(c) *If* $\sigma(\xi_i) = \pi_{0,\eta}(\xi_i)$ *for* $i < n < \omega$, *then* $\sigma(\xi_n) \geq \pi_{0,\eta}(\xi_n)$.

As before, this follows from:

Lemma 3.5.12. *Let* $M, \langle \xi_i | i < \omega \rangle$ *be as above. There is an* $\omega_1 + 1$*–successful* n*–full iteration strategy* S *to* M *such that whenever* I *is an* S*–conforming* n *to* n*–full iteration from* M *to* M' *and* $\sigma : M \to_{\Sigma(n)} M'$, *then:*

(a) *No* $i < \mathrm{lh}(I)$ *is a truncation point. (Hence the map* $\pi = \pi^{(M,I)}$ *is a total function on* M.*)*

(b) *If* $\sigma(\xi_i) = \pi(\xi_i)$ *for* $i < n$, *then* $\sigma(\xi_n) \geq \pi(\xi_n)$.

The proofs are virtually unchanged.

3.6 Verifying full iterability

3.6.1 Introduction

As we said, full iterability is a difficult property to verify. A theorem that every normally iterable mouse is fully iterable would be useful, if true, but seems unlikely. We can, however, prove the following pair of theorems:

Theorem 3.6.1. *If* M *is smoothly* α*–iterable, then it is fully* α*–iterable.*

Theorem 3.6.2. *Let* $\kappa > \omega$ *be regular and let* M *be uniquely normally* $\kappa + 1$ *iterable. Then* M *is smoothly* $\kappa + 1$*–iterable.*

The proofs of these theorems are quite complex. To prove theorem 3.6.1, we redo much of chapter 2, developing a theory of embeddings which are Σ^*–preserving *modulo pseudo projecta*, which may not be the real projecta, but behave simiarly. The proof of theorem 3.6.2 requires us, in addition, to delve rather deeply into the combinatorics of normal iteration, using technique which, essentially, were developed by John Steel and Farmer Schlutzenberg.

This section (§3.6) is devoted to the proof of theorem 3.6.1. The following section brings the proof of theorem 3.6.2. In later chapters we shall make frequent use of both these theorems, but will seldom, if ever, refer to their proofs. Hence it would be justifiable for a first time reader of this this book to skip §3.6 and §3.7, taking the above theorems for granted and deferring their proofs until later.

3.6.2 Pseudo projecta

In order to prove theorem 3.6.1, we must redo §2.6, allowing "pseudo projecta" to play the role of the real projecta.

Definition 3.6.1. Let $M = \langle J_\alpha^A, B\rangle$ be acceptable. Then $\rho = \langle \rho_i | i < \omega\rangle$ is a *good sequence of pseudo projecta for M* iff the following hold:

(a) ρ_i is p.r. closed if $i > 0$.

(b) $\omega \leq \rho_{i+1} \leq \rho_i \leq \rho_M^i$ for $i < \omega$.

(c) $J_{\rho_i}^A$ is cardinally absolute in M (i.e. if $\gamma \in J_{\rho_i}^A$ is a cardinal in $J_{\rho_i}^A$, then it is a cardinal in M).

Note. $\rho_0 < \rho_M^0 = \mathrm{On}_M$ is not excluded. Moreover, ρ_i itself need not be a cardinal in M.

We shall generally write "ρ is good for M" instead of "ρ is a good sequence of pseudo projecta for M^i".

Definition 3.6.2. Let ρ be good for $M = J_\alpha^A$. $H_i = H_i(M, \rho) =: |J_{\rho_i}^A|$ for $i < \omega$.

We adopt the same language with typed variables $v^i (i < \omega)$ as before. The formula classes $\Sigma_h^{(n)} (h, n < \omega)$ are defined exactly as before. The satisfaction relation:

$$M \models \varphi[x_1, \ldots, x_n] \mod \rho$$

is defined as before except that the variables v^i now range over $H_i = H_i(M, \rho)$ instead of $H^i = H_M^i$. A relation $R(x_1^{i_1}, \ldots, x_n^{i_n})$ is $\Sigma_j^{(n)}(M, \rho)$ (or $\Sigma_j^{(n)}(M)$ mod ρ) iff it is M–definable mod ρ by a $\Sigma_j^{(n)}$ formula.
Similarly for $\underline{\Sigma}_j^{(n)}, \Sigma^*, \underline{\Sigma}^*$. We then define:

Definition 3.6.3. $\sigma : M \to_{\Sigma_j^{(n)}} M' \mod (\rho, \rho')$ iff the following hold:

(a) ρ is good for M and ρ' is good for M'.

(b) $\sigma'' H_i \subset H'_i$ for $i < \omega$, where $H_i = H_i(M, \rho), H'_i = H_i(M', \rho')$.

(c) Let φ be $\Sigma_i^{(n)}, \varphi = \varphi(v_1^{i_1}, \ldots, v_p^{i_p})$ where $i_1, \ldots, i_p \leq n$. Then:

$$M \models \varphi[\vec{x}] \mod \rho \leftrightarrow M' \models \varphi[\sigma(\vec{x})] \mod \rho'$$

for all $x_1, \ldots, x_p \in M$ such that $x_i \in H_{i_l} (l = 1, \ldots, p)$.

We also define:

Definition 3.6.4. $\sigma : M \rightarrow_{\Sigma^*} M' \mod (\rho, \rho')$ iff

$$\sigma \text{ is } \Sigma_0^{(n)}\text{–preserving} \mod (\rho, \rho') \text{ for } n < \omega.$$

As before, this is equivalent to:

$$\sigma \text{ is } \Sigma_1^{(n)}\text{–preserving} \mod (\rho, \rho') \text{ for } n < \omega.$$

We also write:

$$\sigma : M \rightarrow_{\Sigma_j^{(n)}} M' \mod \rho'$$

to mean

$$\begin{cases} \sigma : M \rightarrow_{\Sigma_j^{(n)}} M' \mod (\rho, \rho'), \\ \text{where } \rho_i = \rho_M^i \text{ for } i < \omega. \end{cases}$$

(Similarly for $\sigma : M \rightarrow_{\Sigma^*} M' \mod \rho'$.)

