However, the canonical embedding can only be shown to be Σ_0–preserving. If, however, M is acceptable and $\kappa < \rho^n_M$, the methods of §2.7.8 suggest another type of ultrapower with a $\Sigma^{(n)}_0$–preserving map. We define:

Definition 3.2.15. Let M be acceptable. Let F be an extender at κ on M. Let $H = H^M_\tau$ be the base of F and let $\langle H', \pi'\rangle$ be the extension of H by F. Let $\rho^n_M > \kappa$ (hence $\rho^n_M \geq \tau$). We call $\langle N, \pi\rangle$ the $\Sigma^{(n)}_0$*–extension of M by F* (in symbols: $\pi : M \to^{(n)}_F N$) iff $\langle N, \pi\rangle$ is the $\Sigma^{(n)}_0$ liftup of $\langle M, \pi'\rangle$.

The extension we originally defined is then the Σ_0 ultrapower (or $\Sigma^{(0)}_0$ ultrapower). The $\Sigma^{(n)}_0$ analogues of Lemma 3.2.4 and Lemma 3.2.5 are obtained by a virtual repetition of our proofs, which we leave to the reader.

Letting $\Gamma^n = \Gamma^n(\tau, M)$ be defined as in §2.7.2 we get the analogue of Lemma 3.2.4.

Lemma 3.2.9. *Let F be an extender at κ on M of length λ. Let $\rho^n_M > \kappa$ and let $\langle N, \pi\rangle$ be the $\Sigma^{(n)}_0$ extension of M by F. Then every element of N has the form $\pi(f)(\alpha)$ where $\alpha < \lambda$ and $f \in \Gamma^n$ such that* $\mathrm{dom}(f) = \kappa$.

Lemma 3.2.10. *Let M, F, λ, N, π be as above. Let $\alpha_1, \ldots, \alpha_m < \lambda$ and let $f_i \in \Gamma^n$ such that* $\mathrm{dom}(f_i) = \kappa^m$ *for $i = 1, \ldots, p$. Let φ be a $\Sigma^{(n)}_0$ formula. Then:*

$$N \models \varphi[\pi(\vec{f})(\vec{\alpha})] \leftrightarrow \{\langle\vec{\xi}\rangle | M \models \varphi[\vec{f}(\vec{\xi})]\} \in F_{\vec{\alpha}}.$$

Note. We remind the reader that an element f of Γ^n is not, in general, an element of M. The meaning of $\pi(f)$ is explained in §2.7.2.

Using Lemma 2.7.22 we get:

Lemma 3.2.11. *Let $\pi^* : M \to_{\Sigma^{(n)}_0} M^*$ where $\kappa =$* $\mathrm{crit}(\pi^*)$ *and $\pi^*(\kappa) \geq \lambda$, where λ is Gödel closed. Assume: $\mathbb{P}(\kappa) \cap M \in M$. Set:*

$$F(X) =: \pi^*(X) \cap \lambda \textit{ for } X \in \mathbb{P}(\kappa) \cap M.$$

Then:

(a) *F is an extender at κ of length λ on M.*

(b) *The $\Sigma_0^{(n)}$ extension $\langle M', \pi\rangle$ of M by F exists.*

(c) *There is a unique $\sigma : M' \to_{\Sigma_0^{(n)}} M^*$ such that $\sigma' \restriction \lambda = \mathrm{id}$ and $\sigma\pi = \pi^*$.*

Proof: Let $H = H_\tau^M$, $H^* = \pi^*(H)$. Then H is a base for κ and $\pi^* \restriction H : H \prec H^*$. Hence by Lemma 3.2.3 F is an extender at κ with base H and extension $\langle H', \pi'\rangle$. Moreover, there is a unique $\sigma' : H' \prec H^*$ such that $\sigma' \restriction \lambda = \mathrm{id}$ and $\sigma'\pi' = \pi^* \restriction H$. But by Lemma 2.7.22 the $\Sigma_0^{(n)}$ liftup $\langle M', \pi\rangle$ of $\langle M, \pi'\rangle$ exists. Moreover, there is a unique $\sigma : M' \to_{\Sigma_0^{(n)}} M^*$ such that $\sigma \restriction H' = \sigma'$ and $\sigma\pi' = \pi^*$. In particular, $\sigma \restriction \lambda = \mathrm{id}$. But σ is then unique with these properties, since if $\tilde{\sigma}$ had them, we would have:

$$\tilde{\sigma}(\pi(f)(\alpha)) = \pi^*(f)(\alpha) = \sigma(\pi(f)(\alpha))$$

for $f \in \Gamma^n, \mathrm{dom}(f) = \kappa, \alpha < \lambda$. QED (Lemma 3.2.11)

By Lemma 2.7.21 we get:

Lemma 3.2.12. *Let $\pi : M \longrightarrow_F^{(n)} N$. Let $i < n$. Then:*

(a) *π is $\Sigma_2^{(i)}$ preserving.*

(b) *$\pi(\rho_M^i) = \rho_{M'}^i$ if $\rho_M^i \in M$.*

(c) *$\rho_{M'}^i = \mathrm{On} \cap M'$ if $\rho_M^i = \mathrm{On} \cap M$.*

The following definition expresses an important property of extenders:

Definition 3.2.16. Let F be an extender at κ of length λ with base S. F is *weakly amenable* iff whenever $X \in \mathbb{P}(\kappa^2) \cap S$, then $\{\nu < \kappa | \langle \nu, \alpha\rangle \in F(X)\} \in S$ for $\alpha < \lambda$.

Lemma 3.2.13. *Let F be an extender at κ with base S and extension $\langle S', \pi\rangle$. Then F is weakly amenable iff $\mathbb{P}(\kappa) \cap S' \subset S$.*

Proof:

($\rightarrow$) Let $Y \in \mathbb{P}(\kappa) \cap S'$, $Y = \pi(f)(\alpha), \alpha < \lambda$. Set $X = \{\langle \nu, \xi\rangle \in \kappa^2 | \nu \in f(\xi)\}$. Then $\pi(f)(\alpha) = \{\nu < \kappa | \langle \nu, \alpha\rangle \in F(X)\} \in S$, since $F(X) = \pi(X) \cap \lambda$.

($\leftarrow$) Let $X \in \mathbb{P}(\kappa^2) \cap S$, $\alpha < \lambda$. Then $\{\nu < \kappa | \langle \nu, \alpha\rangle \in \pi(X)\} \in \mathbb{P}(\kappa) \cap S' \subset S$.

QED (Lemma 3.2.13)

Corollary 3.2.14. *Let M be acceptable. Let F be a weakly amenable extender at κ on M. Let $\langle N, \pi \rangle$ be the $\Sigma_0^{(n)}$ extension of M by F. Then $\mathbb{P}(\kappa) \cap N \subset M$.*

Proof: Let $H = H_\tau^M$, $\tilde{H} = \bigcup_{u \in H} \pi(u)$, $\tilde{\pi} = \pi \restriction H$. Then H is the base for F and $\langle \tilde{H}, \tilde{\pi} \rangle$ is the extension of H by F. Hence $\mathbb{P}(\kappa) \cap \tilde{H} \subset H \subset M$. Hence it suffices to show:

Claim $\mathbb{P}(\kappa) \cap N \subset \tilde{H}$.