Lemma 3.6.3. *Let $\sigma : M \rightarrow_{\Sigma_j^{(n)}} M'$. Let ρ be good for M and define ρ' by:*

$$\rho'_i = \begin{cases} \sigma(\rho_i) & \textit{if } \rho_i < \rho_M^i \\ \rho_M^i & \textit{if not.} \end{cases}$$

Then $\sigma : M \rightarrow_{\Sigma_j^{(n)}} M' \mod (\rho, \rho')$.

(Hence, if σ is fully Σ^*–preserving, it is also Σ^*–preserving modulo (ρ, ρ').)

Proof: Clearly ρ' is good for M'. Now let $R(x_1^{i_1}, \ldots, x_p^{i_p})$ be $\Sigma_j^{(n)}(M, \rho)$, where $i_1, \ldots, i_p \leq n$. By an induction on n, R is uniformly $\Sigma_j^{(n)}(M)$ in the parameter $u = \langle \rho_i : l \leq n \wedge \rho_l < \rho_M^l \rangle$. (We leave the detail to the reader.)

But then, if R' is $\Sigma_i^{(n)}(M', \rho')$ by the same definition, it is $\Sigma_j^{(n)}(M')$ in $\sigma(u)$ by the same definition. QED (Lemma 3.6.3)

Lemma 3.6.4. *Let $\sigma : M \rightarrow_{\Sigma^*} M'$ and let ρ, ρ' be as in lemma 3.6.3. Let $\kappa = \text{crit}(\sigma)$, where $\rho_{i+1} \leq \kappa < \rho_i$. Define ρ'' by:*

$$\rho''_j =: \rho'_j \textit{ for } j \neq i, \rho''_i =: \sup \sigma'' \rho_i.$$

Then:

$$\sigma : M \rightarrow_{\Sigma^*} M' \mod (\rho, \rho'').$$

Proof: ρ'' is still good for M'. By induction on n it then follows that σ is $\Sigma_1^{(n)}$–preserving modulo (ρ, ρ''). QED (Lemma 3.6.4)

One might expect that most of §2.6 will not go through with pseudo projecta in place of projecta, since $\langle H_i, B\rangle$ is *not* necessarily amenable when B is $\Sigma_0^{(i)}(M, \rho)$. As it turns out, however, a great many proofs in §2.6 do not use this property (in contrast to the treatment in §2.5). In particular, lemmas 2.6.3 – 2.6.16 go through without change. Similarly, the definition of a good function can be relativized to a good ρ in place of $\langle \rho_M^n | n < \omega\rangle$. We define

$$\mathbb{G}_n = \mathbb{G}_n(M, \rho); \mathbb{G}^* = \mathbb{G}^*(M, \rho)$$

exactly as before with ρ in place of $\langle \rho_M^i | i < \omega\rangle$. Lemma 2.6.22 — 2.6.25 then go through exactly as before. Leaving the definition of *good* $\Sigma_1^{(n)}$ *definition* unchanged, we get the following version of Lemma 2.6.27: Let F be a good $\Sigma_1^{(n)}$ function mod ρ. There is a good $\Sigma_1^{(n)}$ definition which defines F mod ρ.

Even some of §2.7 remains valid for pseudo projecta. In §2.7.1 we define $\Gamma^0(\tau, M)$ (τ being a cardinal in M) as the set of maps $f \in M$ such that $\mathrm{dom}(f) \in H = H_\tau^M$. In §2.7.2 we then introduce $\Gamma^n = \Gamma^n(\tau, M)$ for the case that $n > 0$ and $\tau \leq \rho_M^n$, defining Γ^n to be the set of f such that:

(a) $\mathrm{dom}(f) \in H = H_\tau^M$.

(b) For some $i < n$ there is a good $\Sigma_1^{(i)}(M)$ function G and a parameter $p \in M$ such that:

$$f(x) = G(x, p) \text{ for all } x \in \mathrm{dom}(f).$$

Lemma 2.7.10 then told us that, whenever $\pi : M \to_{\Sigma_0^{(n)}} M'$, there is a canonical way of assigning to each $f \in \Gamma^n$ a definable partial map $\pi'(f)$ on M'. This continues to hold if $\pi : M \to_{\Sigma_0^{(n)}} M'$ mod ρ. The extended version of 2.7.10 reads:

Lemma 3.6.5. *Let $\pi : M \to_{\Sigma_0^{(n)}} M'$ mod ρ. There is a unique map π' which assigns to each $f \in \Gamma^n(\tau, M)$ a function $\pi'(f)$ with the following property:*

(*) $\pi'(f) : \pi(\mathrm{dom}(f)) \to M'$. Moreover, if $f(x) = G(x, p)$ for all $x \in \mathrm{dom}(f)$, where G is a good $\Sigma_1^{(i)}(M)$ function for an $i < n$ and $p \in M$, then

$$\pi'(f)(x) = G'(x, \pi(p)) \text{ for } x \in \pi(\mathrm{dom}(f)),$$

where G' is a good $\Sigma_1^{(i)}(M', \rho)$ function by the same good definition.

The proof is exactly as before. As before we get:

Lemma 3.6.6. *Let u, τ, π, π' be as above. Then $\pi'(f) = \pi(f)$ for $f \in \Gamma^0(\tau, M)$.*

Thus, again, we could unambiguously write $\pi(f)$ instead of $\pi'(f)$ for f. *However*, this is only unambiguous if we have previously specified the good sequence ρ. π' depends not only on π but also on the good sequence ρ. For this reason we shall write: $\pi_\rho(f)$ for $\pi'(f)$. We can omit the subscript ρ if the good sequence is clear from the context.

In §3.2 we then considered the special case that $\tau = \kappa^{+M}$ where κ is a cardinal in$M$. (This is mainly of interest when there is an extender F on M at κ.) We then set:

$$\Gamma^n_*(\kappa, M) =: \{f \in \Gamma^n(\kappa, M) | \operatorname{dom}(f) = \kappa\}.$$

We also set:

$$\Gamma^*(\kappa, M) =: \Gamma^n_*(\kappa, M) \text{ where } n \leq \omega \text{ is maximal such that } \kappa < \rho^n_M.$$

Let us call p a *defining parameter* for $f \in \Gamma^*(\kappa, M)$ iff either $p = f$ or else:

$$f(\xi) = G(\xi, p) \text{ for all } \xi < \kappa$$

where G is a good $\Sigma_1^{(i)}(M)$ function for an $i < n$. By lemma 2.6.25 we can then conclude:

Fact 1 Let $R(\vec{x}, y_1, \ldots, y_r)$ be a $\Sigma_0^{(n)}(M)$ relation. Let $f_i \in \Gamma^n_*(\kappa, M)$ have a defining parameter p_i for $i = 1, \ldots, r$. Then the relation:

$$Q(\vec{x}, \vec{\xi}) \longleftrightarrow: R(\vec{x}, f_1, (\xi_1), \ldots, f_r(\xi)$$

is $\Sigma_0^{(n)}(M)$ in the parameters $\kappa, p_1, \ldots, p_r$.

Moreover, if:

$$\sigma : M \to_{\Sigma_0^{(n)}} M' \mod \rho.$$

and R' has the same $\Sigma_0^{(n)}(M, \rho)$ definition, then the relation:

$$Q'(\vec{x}, \vec{\xi}) \leftrightarrow: R'(\vec{x}, \sigma_\rho(f_1)(\xi_1), \ldots, \sigma_\rho(f_r)(\xi_r))$$

is $\Sigma_1^{(n)}(M', \rho)$ in $\kappa, \sigma(p_1), \ldots, \sigma(p_r)$ by the same definition as Q.

Now let $a_1, \ldots, a_m \in M$ and set:

$$X = \{\langle\vec{\xi}\rangle | R(\vec{a}, \vec{f}(\xi))\}.$$

Then $X \in H^n_M$ and $\langle H^n_M, Q\rangle$ is amenable.

Fact 2 Let $R, R', Q, Q', f_1, \ldots, f_r, \sigma, M, M'$ be as in Fact 1. Let $\vec{a}, X$ be as above. Then:

$$\sigma(X) = \{\prec \vec{\xi} \succ \in \sigma(\kappa) | R'(\sigma(\vec{a}), \sigma_\rho(\vec{f})(\vec{\xi}))\}.$$

Proof (sketch)

We know:

$$\bigwedge \vec{\xi} < \kappa(\prec \vec{\xi} \succ \in X \leftrightarrow Q(\vec{a}, \vec{\xi}))$$

which is $\Pi_0^{(n)}(M)$ in the parameters $H_\kappa^M, \vec{a}, \vec{p}$. (We use here the fact that κ and the Gödel ν–tuple function on κ are H_κ^M–definable.) But then the corresponding $\Pi_0^{(n)}(M', \rho)$ statement holds of $H_n(M', \rho), \sigma(\vec{a})$, $\sigma(\vec{\alpha}), \sigma(\vec{p})$. QED (Fact 2)

Note. σ is Σ_1 preserving $\mod \rho$, if $n > 0$. But then $\kappa' = \sigma(\kappa)$ is a cardinal in M', since it is a cardinal in $H_0 = H_0(M', \rho)$ and ρ_0 is cardinally absolute in M'.

We now recall the Q–quantifier:

$$Qz^i \varphi(z^i) =: \bigwedge u^i \bigvee v^i (v^i \supset u^i \wedge \varphi(v^i)).$$

By a $Q^{(i)}$ formula we mean any formula of the form $Qz' \varphi(z^i)$, where $Q(\nu^i)$ is $\Sigma_1^{(i)}$. We write:

$$\sigma : M \to_{Q^*} N \mod (\rho, \rho')$$

to mean that σ is elementary $\mod (\rho, \rho')$ with suspect to $Q^{(n)}$ formulae for all $n < \omega$. Clearly, if σ is Q^* preserving $\mod (\rho, \rho')$, then it is Σ^*–preserving $\mod (\rho, \rho')$. If $\rho = \langle \rho_M^i | i < \omega \rangle$, we write:

$$\sigma : M \to_{Q^*} N \mod \rho.$$

In the following assume:

(1) $\sigma : M \to_{\Sigma^*} N \mod \rho'$.

We define a *minimal* good sequence:

$$\rho = \min \rho' = \min(\sigma, N, \rho')$$

with the following properties:

(a) $\sigma : M \to_{Q^*} N \mod \rho$.

(b) $\sup \sigma'' \rho_M^i \leq \rho_i \leq \rho_i'$ for $i < \omega$.

(c) Let φ be $\Sigma_0^{(i)}$. Let $x \in M, z_1, \ldots, z_p \in H_i(N, \rho)$. Then:

$$N \models \varphi[\vec{z}, \sigma(x)] \mod \rho \leftrightarrow N \models \varphi[\vec{z}, \sigma(x)] \mod \rho'.$$

(d) $\rho = \min \rho$.

We define ρ as follows:

Definition 3.6.5. Let $\sigma : M \to_{\Sigma^*} N \mod \rho'$. We define:

- $\rho_i(0) =: \sup \sigma'' \rho^i_M$.
- $\rho_i(n+1) =:$ the supremum of all $F(\eta)$ such that $\eta < \rho_{i+1}(n)$ and F is a $\Sigma^{(i)}_1(N, \rho')$ map to ρ'_i in parameters from $\mathrm{rng}(\sigma)$.
- $\rho_i =: \sup\limits_{n<\omega} \rho_i(n)$.
- $\rho = \langle \rho_i | i < \omega \rangle$.

Lemma 3.6.7. $\rho_i(n) \leq \rho_i(n+1)$.

Proof: We show by induction on n that it holds for all $i \leq \omega$.

Case 1 $n = 0$.
If $\xi < \rho^i_M$, then $\sigma(\xi) = F(0)$, where $F =$ the constant function $\sigma(\xi)$. But then F is $\Sigma^{(i)}_1(N, \rho')$ in $\sigma(\xi)$. Hence $\sigma(\xi) < \rho_i(1)$.

Case 2 $n > 0$.
Then $\rho_{i+1}(n) \geq \rho_{i+1}(n-1)$. Hence:

$$F'' \rho^{(n)}_{i+1} \supset F'' \rho^{(n-1)}_{i+1}$$

for all F which is a $\Sigma^{(i)}_1(N, \rho')$ map to ρ'_i.

The conclusion is immediate. QED (Lemma 3.6.7)

Lemma 3.6.8. *$\rho_i(n)$ is p.r. closed for $i > 0$.*

Proof: We show by induction on n that it holds for all $i > 0$.

Case 1 $n = 0$.
$\sigma \upharpoonright J^A_{\rho^i_M} : J^A_{\rho^i_M} \to_{\Sigma_0} J^A_{\rho_i}$ cofinally, where ρ^i_M is p.r. closed.