Proof: Since $\pi(\kappa) > \kappa$ is a cardinal in N and N is acceptable, we have:

$$\mathbb{P}(\kappa) \cap N \subset H_{\pi(\kappa)}^N = \pi(H_\kappa^M) \in \tilde{H}.$$

QED (Corollary 3.2.14)

Corollary 3.2.15. *Let M, F, N, π be as above. Then κ is inaccessible in M (hence in N by Corollary 3.2.14).*

Proof:

(1) κ is regular in M.
Proof: If not there is $f \in M$ mapping a $\gamma < \kappa$ cofinally to κ. But then $\pi(f)$ maps γ cofinally to $\pi(\kappa)$. But $\pi(f)(\xi) = \pi(f(\xi)) = f(\xi) < \kappa$ for $\xi < \gamma$. Hence $\sup\{\pi(f)(\xi) | \xi < \gamma\} \subset \kappa$. Contradiction!

(2) $\kappa \neq \gamma^+$ in M for $\gamma < \kappa$.
Proof: Suppose not. Then $\pi(\kappa) = \gamma^+$ in N where $\pi(\kappa) > \kappa$. Hence $\overline{\overline{\kappa}} = \gamma$ in N and N has a new subset of κ. Contradiction!

QED (Corollary 3.2.15)

By Corollary 3.2.14 and Lemma 2.7.23 we get:

Lemma 3.2.16. *Let $\pi : M \to_F^{(n)} N$ where F is weakly amenable. Let n be maximal such that $\rho_M^n > \kappa$. Then $\rho_N^n = \sup \pi"\rho_M^n$. (Hence π is $\Sigma_1^{(n)}$ preserving.)*

With further conditions on F and n we can considerably improve this result. We define:

Definition 3.2.17. Let F be an extender at κ on M of length λ. F is *close to* M if F is weakly amenable and F_α is $\underline{\Sigma}_1(M)$ for all $\alpha < \lambda$.

This very important notion is due to John Steel. Using it we get the following remarkable result:

Theorem 3.2.17. *Let M be acceptable. Let F be an extender at κ on M which is close to M. Let $n \leq \omega$ be maximal such that $\rho^n > \kappa$ in M. Let $\langle N, \pi\rangle$ be the $\Sigma_0^{(n)}$ extension of M by F. Then π is Σ^* preserving.*

Proof: If $n = \omega$ this is immediate, so let $n < \omega$. Then $\rho^{n+1} \subseteq \kappa < \rho^n$ in M. By the previous lemma π is Σ_1–preserving. Hence $\pi(\kappa)$ is regular in N. Set: $H = H_\kappa^M$. Then $H = H_\kappa^N$ by Corollary 3.2.14.

(1) Let $D \subset H$ be $\underline{\Sigma}_1^{(n)}(N)$. Then D is $\underline{\Sigma}_1^{(n)}(M)$.
Proof: Let:

$$D(z) \leftrightarrow \bigvee x^n D'(x^n, z, \pi(f)(\alpha))$$

where $\alpha < \lambda$, $f \in \Gamma^n$ such that $\mathrm{dom}(f) = \kappa$, and D' is $\Sigma_0^{(n)}$. Then by Lemma 3.2.16:

$$\begin{aligned} D(z) &\leftrightarrow \bigvee u \in H_M^n \bigvee x \in \pi(u) D'(x, z, \pi(f)(\alpha)) \\ &\leftrightarrow \bigvee u \in H_M^n \alpha \in \pi(e) \\ &\leftrightarrow \bigvee u \in H_M^n e \in F_\alpha \end{aligned}$$

where $e = \{\xi | \bigvee x \in u \overline{D}(x, z, f(\xi))\}$ where $\overline{D}$ is $\Sigma_0^{(n)}(M)$ by the same definition as D' over N. QED (1)

By induction on $m > n$ we then prove:

(2) (a) $H_M^m = H_N^m$
(b) $\underline{\Sigma}_1^{(m)}(M) \cap \mathbb{P}(H) = \underline{\Sigma}_1^{(m)}(N) \cap \mathbb{P}(H)$
(c) π is $\Sigma_1^{(m)}$–preserving.

Proof:

Case 1 $m = n + 1$

(a) Let $M = \langle J_\alpha^A, B\rangle$, $N = \langle J_{\alpha'}^{A'}, B'\rangle$. Then: $H = J_\kappa^A = J_\kappa^{A'}$. But

$$\mathbb{P}(\rho) \cap M = \mathbb{P}(\rho) \cap N = \mathbb{P}(\rho) \cap H \text{ for } \rho \leq \kappa.$$

But then in M and N we have:

$$\rho^m = \text{ the least } \rho < \kappa \text{ such that } D \cap J_\rho^A \notin H \text{ for } D \in \underline{\Sigma}_1^{(n)}$$
$$\text{and } H^m = J_{\rho^m}^A.$$

Hence $\rho_M^m = \rho_N^m, H_M^m = H_N^m$. QED (a)

(c) Let $\overline{A}(\vec{x}^m, x_{i_1}, \ldots, x_{i_p})$ be $\Sigma_1^{(m)}(M)$, where $i_1, \ldots, i_p \leq n$. Let A be $\Sigma_1^{(m)}(N)$ by the same definition. Then there are $\Sigma_1^{(m)}(M)$ relations $\overline{B}^j(\vec{x}^m, \vec{x})(j = 1, \ldots, q)$ and a Σ_1 formula φ such that

$$\overline{A}(\vec{x}^m, \vec{x}) \leftrightarrow \overline{H}_{\vec{x}}^m \models \varphi[\vec{x}^m]$$

where $\overline{H}_{\vec{x}}^m = \langle H^m, \overline{B}_{\vec{x}}^1, \ldots, \overline{B}_{\vec{x}}^q\rangle$ and

$$\overline{B}_{\vec{x}}^j = \{\langle \vec{x}^{\,m}\rangle | \overline{B}^j(\vec{z}^{\,m}, \vec{x})\}(j = 1, \ldots, q).$$

Let $B^j(z^m, \vec{x})$ have the same $\Sigma_1^{(n)}$ definition over N. Define $H_{\vec{x}}^m$ the same way, using $B^1, \ldots, B^q$ in place of $\overline{B}^1, \ldots, \overline{B}^q$. Then

$$A(\vec{x}^{\,m}, \vec{x}) \leftrightarrow H_{\vec{x}}^m \models \varphi[\vec{x}^m].$$