Case 2 $n > 0$. Let $n = m + 1$.
Then $\rho_i(m)$ is p.r. closed. Let f be a monotone p.r. function on On. It suffices to show:

Claim $f``\rho_i(n) \subset \rho_i(n)$.

Let $\nu < \rho_i(n)$. Then $\nu < F(\eta)$ where $\eta < \rho_{i+1}^{(m)}$ and F is $\Sigma_1^{(i)}(N, \rho')$ to ρ_i' in $\sigma(x)$. But then $f \circ F$ is $\Sigma_1^{(i)}(N, \rho')$ to ρ_i', since ρ_i' is p.r. closed. Hence $f(\nu) < f \cdot F(\eta) < \rho_i(n)$. QED (Lemma 3.6.8)

Corollary 3.6.9. *ρ_i is p.r. closed for $i > 0$.*

Definition 3.6.6.

$$H_i(n) = H_i(N, \sigma, \rho_i(n)) =: |J^{A^N}_{\rho_i(n)}|$$
$$H_i = H_i(N, \rho) =: |J^{A^N}_{\rho_i}|$$

Lemma 3.6.10. *(a) $H_i(0) = \bigcup \sigma'' H_M^i$.*

(b) $H_i(n+1) =$ the union of all $F(x)$ such that $x \in H_{i+1}^{(n)}$ and F is $\Sigma_1^{(i)}(n, \rho')$ to ρ_i' in parameters from $\mathrm{rng}(\sigma)$.

(c) $H_i = \bigcup\limits_n H_i(n)$.

Proof: (c) is immediate. (a) is immediate since:

$$\sigma \upharpoonright H_M^i : H_M^i \rightarrow_{\Sigma_0} H_i(0) \text{ cofinally.}$$

We prove (b). Let $y = F(x)$, where F, x are as in (b).

Claim $y \in H_i(n+1)$.

Proof: We recall the function $\langle S_\nu^A | \nu < \infty \rangle$ such that for all limit α:

$$J_\alpha^A = \bigcup_{\nu<\alpha} S_\nu^A \text{ and } \langle S_\nu^A | \nu < \alpha \rangle \text{ is}$$
$$\text{uniformly } \sigma_1(J_\alpha^A).$$

Since $\rho_{i+1}(n)$ is p.r. closed, there is a $\Sigma_1(H_{i+1}(n))$ map f of $\rho_{i+1}(n)$ onto $H_{i+1}(n)$. Set:

$$g(x) =: \text{ the least } \nu \text{ sucht that } x \in S_\nu.$$

Then $\tilde{F}(\xi) \simeq gFf(\xi)$ is a $\Sigma_1^{(i)}(N, \rho')$ map to ρ_i' in parameters from $\mathrm{rng}(\sigma)$. Hence, where $f(\eta) = x$, we have $y \in S^A_{\tilde{F}(\eta)} \subset H_i(n+1)$.

QED (Lemma 3.6.10)

By the definition 3.6.5 and Lemma 3.6.7:

Lemma 3.6.11. *Let $\rho = \min \rho'$. Then:*

- $\sigma"\rho^i_M \subset \rho_i \leq \rho'_0 \leq \rho^0_N$.
- $\rho_i = \sup X$, *where* X *is the set of all* $F(\nu)$ *such that* $\nu < \rho_{i+1}$ *and* F *is a* $\Sigma^{(i)}_1(N, \rho')$ *map to* ρ'_0 *in some* $\sigma(x)$.

Similarly by Lemma 3.6.10.

Lemma 3.6.12. *Let* $\rho = \min \rho'$. *Then:*

- $\sigma'' H^i_M \subset H_i \subset H'_i \subset H^i_N$.
- $H_i = \bigcup X$ *where* S *is the set of all* $F(x)$ *such that* $z = H_{i+1}$ *and* F *is a* $\Sigma^{(i)}_1(N, \rho')$ *map to* H'_i *in some* $\sigma(x)$.

We now can show:

Lemma 3.6.13. ρ *is good for* N.

Proof: By Lemma 3.6.11 we have:

$$\omega \leq \rho_{i+1} \leq \rho_i \leq \rho'_i \leq \rho^i_N.$$

Moreover, ρ_i is p.r. closed for $i > 0$ by Lemma 3.6.8.

It remains only to show:

Claim H_i is cardinally absolute with respect to N.

Proof: We know: $H_i = \bigcup X$, where $X =$ the set of $F(z)$ such that $z \in H_{i+1}$ and F is a $\Sigma^{(i)}_1(N, \rho')$ map to $H'_i = H_i(N, \rho')$. Moreover H'_i is cardinally absolute in N.

(1) Let $\alpha \in X$. Then $\overline{\overline{\alpha}}^N \in X$ and there is $f \in X$ such that $f : \overline{\overline{\alpha}}^N \xrightarrow{\text{onto}} \alpha$.

Proof: Suppose not.

Define a $\Sigma_1(H_i)$ map by:

$$F(\beta) \simeq \text{ the } <_{SA}\text{–least pair } \langle \gamma, f \rangle \text{ such that } \gamma < \beta \text{ and } f : \gamma \xrightarrow{\text{onto}} \beta.$$

Then $F''X \subset X$. Set:

$$\alpha_0 = \alpha_i \alpha_{i+1} \simeq (F(\alpha_i))_0.$$

By induction on i it follows that α_i exists and $\alpha_i \in X$. But then $\alpha_{i+1} < \alpha_i$ for $i < \omega$. Contradiction! QED (1)

Now let α be a cardinal in H_i but not in N. Then $\alpha \notin X$ by (1). But $\alpha < \beta$ for a $\beta \in X$. Hence $\overline{\overline{\beta}}^N > \alpha$. (Otherwise, letting $\gamma = \overline{\overline{\beta}}^N < \alpha$, we have $\gamma \in X \subset H_i$ and there is $f \in X \subset H_i$ such that $f : \gamma \xrightarrow{\text{onto}} \beta$. Hence there is $g \in H_i$ such that $g : \gamma \xrightarrow{\text{onto}} \alpha$, since $0 < \alpha < \beta$. Hence α is not a cardinal in H_i.) But then, letting $\gamma = \overline{\overline{\beta}}^N$, α is a cardinal in J^A_γ and γ is a cardinal in N. Hence α is a cardinal in N by acceptability. QED (Lemma 3.6.13)

We now verify property (c) for $\rho = \min \rho'$.