But $H_M^m = H_N^m$. Hence, since π is $\Sigma_1^{(n)}$ preserving, we have: $\overline{B}_{\vec{x}}^j = B_{\pi(\vec{x})}^j$. Hence $\overline{H}_{\vec{x}}^m = H_{\pi(\vec{x})}^m$. But then:

$$\begin{aligned} \overline{A}(\vec{x}^m, \vec{x}) &\leftrightarrow \overline{H}_{\vec{x}}^m \models \varphi[\vec{x}^m] \\ &\leftrightarrow H_{\pi(\vec{x})}^m \models \varphi[\vec{x}^m] \\ &\leftrightarrow A(\vec{x}^m, \pi(\vec{x})) \\ &\leftrightarrow A(\pi(\vec{x}^m), \pi(\vec{x})) \end{aligned}$$

since $\pi(\vec{x}^m) = \vec{x}^m$. QED (c)

(b) The direction $\subset$ follows straightforwardly from (c). We prove the direction $\supset$. Let $A(\vec{x}^m, x_{i_1}, \cdots, x_{i_r})$ be $\underline{\Sigma}_1^{(m)}(N)$ such that $A \subset H$. Then there are B^j $(j = 1, \ldots, q)$ such that B^j is $\underline{\Sigma}_1^{(n)}(N)$ and

$$A_{\vec{x}}(x^n) \leftrightarrow H_{\vec{x}}^n \models \varphi[\vec{x}, s]$$

where $s \in H^m$ and φ is a Σ_1 formula and $H_{\vec{x}}^m = \langle H^m, B_{\vec{x}}^1, \ldots, B_{\vec{x}}^q\rangle$. By (1) there are $\overline{B}^j$ $(j = 1, \ldots, q)$ such that $\overline{B}^j$ is $\underline{\Sigma}_1^{(n)}(M)$ and $\overline{B}_{\vec{x}}^j = B_{\vec{x}}^j$ whenever $x_{i_1}, \ldots, x_{i_r} \in H$. The conclusion is immediate.

QED (Case 1)

Case 2 $m = h + 1$ where $h > n$.
This is virtually identical to Case 1 except that we use:

$$\underset{\sim}{\Sigma}_1^{(h)} \cap \mathbb{P}(H_M^h) = \underset{\sim}{\Sigma}_1^{(h)} \cap \mathbb{P}(H_N^h)$$

in place of (1). QED (Theorem 3.2.17)

Theorem 3.2.17 justifies us in defining:

Definition 3.2.18. Let F be an extender at κ on M. Let $n \le \omega$ be maximal such that $\rho_M^m > \kappa$. We call $\langle N, \pi \rangle$ the *Σ^*–extension of M by F* (in symbols $\pi : M \rightarrow_F^* N$) iff F is close to M and $\langle N, \pi \rangle$ is the $\Sigma_0^{(n)}$ extension by F.

As a corollary of the proof of Lemma 3.2.16 we have:

Corollary 3.2.18. *Let $\pi : M \longrightarrow_F^* N$. Let $H = H_\kappa^M$ and $\rho_M^{n+1} \le \kappa$. Then:*

- $H = H_\kappa^N$
- $M \cap \mathbb{P}(H) = N \cap \mathbb{P}(H)$.
- $\Sigma_1^{(n)}(M) \cap \mathbb{P}(H) = \Sigma_1^{(n)}(N) \cap \mathbb{P}(H)$.
- $H_M^{n+1} = H_N^{n+1}$.

3.2.3 n–extendibility

Definition 3.2.19. Let F be an extender of length λ at κ on M. M is *n–extendible* by F iff $\kappa < \rho_M^n$ and the $\Sigma_0^{(n)}$ extension $\langle N, \pi \rangle$ of M by F exists.

$\langle N, \pi \rangle$, if it exists, is the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi' \rangle$ where $H = H_\tau^M$ is the base of F, $\tau = \kappa^{+M}$, and $\langle M', \pi' \rangle$ is the extension of H by F. To analyse this situation we use the term model $\mathbb{D} = \mathbb{D}^{(n)}(\pi', M)$ defined in §2.7.2. The points of $\mathbb{D}$ are pairs $\langle f, z \rangle$ such that $f \in \Gamma^n = \Gamma^n(\tau, M)$ as defined in §2.7.2. and $z \in \pi'(\mathrm{dom}(f))$. The equality and set membership relation of $\mathbb{D}$ are again defined by:

$$\langle f, z \rangle \simeq \langle g, w \rangle \leftrightarrow \langle z, w \rangle \in \pi'(\{\langle x, y \rangle | f(x) = g(y)\})$$
$$\langle f, z \rangle \tilde{\in} \langle g, w \rangle \leftrightarrow \langle z, w \rangle \in \pi'(\{\langle x, y \rangle | f(x) = g(y)\})$$

Set: $\Gamma_*^n = \Gamma_*^n(\kappa, M) =:$ the set of $f \in \Gamma^n$ such that $\mathrm{dom}(f) = \kappa$. Let $\mathbb{D}_* = \mathbb{D}_*^{(n)}(F, M)$ be the restriction of $\mathbb{D}$ to points $\langle f, d \rangle$ such that $f \in \Gamma_*^n$ and $\alpha < \lambda$. The proof of Lemma 3.2.7 tells us that

$$\bigwedge x \in \mathbb{D} \bigvee y \in \mathbb{D}_* x \simeq y.$$

Hence M is $\Sigma_0^{(n)}$ extendable iff the restriction $\in^*$ of the relation $\tilde{\in}$ to $\mathbb{D}_*$ is well founded.

We have:

$$\langle f,\alpha\rangle \in^* \langle g,\beta\rangle \leftrightarrow \langle\alpha,\beta\rangle \in F(\{\langle\xi,\zeta\rangle | f(\xi)\in g(\zeta)\}).$$

Note. When dealing with extenders, we shall again sometimes fail to distinguish notationally between $\Gamma_*^n, \mathbb{D}_*^{(n)}, \in^*$ and $\Gamma^n, \mathbb{D}^{(n)}, \tilde{\in}$.

We now prove:

Lemma 3.2.19. *Let $\langle\pi,g\rangle : \langle\overline{M},\overline{F}\rangle \to \langle M,F\rangle$, where M is m–extendible by F. Let $n\leq m$ and let π be $\Sigma_0^{(n)}$ preserving with $\overline{\kappa} < \rho^m$ in $\overline{M}$, where $\overline{\kappa} = \mathrm{crit}(\overline{F})$. Then $\overline{M}$ is n–extendible by $\overline{F}$. Moreover, if $\langle N,\sigma\rangle$ is the $\Sigma_0^{(m)}$ extension of M by F and $\langle\overline{N},\overline{\sigma}\rangle$ is the $\Sigma_0^{(n)}$ extension of $\overline{M}$ by F, then there is a unique π' such that*

$$\pi' : \overline{N} \to_{\Sigma_0^{(n)}} N, \pi'\overline{\sigma} = \sigma\overline{N}, \pi' \restriction \overline{\lambda} = g.$$

π' is defined by:

$$\pi'(\overline{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha))$$

for $f \in \Gamma_*^n(\overline{\kappa},\overline{M}), \alpha < \overline{\beta}$.

Proof: Let $\in^*$ be the set membership relation of $\overline{\mathbb{D}}_* = \overline{\mathbb{D}}_*(\overline{F},\overline{M})$.

Then:

$$\begin{aligned}\langle f,\alpha\rangle \in^* \langle h,\beta\rangle \quad &\leftrightarrow \langle\alpha,\beta\rangle \in \overline{F}(\{\langle\xi,\zeta\rangle | f(\xi)\in g(\zeta)\})\\ &\leftrightarrow \langle g(\alpha),g(\beta)\rangle \in F(\{\langle\xi,\zeta\rangle | \pi(f)(\xi)\in \pi(h(\zeta)\})\\ &\leftrightarrow \sigma\pi(f)(\alpha) \in \sigma\pi(f)(\beta).\end{aligned}$$

Hence there is $\pi' : \overline{N} \to_{\Sigma_0^{(n)}} N$ defined by:

$$\pi'(\overline{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha)).$$

But any π' fulfilling the above conditions satisfies this definition.

QED (Lemma 3.2.19)

Taking π, g as id, we get:

Corollary 3.2.20. *Let M be $\Sigma_0^{(m)}$ extendible by F. Let $n \leq m$. Then M is $\Sigma_0^{(n)}$ extendible by F. Moreover, if $\sigma : M \to_F^{(m)} N$ and $\overline{\sigma} : M \to_F^{(m)} \overline{N}$, there is $\pi : \overline{N} \to_{\Sigma_0^{(n)}} N$ defined by:*

$$\pi(\overline{\sigma}(f)(\alpha) = \sigma(f)(\alpha) \textit{ for } f \in \Gamma^n, \alpha < \lambda.$$

Lemma 3.2.19 is normally applied to the case $n = m$. The condition $\overline{\kappa} < \rho^n_{\overline{M}}$ will be satisfied if the map π is *strictly* $\Sigma_0^{(n)}$-preserving. However, it does not follows that π' is strictly $\Sigma_0^{(n)}$-preserving. Similarly, even if we assume that π is fully $\Sigma_1^{(n)}$-preserving, we get no corresponding strengthening of π'. We can remedy this situation by strengthening our basic premiss:

$$\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \longrightarrow \langle M, F\rangle$$

We define:

Definition 3.2.20. $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \rightarrow^* \langle M, F\rangle$ iff the following hold:

- $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \rightarrow \langle M, F\rangle$
- $\overline{F}, F$ are weakly amenable
- Let $\alpha < \overline{\lambda} =$ length $(\overline{F})$. Then $\overline{F}_\alpha$ is $\Sigma_1(\overline{M})$ in a parameter $\overline{p}$ and $F_{g(\alpha)}$ is $\Sigma_1(M)$ in $p = \pi(\overline{p})$ by the same definition.

(Hence $\overline{F}$ is close to $\overline{M}$.) Taking $n = m$ in Lemma 3.2.19 we prove:

Lemma 3.2.21. *Let $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \rightarrow^* \langle M, F\rangle$. Let $\sigma : M \rightarrow_F^{(n)} N$ where π is $\Sigma_1^{(n)}$ preserving. Let $\overline{\sigma} : \overline{M} \rightarrow_{\overline{F}}^{(n)} \overline{N}$, $\pi' : \overline{N} \rightarrow N$ be given by Lemma 3.2.19. Then π' is $\Sigma_1^{(n)}$ preserving.*

We derive this from a stronger lemma:

Lemma 3.2.22. *Let $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \rightarrow^* \langle M, F\rangle$. Let $n, \overline{N}, N, \pi'$ be as above, where π is $\Sigma_1^{(n)}$ preserving. Let $\overline{D}(y, x_1, \ldots, x_r)$ be $\Sigma_1^{(n)}(\overline{N})$ and $D(\vec{y}, x_1, \ldots, x_r)$ be $\Sigma_1^{(n)}(N)$ by the same definition. Let $\pi'(\overline{x}_i) = x_i (i = 1, \ldots, r)$. Then*

$$\{\langle \vec{y}\rangle \in H^{\overline{M}}_{\overline{\kappa}} | D(\vec{y}, \overline{x}_1, \ldots, \overline{x}_r)\}$$

is $\Sigma_1^{(n)}(\overline{M})$ in a parameter $\overline{p}$

and:

$$\{\langle \vec{y}\rangle \in H^M_\kappa | D(\vec{y}, x_1, \ldots, x_r)\}$$

is $\Sigma_1^{(n)}(M)$ in $p = \pi(\overline{p})$ by the same definition.

Before proving Lemma 3.2.22 we show that it implies Lemma 3.2.21. Let $\overline{D}(x_1, \ldots, x_r)$ be $\Sigma_1^{(n)}(\overline{N})$ and let $D(x_1, \ldots, x_r)$ be $\Sigma_1^{(n)}(N)$ by the same definition. Set:

$$D'(y, \vec{x}) \leftrightarrow: y = \varnothing \wedge D(\vec{x}); \; \overline{D}'(y, \vec{x}) \leftrightarrow: y = \varnothing \wedge \overline{D}(\vec{x}).$$

Let $\pi'(\overline{x}_i) = x_i$ $(i = 1, \dots, r)$. Applying Lemma 3.2.22 and the $\Sigma_1^{(n)}$ preservation of π we have:

$$\begin{aligned}
\overline{D}(\overline{x}_1, \dots, \overline{x}_r) \quad &\leftrightarrow \varnothing \in \{y \in H_{\kappa}^{\overline{M}} | \overline{D}'(y, \overline{x}_1, \dots, \overline{x}_r)\} \\
&\leftrightarrow \varnothing \in \{y \in H_{\kappa}^{M} | D'(y, x_1, \dots, x_r)\} \\
&\leftrightarrow D(x_1, \dots, x_r).
\end{aligned}$$

QED

We now prove Lemma 3.2.22. For the sake of simplicity we display the proof for the case $r = 1$. Let $\overline{D}(\vec{y}, x)$ be $\Sigma_1^{(n)}(\overline{N})$ and $D(\vec{y}, x)$ be $\Sigma_1^{(n)}(N)$ by the same definition. We may assume:

$$\overline{D}(\vec{y}, x) \leftrightarrow \bigvee z^n \overline{B}(z^n, y, x),\ D(\vec{y}, x) \leftrightarrow \bigvee z^n B(z^n, y, x)$$

where $\overline{B}$ is $\Sigma_0^{(n)}(\overline{N})$ and B is $\Sigma_0^{(n)}(N)$ by the same definition. Let $\overline{A}$ have the same definition over $\overline{M}$ and A the same definition over M. Let $x = \pi'(\overline{x})$. Then $\overline{x} = \overline{\sigma}(f)(\alpha)$ for an $f \in \Gamma^n$ and $\alpha < \overline{\lambda}$. Hence $x = \sigma\pi(f)(g(\alpha))$. Then for $\vec{y} \in H_{\kappa}^{\overline{M}}$:

$$\begin{aligned}
\overline{D}(\vec{y}, \overline{x}) \quad &\leftrightarrow \bigvee z^n \overline{B}(z^n, \vec{y}, \overline{x}) \\
&\leftrightarrow \bigvee u \in H_{\overline{M}}^{n} \bigvee z \in \overline{\sigma}(u) \overline{B}(z^n, \vec{y}, \overline{\sigma}(f)(\alpha)) \\
&\leftrightarrow \bigvee u \in H_{\overline{M}}^{n} \bigvee \{\xi < \overline{\kappa} | \bigvee z \in u\, \overline{A}(z, \vec{y}, f(\xi))\} \in \overline{F}_{\alpha}.
\end{aligned}$$