Lemma 3.6.14. *Let $\overline{B}(\vec{w}^i)$ be $\Sigma_0^{(i)}(M)$ in the parameter $x \in M$. Let $B'(\vec{w}^i)$ be $\Sigma_0^{(i)}(N, \rho')$ in $\sigma(x)$ and $B(\vec{w}^i)$ be $\Sigma_0^{(i)}(N, \rho)$ in $\sigma(x)$ by the same definition. Then:*

$$\bigwedge \vec{z} \in H_i(B(\vec{z}) \leftrightarrow B'(\vec{z})).$$

Proof: By induction on i. The case $i = 0$ is trivial. Now let it hold for h where $i = h + 1$. It suffices to prove the claim for $\overline{B}$ which is $\Sigma_1^{(h)}(M)$ in x. We than have:

$$\overline{B}(\vec{z}) \leftrightarrow \bigvee a^h D(a^h, \vec{z})$$

where $\overline{D}$ is $\Sigma_0^{(h)}(M)$ in x;

$$B'(\vec{z}) \leftrightarrow \bigvee a^h D'(a^h, \vec{z})$$

where D' is $\Sigma_0^{(h)}(N, \rho')$ in $\sigma(x)$ by the same definition, and:

$$B(\vec{z}) \leftrightarrow \bigvee a^h D(a^h, \vec{z})$$

where D is $\Sigma_0^{(h)}(N, \rho)$ in $\sigma(x)$ by the same definition.

Define a map F to ρ'_h which is $\Sigma_1^{(h)}(N, \rho')$ in $\sigma(x)$ by:

$$\begin{aligned} \xi = F(\vec{z}) \quad & \leftrightarrow (\vee u \in S_\xi D'(u\vec{z}) \cap \\ & \wedge \xi' < \xi \wedge u \in S_\xi, \neg D'(u, \vec{z}) \end{aligned}$$

Hence for $\vec{z} \in H_i$:

$$\begin{aligned} B'(\vec{z}) \quad & \leftrightarrow \vee u \in H_h D'(u, \vec{z}) \\ & \leftrightarrow \vee u \in S_{F(\vec{z})} D'(u, \vec{z}) \\ & \leftrightarrow \vee u \in H_h D'(u, \vec{z}) \\ & \leftrightarrow \vee u \in H_h D(u, \vec{z}) \leftrightarrow B(\vec{z}) \end{aligned}$$

(by the induction hypothesis). QED (Lemma 3.6.14)

Since $\sigma : M \to_{\Sigma^{(i)}} N \mod \rho'$, we conclude that $\sigma : M \to_{\Sigma^{(i)}} N \mod \rho$.

Since this holds for all $i < \omega$, we conclude:

Corollary 3.6.15. $\sigma : M \to_{\Sigma^*} N \mod \rho$.

Another immediate corollary is:

Corollary 3.6.16. $\rho = \min(N, \sigma, \rho)$.

It remains only to prove:

Lemma 3.6.17. $\sigma : M \to_{Q^*} N \mod \rho$.

Proof:

Assume: $M \models Qu^i\varphi(u^i, x)$ where φ is $\Sigma_1^{(i)}$.

Claim $N \models Qu^i\varphi(u^i, x) \mod \rho$.

Let $v \in H_i$. Then $v \subset w = G(\overline{w})$, where $\overline{w} \in H_{i+1}$. Then $v \subset w = G(\overline{w})$, where $\overline{w} \in H_{i+1}$ and G is $\Sigma_1^{(i)}(N, \rho)$ map to H_i in parameter from $\operatorname{rng}\sigma$. Let:

$$\varphi = \bigvee z^i\psi(z^i, u^i, x) \text{ where } \psi \text{ is } \Sigma_0^{(i)}.$$

Define a $\Sigma_1^{(i)}(N, \rho)$ map to H_i in $\sigma(x)$ by:

$$\begin{aligned} &F(w) \simeq \text{ the } N\text{–least } \langle z, u\rangle \in H^i \text{ such that} \\ &z \subset u \wedge \psi(z, u, \sigma(x)). \end{aligned}$$

The $\Pi_1^{(i+1)}$–statement:

$$\bigwedge a^{i+1}(a^{i+1} \in \operatorname{dom}(G) \to a^{i+1}) \in \operatorname{dom}(F \circ G))$$

holds in N, since the corresponding statement holds in M by our assumption. Let $\langle z, u\rangle = FG(\overline{w}) = F(w)$. Then $v \subset w \subset u$ and $\psi(z, u, \sigma(x))$. Hence:

$$N \models Qu\varphi(u, \sigma(x)) \mod \rho.$$

QED (Lemma 3.6.17)

Then $\rho = \min \rho'$ possess all the properties that we ascribed to it.

As a corollary of Lemma 3.6.17 we get:

Corollary 3.6.18. *Let B be $\Sigma_1^{(i)}(N, \rho)$ in parameters from $\operatorname{rng} \sigma$. Then $\langle H_i, B\rangle$ is amenable.*

Proof: Let $\overline{B}$ be $\Sigma_1^{(i)}(M)$ in x and B be $\Sigma_1^{(i)}(N, \rho)$ in the same definition. Since $\langle H_M^i, \overline{B}\rangle$ is amenable, we have:

$$Qu^i \bigvee y^i \; y^i = u^i \cap \overline{B} \text{ in } M.$$

But then:

$$Qu^i \bigvee y^i \; y^i = u^i \cap B \text{ in } N \mod \rho.$$

Let $u \in H_i$. There is then $v \supset u, v \in H_i$ such that $v \cap B \in H_i$. Hence $u \cap B = u \cap v \in H_i$. QED (Corollary 3.6.18)

Definition 3.6.7. $\sigma : M \to_{\Sigma^*} N \min \rho$ iff

$$[\sigma : M \to_{\Sigma^*} N \mod \rho] \wedge [\rho = \min(N, \sigma, \rho)].$$

(Similarly for $\Sigma_j^{(n)}, Q_j^{(n)}, Q^*$ etc.)

In the following we shall always assume that M is acceptable, $\kappa \in M$ is inaccessable in M, and that $\tau = \kappa^{+M} \in M$.