Similarly for $\vec{y} \in H$ we get:

$$\overline{D}(\vec{y}, \overline{x}) \leftrightarrow \bigvee u \in H_M^n \{\xi < \kappa | \bigvee z \in uA(z, \vec{y}, \pi(f)(\xi))\} \in F_{g(\alpha)}.$$

$\overline{F}_{\alpha}$ is $\Sigma_1(\overline{M})$ in a parameter $\overline{p}$ and $F_{g(\alpha)}$ is $\Sigma_1(M)$ in a parameter $p = \pi(\overline{p})$. But by the definition of Γ^n we know that there are $\overline{q}, q$ such that either:

$$f = \overline{q} \in H_{\overline{M}}^{n} \text{ and } q = \pi(f)$$

or:

$$f(\xi) \simeq \overline{G}(\xi, \overline{q}) \text{ where } \overline{G} \text{ is a good } \Sigma_1^{(i)}(\overline{M}) \text{ map}$$

and:

$$\pi(f)(\xi) \simeq G(\xi q) \text{ where } G \text{ has the same good definition over } M.$$

Hence:

$$\{\langle \vec{y} \rangle \in H_{\kappa}^{\overline{M}} | \overline{D}(\vec{y}, \overline{x})\}$$

is $\Sigma_1^{(n)}(\overline{M})$ in $\overline{\kappa}, \overline{q}, \overline{p}$ and:

$$\{\langle\, \vec{y} \in \rangle H_{\kappa}^{M} | D(\vec{y}, x)\}$$

is $\Sigma_1^{(m)}(M)$ in κ, q, p by the same definition. QED (Lemma 3.2.22)

3.2.4 $*$–extendability

Definition 3.2.21. Let F be an extender of length λ at κ on M. M is $*$*–extendible* by F iff F is close to M and M is n–extendible by F, where $n \leq w$ is maximal such that $\kappa < \rho^n_M$.

(Hence $\pi : M \to^*_F N$ where $\langle N, \pi\rangle$ is the $\Sigma^{(n)}_0$–extension.)

Lemma 3.2.23. *Assume $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \to^* \langle M, F\rangle$ where M is $*$–extendible by F. Assume that π is Σ^* preserving. Then $\overline{M}$ is $*$–extendible by E. Moreover, if $\overline{\sigma} : \overline{M} \to^*_{\overline{F}} \overline{N}$ and $\sigma : M \to^*_F N$, there is a unique $\pi' : \overline{N} \to_{\Sigma^*} N$ such that $\pi'\overline{\sigma} = \sigma\pi$ and $\pi' \restriction \overline{\lambda} = g$.*

Proof: Let n be maximal such that $\kappa < \rho^n_M$. Let $\sigma : M \to^{(n)}_F N$. By Lemma 3.2.21 we have $\overline{\kappa} < \rho^n_{\overline{M}}$ and there is $\overline{\sigma} : \overline{M} \to^{(n)}_{\overline{F}} M$. Moreover there is $\pi' : \overline{N} \to_{\Sigma^{(n)}_1} N$ such that $\pi'\overline{\sigma} = \sigma\pi$ and $\pi' \restriction \overline{\lambda} = g$.

Claim 1 n is maximal such that $\overline{\kappa} < \rho^n_{\overline{M}}$.

Proof: If not, then $n < w$ and $\rho^{n+1}_M \leq \kappa < \rho^n_M$. Hence

$$\bigwedge z^{n+1} z^{n+1} \neq \kappa \text{ holds in } M.$$

Thus $\bigwedge z^{n+1} z^{n+1} \neq \overline{\kappa}$ in $\overline{M}$, since π is $\Sigma^{(n+1)}_0$ preserving. Hence $\rho^{n+1}_{\overline{M}} \leq \overline{\kappa} < \rho^n_{\overline{M}}$. (QED Claim 1)

Note. In the case $n < w$ we needed only the $\Sigma^{(n+1)}_0$ preservation of π to establish Claim 1.

By Claim 1 we then have:

(1) $\pi : \overline{M} \to^*_{\overline{F}} \overline{N}$.

Hence $\overline{M}$ is $*$–extendible by $\overline{F}$. It remains only to show:

Claim 2 π' is Σ^* preserving.

Proof: If $n = w$, there is nothing to prove, so assume $n < w$. We must show that π' is $\Sigma^{(m)}_0$ preserving for $n < m < w$. Let $n < m < w$. Since $\sigma : M \to^*_F N$, we know that:

(2) $\rho^m_M = \rho^m_N$ and $\sigma \restriction \rho^m_M = \text{id}$.

By Claim 1 an (1) we similarly conclude:

(3) $\rho^m_{\overline{M}} = \rho^m_{\overline{N}}$ and $\overline{\sigma} \upharpoonright \rho^m_{\overline{M}} = \mathrm{id}$.

Using (2), (3) and Lemma 3.2.22 we can then show:

(4) Let $\overline{D}(\vec{y}^m, \vec{x})$ be $\Sigma^{(m)}_j(\overline{N})$. Let $D(\vec{y}^m, \vec{x})$ be $\Sigma^{(m)}_j(N)$ by the same definition. Let

$$\pi'(\overline{x}_i) = x_i (i = 1, \ldots, r).$$

Then:

$$\overline{D}_{\overline{x}_1,\ldots,\overline{x}_r} =: \{\langle \overline{y}_m \rangle \upharpoonright \overline{D}(\vec{y}^m, \overline{x}_1, \ldots, \overline{x}_r)\}$$

is $\Sigma^{(m)}_j(\overline{M})$ in a parameter $\overline{p}$ and:

$$D_{x_1,\ldots,x_r} =: \{\langle \vec{y}_m \rangle | D(\vec{y}_m, x_1, \ldots, x_r)\}$$

is $\Sigma^{(m)}_j(M)$ in $p = \pi(\overline{p})$ by the same definition.

Proof: By induction on m.

Case 1 $m = n + 1$

We know:

$$\overline{D}(\vec{y}_m, \vec{x}) \leftrightarrow \overline{H}^m_{\vec{x}} \models \varphi[\vec{y}^m]$$

where φ is Σ_j and

$$\overline{H}^m_{\vec{x}} = \langle H^m_{\overline{M}}, \overline{B}^1_{\vec{x}}, \ldots, \overline{B}^q_{\vec{x}} \rangle$$

where $\overline{B}^i_{\vec{x}} = \{\langle \vec{z}^m \rangle | \overline{B}^i(\vec{z}^m, x)\}$ and $\overline{B}^i$ is $\Sigma^m_1(\overline{N})$ for $i = 1, \ldots, q$. Since $D(y^m, \vec{x})$ has the same $\Sigma^{(m)}_j$ definition, we can assume

$$D(\vec{y}m, \vec{x}) \leftrightarrow H^m_{\vec{x}} \models \varphi[\vec{y}m]$$

where:

$$H^m_{\vec{x}} = \langle H^m_M, B^1_{\vec{x}}, \ldots, B^q_{\vec{x}} \rangle$$

where $B^i_{\vec{x}} = \{\langle z^m \rangle | B^i(\vec{z}^m, x)\}$ and B^i is $\Sigma^{(n)}_1(N)$ by the same definition as $\overline{B}^i$ over $\overline{N}$. Letting $\pi'(\overline{x}_i) = x_i$ $(i = q, \ldots, r)$, we know by Lemma 3.2.22 that each of $\overline{B}^i_{\overline{x}_1,\ldots,\overline{x}_r}$ is $\Sigma^{(n)}_1(\overline{M})$ in a parameter $\overline{p}$ and $B^i_{x_1,\ldots,x_r}$ is $\Sigma^{(n)}_1(M)$ in $p = \pi(\overline{p})$ by the same definition. (We can without loss of generality assume that $\overline{p}$ is the same for $i = 1, \ldots, r$.) But then $\overline{D}_{\overline{x},\ldots,\overline{x}_r}$ is $\Sigma^{(m)}_j(\overline{M})$ in $\overline{p}$ and $D_{x_1,\ldots,x_r}$ is $\Sigma^{(m)}_j(M)$ in $p = \pi(p)$ by the same definition. QED (Case 1)

Case 2 $m = h + 1$ where $h > n$.

We repeat the same argument using the induction hypothesis in place of Lemma 3.2.22. QED (4)

But Claim 2 follows easily from Claim 4 and the fact that π is Σ^* preserving. Let $\overline{D}(\vec{x})$ be $\Sigma_0^{(m)}(\overline{N})$ and $D(\vec{x})$ be $\Sigma_0^{(m)}(N)$ by the same definition. Set:

$$\overline{D}'(y, \vec{x}) \leftrightarrow: y = 0 \wedge \overline{D}(\vec{x})$$
$$D'(y, \vec{x}) \leftrightarrow: y = 0 \wedge D(\vec{x})$$

By (4) we have:

$$\overline{D}(\vec{x}) \leftrightarrow 0 \in \overline{D}_{\vec{x}} \leftrightarrow 0 \in D_{\pi'(\vec{x})} \leftrightarrow D(\pi'(\vec{x}))$$

for $x_1, \ldots, x_r \in \overline{M}$, using the $\Sigma_0^{(m)}$ preservation of π and $\pi(0) = 0$.

QED (Lemma 3.2.23)

Note. The last part of the proof also shows that π' is $\Sigma_j^{(m)}$ preserving if π is.

As a corollary of the proof we also get:

Lemma 3.2.24. *Let $\langle \pi, g \rangle : \langle \overline{M}, \overline{F} \rangle \longrightarrow \langle M, F \rangle$. Let M be $*$-extendible by F. Let n be the maximal n such that $\kappa = \mathrm{crit}(F) < \rho_M^n$. Let $n < r < \omega$ and suppose that π is $\Sigma_j^{(r)}$ preserving, where $j < \omega$. Then:*

(a) *n is maximal such that $\overline{\kappa} = \mathrm{crit}(F) < \rho_{\overline{M}}^n$.*

(b) *$\overline{M}$ is $*$-extendible by $\overline{F}$.*

(c) *Let π' be the unique $\pi' : \overline{N} \longrightarrow_{\Sigma_0} N$ such that $\pi'\overline{\sigma} = \sigma\pi$ and $\pi' \restriction \overline{\lambda} = g$. Then π' is $\Sigma_j^{(r)}$ preserving.*

Proof. (a) follows by the proof of Claim 1 in Lemma 3.2.23, since that only need that π is Σ_0^{n+1}-preserving. (1) then follows as before. Hence $\overline{M}$ is $*$-extendible by $\overline{F}$. (2) and (3) follows for $r \geq m > n$, using the $\Sigma_0^{(r)}$ preservation of π. Hence (4) follows as before and we can conclude that π' is $\Sigma_j^{(n)}$ preserving as before.

QED(Lemma 3.2.24)

Notation. $\Gamma_*^n(\kappa, M) = \{f \in \Gamma^n(\tau, M) : \mathrm{dom}(f) = \kappa\}$ and $\Gamma^*(\kappa, M) = \Gamma_*^n(\kappa, M)$ where $n \leq \omega$ is maximal such that $\kappa < \rho_M^n$.

3.2.5 Good Parameters

We now recall some concepts which were developed in §2.5. Let $M = \langle J_\alpha^E, B\rangle$ be acceptable. The set P_M^{n+1} of $n+1$-*good parameters* can be defined by:

> $a \in P_M^{n+1}$ iff $a \in [\mathrm{On}_M]^{<\omega}$ and there is an $A \subset H_M^n$ which is $\Sigma_1^{(n)}(M)$ in parameters from $\rho^{n+1} \cup a$ such that $A \cap H^{n+1} \notin M$.

We then say that A *confirms* $a \in P^{n+1}$. We also set: $P_M^0 = [\mathrm{On}_M]^{<\omega}$. It is not hard to prove:

Fact 1. Let $a \in P^n$. Then:

- $a \subset b \in [\mathrm{On}_M]^{<\omega} \longrightarrow b \in P^M$.
- $a \setminus \rho^n \in P^n$.

The definition of P_M^{n+1} is equivalent to that given in §2.5. However, we thus required $a \in P_M^n$ in place of $a \in [\mathrm{On}_M]^{<\omega}$. To show the equivalence of these definitions, we must prove: $P_M^{n+1} \subset P_M^n$ $(n < \omega)$. With a view to proving this we recall the following definition, which was stated in an equivalent form in §2.5.

With a view to proving this we recall the following definition, which was stated in an equivalent form in §2.5.

Definition 3.2.22. Let $M = \langle J_\alpha^A, B\rangle$ be acceptable. Let $a \in [\alpha]^{<\omega}$. For $n < \omega$ we define the n-th reduct $M^{n,a}$ and the n-th standard predicate $T_M^{n,a}$ with respect to a:

$$T^0 = B, M^n = \langle J_{\rho^M}^A,\, T^n\rangle,$$

$$T^{n+1} = \{\langle i, x\rangle : i < \omega \wedge M^n \models \varphi_i[x, a^{(n)}]\}$$

where $a^(n) = a \cap \rho^n$ and $\langle \varphi_i : i < \omega\rangle$ enumerates recursively all Σ_1 formulae $\psi = \varphi(v_0, v_1)$ with at most the free variables v_0, v_1 in the language of M.