Lemma 3.6.19. *Let $\pi : M \to_{\Sigma^*} M'$. Let $\kappa = \operatorname{crit}(\pi)$, $\lambda \leq \pi(\kappa)$, and suppose an extender F at κ, λ on M to be defined by:*

$$F(X) = \lambda \cap \pi(X) \text{ for } X \in \mathbb{P}(\kappa) \cap M.$$

Let $\sigma : \overline{M} \to_{\Sigma^*} M \min \rho$, where $\sigma(\overline{\kappa}) = \kappa$. Let F be a weakly amenable extender at $\overline{\kappa}, \overline{\lambda}$ on $\overline{M}$. Assume:

$$\langle \sigma, g\rangle : \langle \overline{M}, \overline{F}\rangle \to \langle M, F\rangle, \text{ where } g : \overline{\lambda} \to \lambda.$$

Let $n \leq w$ be maximal such that $\overline{\kappa} < \rho_{\overline{M}}^n$.

Define a good sequence ρ^* for M' by:

$$\rho_i^* = \begin{cases} \sup \pi''\rho_n \text{ if } i = n \\ \pi(\rho_i) \text{ if } i \neq n \text{ and } \rho_i < \rho_M^i \\ \rho_{M'}^i \text{ if } i \neq n \text{ and } \rho_i = \rho_M^i. \end{cases}$$

(Hence $\pi : M \to_{\Sigma^*} M' \mod (\rho, \rho^*)$ by Lemma 3.6.3 and 3.6.4.) Then:

(a) $\overline{M}$ is n–extendible by $\overline{F}$.

(b) Let $\overline{\pi} : \overline{M} \rightarrow^{(n)}_{\overline{F}} \overline{M}'$. There is a map σ' such that

$$\sigma' : \overline{M}' \rightarrow_{\Sigma_0^{(n)}} M' \quad \text{mod } \rho^* \text{ and } \sigma'\overline{\pi} = \pi\sigma, \sigma' \upharpoonright \overline{\lambda} = g.$$

Moreover, σ' is defined by:

$$\sigma'(\overline{\pi}(f)(\alpha)) = ((\pi\sigma)_{\rho^*}(f))(g(\alpha))$$

for $f \in \Gamma^*(\overline{\kappa}, \overline{M})$, $\alpha < \lambda$.

Proof: We obviously have:

$$\pi\sigma : \overline{M} \rightarrow_{\Sigma^*} M' \quad \text{mod } \rho^*.$$

It is also clear that n is maximal such that $\kappa < \rho_n$ and also maximal such that $\kappa' = \pi(\kappa) < \rho^*_n$.

We now prove (a). We must show that the $\in$–relation $\in^*$ of $\mathbb{D}^*(\overline{F}, \overline{M})$ is well founded. Let $\langle f, \alpha\rangle, \langle f', \alpha'\rangle \in \mathbb{D}^*$. Set:

$$e = \{\prec \xi, \zeta \succ < \overline{\kappa} | f(\xi) \in f'(\zeta)\}.$$

Then:

$$\begin{aligned} \langle f, \alpha\rangle \in^* \langle f', \alpha'\rangle &\longleftrightarrow \langle a, \alpha'\rangle \in \overline{F} \\ &\longleftrightarrow \prec g(\alpha), g(\alpha') \succ \in F(\sigma(e)) \\ &\longleftrightarrow \prec g(\alpha), g(\alpha') \succ \in \pi\sigma(e) \\ &\longleftrightarrow (\pi\sigma)_{\rho^*}(f)(g(\alpha)) \in (\pi\sigma)_{f^*}(f')(g(\alpha)) \end{aligned}$$

(The second line rises the assumption: $\langle \sigma, g\rangle : \langle \overline{M}, \overline{F}\rangle \rightarrow \langle M, F\rangle$. The third uses: $F(X) = \lambda \cap \pi(X)$. The fourth uses Fact 2, which we established earlier in the section. QED (a)

We now prove (b). Let $\overline{R}'$ be a $\Sigma_0^{(n)}(\overline{M}')$ relation and let R' be $\Sigma_0^{(n)}(M')$ by the same definition. We claim that: $\sigma' : \overline{M}' \rightarrow_{\Sigma_0^{(n)}} M'$ where σ' is defined by:

$$\sigma'(\overline{\pi}(f)(\alpha)) = (\pi\sigma)_{\rho^*}(f)(g(\alpha))$$

for $f \in \Gamma^*(\overline{u}, \overline{M}), \alpha < \lambda$.

Let $\overline{R}'$ be a $\Sigma_0^{(n)}(\overline{M}')$ relation and let R' be $\Sigma_0^{(n)}(M', \rho^*)$ by the same definition. Let $\alpha_1, \ldots, \alpha_m < \overline{\lambda}$ and $f_1, \ldots, f_m \in \Gamma^*(\overline{u}, \overline{M})$. Writing e.g. $\vec{f}(\vec{\alpha})$ for $f_1(\alpha_1), \ldots, (\alpha_m)$, it suffices to show:

Claim $\overline{R}'(\overline{\pi}(\vec{f})(\vec{\alpha})) \leftrightarrow R'(\pi\sigma(\vec{f}), g(\vec{\alpha}))$.

Proof: Let $\overline{R}$ be $\Sigma_0^{(n)}(\overline{M})$ and R be $\Sigma_0^{(n)}(M, \rho)$ by the same definition. Set:

$$e = \{\prec \vec{\xi} \succ | \overline{R}(\vec{f}(\vec{\overline{\xi}})\}.$$

Then:

$$\begin{aligned}
\overline{R}'(\overline{\pi}(\vec{f})(\vec{\alpha})) &\longleftrightarrow \prec \vec{\alpha} \succ \in \overline{F}(e) \\
&\longleftrightarrow \prec g(\vec{\alpha}) \succ \in F(\sigma(e)) \\
&\longleftrightarrow \prec g(\vec{\alpha}) \succ \in \pi\sigma(e) \\
&\longleftrightarrow R'((\pi\sigma)_{\rho^*}(\vec{f})(g(\vec{\alpha})))
\end{aligned}$$

QED (Lemma 3.6.19)

We would like to prove something stronger namely that $\overline{M}$ is $*$–extendible by $\overline{F}$ and that:

$$\sigma' : \overline{M}' \rightarrow_{\Sigma^*} M' \quad \text{mod } \rho^*.$$

For this we must strengthen the condition:

$$\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \rightarrow \langle M, F \rangle.$$

In §3.2 we helped ourselves in a similar situation by strengthening the relation $\rightarrow$ to $\rightarrow^*$. However $\rightarrow^*$ is too strong for our purposes and we adopt the following weakening:

Definition 3.6.8. $\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \rightarrow^{**} \langle M, F \rangle \quad \text{mod } \rho$ iff the following hold:

(a) $\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \rightarrow \langle M, F \rangle$

(b) $\sigma : \overline{M} \rightarrow_{\Sigma_0} M \quad \text{mod } \rho$

(c) Let $\overline{\alpha} < \text{lh}(\overline{F}), \alpha = g(\overline{\alpha})$. There are $\overline{G}, G, \overline{H}, H$ such that letting

$$\overline{\kappa} = \text{crit}(\overline{F}), \kappa = \text{crit}(F)$$

we have:

(i) $\overline{G}, \overline{H}$ are $\Sigma_i(\overline{M})$ in a $\overline{q} \in \overline{M}$ and G, H are $\Sigma_1(M, \rho)$ in $q = \sigma(\overline{q})$ by the same definition.

(ii) $\overline{G} = \overline{F}_{\overline{\alpha}}, \overline{H} = \overline{M} \cap ({}^{\overline{\kappa}}\mathbb{P}(\overline{u}))$

(iii) $G \subset F_\alpha$

(iv) $H \subset \{X \in {}^{\kappa}\mathbb{P}(u) | \bigwedge \xi < \kappa(X_\xi \text{ or } \kappa \setminus X_\xi \in G)\}$

Note. Actually, only the first pseudo projectum ρ_0 is relevant in this definition. (b)says merely that ρ is good for M and that σ is a Σ_0-preserving map into M with $\sigma'' \mathrm{On}_{\overline{M}} \leq \rho_0$. In (c) the statement "$G, H$ are $\Sigma_1(M, \rho)$ in q by the same definition" can be rephrased as: "G, H are $\Sigma_1(M|\rho_0)$ in q by the same definition", where $M|\eta =: \langle J^A_\eta, B \cap J^A_\eta\rangle$ for $M = \langle J^A_\alpha, B\rangle$.

(Note that $M|\eta$ is not necessarily amenable.) We set:

Definition 3.6.9. $\langle\sigma, g\rangle : \langle\overline{M}, \overline{F}\rangle \rightarrow^{**} \langle M, F\rangle$ iff:

$$\langle X, g\rangle : \langle\overline{M}, \overline{F}\rangle \rightarrow^{**} \langle M, F\rangle \quad \mathrm{mod}\ (\langle\rho^n_M | n < w\rangle).$$

Note. This always holds if $\rho_0 = \mathrm{On}_M$.

Note. Let $\sigma : \langle\overline{M}, \overline{F}\rangle \rightarrow^{**} \langle M, F\rangle$ mod ρ. Let $\overline{X} \in \overline{M} \cap ({}^{\overline{\kappa}}\mathbb{P}(\overline{\kappa}))$. If $X = \sigma(\overline{X})$, then $X \in M$ and hence $\bigwedge \xi < \kappa(X_\xi$ or $(\kappa \setminus X_\xi) \in G)$.

Note. Let $\sigma : \langle\overline{M}, \overline{F}\rangle \rightarrow^{*} \langle M, F\rangle$. It follows easily that:

$$\sigma : \langle\overline{M}, \overline{F}\rangle \rightarrow^{**} \langle M, F\rangle.$$

Note. Suppose that $\sigma : \overline{M} \rightarrow_{\Sigma^*} M \min \rho$. Set $M|\rho_0 = \langle J^A_{\rho_0}, B \cap J^A_{\rho_0}\rangle$, where $M = \langle J^A_\gamma, B\rangle$. Then $M|\rho_0$ is amenable by Corollary 3.6.18. Clearly $\tau = \kappa^{+M} \in M|\rho_0$ since $\overline{\tau} = \kappa^{+\overline{M}} \in \overline{M}$. Hence $\mathbb{P}(\kappa) \cap M \subset M|\rho_0$. But then F is an extender at κ on $M|\rho_0$ and it makes sense to write:

$$\langle\sigma, g\rangle : \langle\overline{M}, \overline{F}\rangle \rightarrow^{**} \langle M|\rho_0, F\rangle.$$

But this means exactly the same thing as:

$$\langle\sigma, g\rangle : \langle\overline{M}, \overline{F}\rangle \rightarrow^{**} \langle M, F\rangle \quad \mathrm{mod}\ \rho.$$

We are now ready to prove:

Lemma 3.6.20. *Let $\pi, \sigma, \overline{M}, M, \overline{M}', M', \rho, \rho^*, \overline{\tau}, \tau, \overline{\pi}, \sigma', g$ be as in lemma 3.6.19. Assume:*

$$\langle\sigma, g\rangle : \langle\overline{M}, \overline{F}\rangle \rightarrow^{**} \langle M, F\rangle \quad \mathrm{mod}\ \rho.$$

Then $\overline{M}$ is $*$–extendible by $\overline{F}$ and:

$$\sigma' : \overline{M}' \rightarrow_{\Sigma^*} M' \quad \mathrm{mod}\ \rho^*.$$

Proof: $\overline{F}$ is then close to $\overline{M}$. Hence $\overline{M}$ is $*$–extendible by $\overline{F}$. By induction on i we now show:

Claim $\sigma' : \overline{M}' \to_{\Sigma_1^{(i)}} M' \mod \rho^*$.

For $i < n$ this is given. Now let $i = n$. We prove a somewhat stronger claim:

Subclaim 1 Let $\overline{A} \subset \overline{\kappa}$ be $\Sigma_1^{(n)}(\overline{M}')$ in $\overline{a} \in \overline{M}'$ and $A \subset \kappa$ be $\Sigma_1^{(n)}(M', \rho^*)$ in $a = \sigma'(\overline{a})$ by the same definition. There is $\overline{r} \in \overline{M}$ such that $\overline{A}$ is $\Sigma_1^{(n)}(\overline{M})$ in $\overline{r}$ and A is $\Sigma_1^n(M, \rho)$ in $r = \sigma(\overline{r})$ by the same definition.

(As we shall see, this proves the claim for the case $i = n$.)