By induction on n we get:

Fact 2. Let $a \in [\mathrm{On}_M]^{<\omega}$. Then:

- $T^{n,a}$ is $\Sigma_1^{(n)}(M)$ in a.

- Let $A \subset H^n$ be $\Sigma_1^{(n)}(M)$ in a. There is an $i < \omega$ such that

$$Ax \longrightarrow \langle i, x \rangle \in T^{n,a}$$

From this it follows that:

Fact 3. $a \in P^{n+1} \leftrightarrow T^{n,a}$ confirms $a \in P^{n+1}$. But then:

Fact 4. $P^{n+1} \subset P^n$.

Proof. For $n = 0$ this is trivial. Now let $n = m + 1$. Let $a \in P^{n+1}$. Then $T^{n,a} \cap H^{n+1} \notin M$.

Claim. $T^{m,a} \cap H^n \notin M$.

Suppose not. If $\rho^n \in M$, then:

$$\langle H^n, T^{m,a} \cap H^n \rangle \in M$$

Hence $T^{n,a} \in M$ and $H^{n+1} \cap T^{n,a} \in M$. Contradiction! Now let $\rho^n = \rho^0$. Then for each $x \in M$, there is $i \leq \omega$ such that $\langle i, x \rangle \in T^{m,a}$. If $T^{m,a} \cap H^n = T^{m,a} \in M$, then $\langle i, T^{m,a} \rangle \in T^{m,a}$. Contradiction!

QED(Fact 4.)

We also mention:

Fact 5. $a \in P^{n+1}$ iff there is A which is $\Sigma_1^{(n)}(M)$ in a such that $A \cap \rho^{n+1} \notin M$.

Proof. (Sketch) If $\rho^{n+1} = \rho^0$, take $A = \rho^0$. Now let $\rho^{n+1} < \rho^0$. Then $H^{n+1} = |J^E_{\rho^{n+1}}|$ is a ZFC^- model. Note that for any $N = J^E_\alpha$, the function f_N is uniformly $\Sigma_1(N)$, where

$$f_N(\alpha) = \text{ the } \alpha\text{-th element of } N \text{ in the ordering } <_E .$$

Let A be $\Sigma_1^{(n)}(M)$ such that $A \subset H^n$ and $A \cap H^{n+1} \notin M$. Set:

$$A' = \{\alpha < \rho^n : f(\alpha) \in A\}$$

where $f = f_{J^E_{\rho^n}}$. Then $f \upharpoonright \rho^{n+1} = f_{J^E_{\rho^n}}$ maps ρ^{n+1} onto H^{n+1}. Hence, if $A' \cap \rho^{n+1} \in M$, we have $f``(A' \cap \rho^{n+1}) = A \cap H^{n+1} \in M$. Contradiction!

QED(Fact 5)

Thus $A \cap H^{n+1}$ could have been replaced by $A \cap \rho^n$ in the original definition of P^n.

We now define:

Definition 3.2.23. π is a *strongly* Σ^**-preserving map* of M to N (in symbols: $\pi : M \longrightarrow_{\Sigma^*} N$ strongly) iff the following hold:

- $\pi : M \longrightarrow_{\Sigma^*} N$
- If $\rho^{n+1} = \rho^\omega$ in M, then $\rho^{n+1} = \rho^\omega$ in N.
- If $\rho^{n+1} = \rho^\omega$ in M, A confirms $a \in P^{n+1}$ in M, and A' is $\Sigma_1^{(n)}(N)$ in $\pi(a)$ by the same definition, then A' confirms $\pi(a) \in P^{n+1}$ in N.

By Fact 3 and Fact 4 we conclude:

Lemma 3.2.25. *Let $\pi : M \longrightarrow_{\Sigma^*} N$ strongly. Let $\rho^{n+1} = \rho^\omega$ in M. Let $a \in P^{n+1}$ in M. Then $T^{i,a}$ confirms $a \in P^{i+1}$ in M and $T^{i,\pi(a)}$ confirms $\pi(a) \in P^{i+1}$ in N for $i \leq n$.*

We now prove:

Lemma 3.2.26. *Let $\pi : M \longrightarrow_F^* N$. Then $\pi : M \longrightarrow_{\Sigma^*} N$ strongly.*

Proof. Let $\kappa = \mathrm{crit}(F)$. We consider two cases.

Case 1. $\rho_M^\omega \leq \kappa$.

The conclusion is immediate by Corollary 3.2.18.

Case 2. $\kappa < \rho_M^\omega$.

We show that for any $n < \omega$, if A confirms $a \in P^{n+1}$ in M, then A' confirms $\pi(a) \in P^{n+1}$ in N. Suppose not. Let $A' \cap H_M^{n+1} \in N$. Let $y = A' \cap H_N^{n+1}$. Then $y \in H_N^n$ and in N we have:

$$\bigwedge z^{n+1}(z^{n+1} \in y \longleftrightarrow z^{n+1} \in A'),$$

which is a $\Pi_1^{(n+1)}$ statement in $\pi(a), y$. Let $y = \pi(f)(\alpha)$, where $\alpha < \lambda = \lambda_F$ and $f \in \Gamma^*(\kappa, M)$. Thus $\mathrm{dom}(f) = \kappa$ and:

$$f(\xi) = G(\xi, q)$$

where $q \in H_{\kappa^+}^M$ and G is a good $\Sigma_1^{(m)}$ function to H^n for an $m < \omega$. Assume without lose of generality $m > n + 1$.

The statement:

$$\wedge z^{n+1}(z^{n+1} \in f(\xi) \longleftrightarrow z^{n+1} \in A)$$

is then $\Sigma_1^{(m)}(M)$ in q, a, ξ. Hence it is $\Sigma_0^{m+1}(M)$ in q, a, ξ. Set:

$$X = \{\xi < \kappa : \bigwedge z^{n+1}(z^{n+1} \in f(\xi) \longleftrightarrow z^{n+1} \in A)\}.$$

Then $X \in M$. But $\alpha \in \pi(X)$. This is a contradiction, since $X = \pi(X) = \varnothing$ by the fact that $A \cap H_M^{n+1} \notin M$.

Finally we note that for all $n < \omega$ we have $\kappa < \rho_M^{n+1}$. Hence: $\rho_M^n = \pi(\rho_M^n)$ if $\rho_M^n \in M$ and otherwise $\rho_N^n = \mathrm{On}_N$. Thus:

$$\rho_M^{n+1} = \rho_M^\omega \longrightarrow \rho_N^{n+1} = \rho_N^\omega.$$

QED(Lemma 3.2.26)

Obviously we have:

Lemma 3.2.27. *If $\pi_0 : M_0 \longrightarrow_{\Sigma^*} M_1$ strongly and $\pi_1 : M_1 \longrightarrow_{\Sigma^*} M_2$ strongly, then $\pi_1\pi_0$ is a strong Σ^*-preserving map from M_0 to M_2.*

We now prove:

Lemma 3.2.28. *Let $\pi_{ij} : M_i \longrightarrow_{\Sigma^*} M_j$ strongly $(i \leq j < \lambda)$ where the π_{ij} commute. Suppose that:*

$$\langle M_i : i < \lambda\rangle, \langle \pi_{ij} : i \leq j < \lambda\rangle$$

has a transitivized direct limit:

$$M, \langle \pi_i : i < \lambda\rangle.$$

Then $\pi_i : M_i \longrightarrow_{\Sigma^*} M$ strongly for $i < \lambda$.

Proof. π_i is Σ_1-preserving, since each π_{ij} is. Hence $M = \langle J_\alpha^E, B\rangle$ is acceptable. If we set:

$$\rho_n = \bigcup_{i<\lambda} \pi_i\text{''}\rho_{M_i}^n,\ H_n = \bigcup_{i<\lambda} \pi_i\text{''}H_n,$$

it follows that $H_n = H_{\rho_n}^M = |J_{\rho_n}^E|$. By induction on n we prove:

Claim. $\rho_n = \rho_M^n$ and $\pi_i : M_i \longrightarrow_{\Sigma_1^{(n)}} M$.

Proof.

Case 1. $n = 0$ is trivial.

Case 2. $n = m + 1$.

Let $r \geq n$ such that $\rho^r_{M_0} = \rho^\omega_{M_0}$. Let $a \in P^r_{M_0}$. Then $T^{m,a_i}_{M_i}$ verifies $a_i \in P_{M_i}$ for $i < \lambda$ where $\pi_{0i}(a_0) = a_i$. Let $a = \pi_i(a_i)$ $(i < \lambda)$. By the induction hypothesis π_i is $\Sigma^{(m)}_1$-preserving. Hence

$$x \in T^{m,a_i}_{M_i} \longleftrightarrow \pi_i(x) \in T^{m,a}_M.$$

Claim. $T^{m,a}_M \cap H_n \notin M$.

Proof. Suppose not. Let $y = T^{m,a}_M \cap H_n$. Let $i < \lambda$ such that $\pi(y_i) = y$. For $x \in H^n_{M_i}$ we have:

$$\begin{aligned} x \in T^{m,a_i}_{M_i} &\longleftrightarrow \pi_i(x) \in T^{m,a}_M \cap H_n \\ &\longleftrightarrow \pi(x) \in \pi(y) \\ &\longleftrightarrow x \in y_i. \end{aligned}$$

Hence $T^{m,a_i}_{M_i} \cap H^n_{M_i} = y_i \cap H^n_{M_i} \in M_i$. Contradiction!

QED(Claim 1)

Claim 2. Let $A \subset H_n$ be $\underline{\Sigma}^{(m)}_1(M)$. Then $\langle H_n, A\rangle$ is amenable.

Proof. Let A be $\Sigma^{(m)}_1(M)$ in q. For i such that $q \in \text{rng}(\pi_i)$, let $q_i = \pi^{-1}_i(q)$ and let A_i be $\Sigma^{(m)}_1(M)$ in q_i by the same definition. Now let $x \in H_n$. We claim that $x \cap A \in H_n$. Let i be large enough that $q \in \text{rng}(\pi_i)$. Set $x_i = \pi^{-1}_i(x)$. Let $z_i = A_i \cap x_i$. Then $x_i \in H^n_{M_i}$ where $\langle H^n_{M_i}, A_i\rangle$ is amenable. Hence $z_i \in H^n_{M_i}$ where $z = \pi_i(z_i) = A \cap x$. Hence $z \in H^n_{M_i}$.

QED(Claim 2)

Hence $\rho^n_M = \rho_n$ and $H^n_M = H_M$. It follows straightforwardly that $\pi_i : M_i \longrightarrow_{\Sigma^{(n)}_1} M$ for $i < \lambda$.

QED(Case 2)

It remains to show:

Claim 3. The embedding π_i is strong.

Proof. Let $\rho^{n+1} = \rho^\omega$ in M_i. Let $A \subset H^n$ confirm $a \in P^{n+1}$ in M_i. Let A_j be $\Sigma^{(n)}_1(M_j)$ in $a_j =: \pi_{ij}(a)$ for $i \leq j < \lambda$. Then $\rho^{n+1} = \rho^\omega$ in M_j and A_j confirms $a_j \in \rho^{n+1}$ in M_j. Let $a' = \pi_i(a)$, and let A' be $\Sigma^{(n)}_1(M)$ in a' by the same definition. We repeat the proof of Claim 1 to show that A' confirms $a' \in P^{n+1}$ in M (i.e. $A' \cap H_{n+1} \notin M$).

QED(Lemma 3.2.28)

3.3 Premice

A major focus of modern set theory is the subject of "strong axioms of infinity". These are principles which posit the existence of a large set or class, not provable in ZFC. Among these principles are the *embedding axioms*, which posit the existence of a non trivial elementary embedding of one inner model into another. The best known example of this is the *measurability axiom*, which posits the existence of a non trivial elementary embedding π of V into an inner model. ("Non trivial" here means simply that $\pi \neq \mathrm{id}$. Hence there is a unique *critical point* $\kappa = \mathrm{crit}(\pi)$ such that $\pi \upharpoonright \kappa = \mathrm{id}$ and $\pi(\kappa) > \kappa$.) The critical point κ of π is then called a *measurable cardinal*, since the existence of such an embedding is equivalent to the existence of an ultrafilter (or *two valued measure*) on κ.

This is a typical example of the recursing case that an axiom positing the existence of a proper class (hence not formulable in ZFC) reduces to a statement about set existence. The weakest embedding axiom posits the existence of a non trivial embedding of L into itself. This is equivalent to the existence of a countable transitive set called $0^{\#}$, which can be coded by a real number. (There are many representations of $0^{\#}$, but all have the same degree of constructability.) The "small" object $0^{\#}$ in fact contains complete information about both the proper class L and an embedding of L into itself. We can then form $L(0^{\#})$, the smallest universe containing the set $0^{\#}$. If $L(0^{\#})$ is embeddable into itself we get $0^{\#\#}$, which gives complete information about $L(0^{\#})$ and its embedding ... etc. This process can be continued very far. Each stage in this progression of embeddings, leading to larger and larger universes, is coded by a specific set, called a *mouse*. $0^{\#}$ and $0^{\#\#}$ are the first two examples of mice. It is not yet known how far this process goes, but it is conjectured that all stages can be represented by mice, as long as the embeddings are representable by extenders. (Extenders in our sense are also called *short extenders*, since one must modify the notion in order to go still further.) The concept of mouse, however hard it is to explicate, will play a central role in this book.

We begin, therefore, with an informal discussion of the *sharp operation* which takes a set a to $a^{\#}$, since applications of this operation give us the smallest mice $0^{\#}, 0^{\#\#}$, etc.

Let a be a set such that $a \in L[a]$. Suppose moreover that there is an elementary embedding π of $L^a = \langle L[a], \in, a\rangle$ into itself such that $a \in L^a_\kappa$,