We now prove the subclaim. Let:

$$\overline{A}(i) \leftrightarrow \bigvee y \overline{P}'(y, i, \overline{a}),$$
$$A(i) \leftrightarrow \bigvee y P'(y, i, a)$$

where $\overline{P}'$ is $\Sigma_0(\overline{M}')$ and P' is $\Sigma_0(M', \rho^*)$ by the same definition.

Let $\overline{P}$ be $\Sigma_0^{(n)}(\overline{M})$ and P be $\Sigma_0^{(n)}(M)$ by the same definition. Let $\overline{a} = \overline{\pi}(f)(\overline{\alpha})$ and $a = \overline{\pi}\sigma(f)(\alpha)$, where $\alpha = g(\overline{\alpha})$. Let $\overline{p}$ be a "defining parameter" for f (i.e. either $\overline{p} = f$ or else $f(\xi) = B(\xi, \overline{p})$ where B is a good $\Sigma_1^{(i)}(\overline{M})$ function for an $i < n$.) Then $p = \sigma(\overline{p})$ is in the same sense a defining parameter for $\sigma(f)$ and $p' = \pi\sigma(\overline{p})$ is a defining parameter for $\pi\sigma(f)$. (The good definition of B remaining unchanged.)

Finally, let $\overline{G}, G, \overline{H}, H$ be as given for $\overline{\alpha}, \alpha = g(\overline{\alpha})$ by the principle:

$$\langle \sigma, q \rangle : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M, F \rangle \mod \rho^*.$$

Since $\langle \overline{M}', \overline{\pi} \rangle$ is the extension of $\langle \overline{M}, \overline{F} \rangle$, we know that: $\overline{\pi}``H^n_{\overline{M}}$ is cofinal in H^n_M.

Thus:

(1)
$$\begin{aligned} \overline{A}(i) &\leftrightarrow \bigvee u \in H^n_{\overline{M}} \bigvee y \in \overline{\pi}(u) \overline{P}'(g, i, \overline{\pi}(f)(\overline{\alpha})) \\ &\leftrightarrow \bigvee u \in H^n_{\overline{M}} \overline{\alpha} \in \overline{\pi}(\overline{X}(i, u)) \\ &\leftrightarrow \bigvee u \in H^n_{\overline{M}} \overline{X}(i, u) \in \overline{G}, \end{aligned}$$

where $\overline{X}(i, u) = \{\xi < \overline{u} | \overline{P}(y, i, f(\xi))\}$.

Thus $\overline{A}$ is $\Sigma_1^{(n)}(\overline{M})$ in $\overline{p}, \overline{q}, \overline{\kappa}$. We now show that A is $\Sigma_1^{(n)}(M)$ in p, q, κ by the same definition. Set:

$$H_n = H_n(M, \rho), \; H'_n = H_n(M', \rho^*).$$

It is easily seen that the relation:

$$Q(u, i, \xi) \longleftrightarrow: (u \in H_n \wedge \bigvee y \in u P(y, i, \sigma_\rho(f)(\xi))$$

is $\Sigma_0^{(n)}(M,\rho)$ in p and the relation:

$$Q'(u,i,\xi) \longleftrightarrow: (u \in H'_n \wedge \bigvee y \in uP'(y,i,(\pi\sigma)_{\rho^*}(\xi))$$

is $\Sigma_0^{(n)}(M',\rho^*)$ in p' by the same definition. Set: $X(u,i) = \{\xi < u | Q(u,i,\xi)\}$. Then $X(u,i) \in H_n$, since $\langle H_n, Q\rangle$ is amenable by lemma 3.6.14 and hence is rud closed. Since $\rho_n^* = \sup \sigma"\rho_n$, we know that $\pi"H_n$ is cofinal in H'_n. Thus:

(2)

$$\begin{aligned} A(i) &\leftrightarrow \bigvee u \in H_n \bigvee y \in \pi(u) P'(y,i,((\pi\sigma)_{\rho^*}(f)(\alpha)) \\ &\leftrightarrow \bigvee u \in H_n Q(\pi(u),i,\alpha) \\ &\leftrightarrow \bigvee u \in H_n \alpha \in \pi(X(u,i)) \cap X \\ &\leftrightarrow \bigvee u \in H_n \alpha \in F(X(u,i)) \\ &\leftrightarrow \bigvee u \in H_n X(u,i) \in F_\alpha. \end{aligned}$$

If $F_\alpha = G$, we would be finished, but G might be a proper subset of F_α. (Moreover, we don't even know that F_α is M–definable in parameters.) However, we can prove:

(3) $A(i) \leftrightarrow \bigvee u \in H_n X(u,i) \in G$,

which establishes subclaim 1. The direction $(\leftarrow)$ is trivial by (2), since $G \subset F_\alpha$. We prove $(\rightarrow)$. Assume $A(i_0)$, where $i_0 < \kappa$. We must show that $u \in H_n$ can be chosen large enough that $X(u,i_0) \in G$. We know that it can be chosen large enough that $X(u,i_0) \in F_\alpha$. Since $\rho = \min(M,\sigma,\rho)$, we also know that the set of $S(\xi)$ such that S is a partial $\Sigma_1^{(n)}(M,\rho)$ map to H_n in a parameter $s = \sigma(\overline{s})$ and $\xi < \rho_{n+1}$ is cofinal in H_n. (This uses Lemma 3.6.12.) Hence we can assume w.l.o.g. that $u = S(\xi_0)$ for a $\xi_0 < \rho_{n+1}$. Now set:

$$Y(v) =: \{x(v,i) | i < u\} \text{ for } v \in H_n.$$

Then $Y(v) \in H_n$ by the rud closure of $\langle H_n, Q\rangle$. Moreover, the function Y is $\Sigma_1(\langle H_n, Q\rangle)$ and hence is a $\Sigma_1^{(n)}(M,\rho)$ function. Hence $Y \circ S$ in $\Sigma_1^{(n)}(M,\rho)$ in s. Let $\overline{S}$ be $\Sigma_1^{(n)}(M)$ is $\overline{s}$ and $\overline{Y}$ be $\Sigma_1^{(n)}(\overline{M})$ by the same definition. The $\Pi^{(n+1)}(M,\rho)$ statement:

$$\bigwedge \zeta < \rho_{n+1}(\zeta \in \mathrm{dom}(Y \cdot S) \rightarrow Y \cdot S(\zeta) \in H)$$

is true, since the corresponding statement:

$$\bigwedge \zeta < \rho_M^{n+1}(\zeta \in \mathrm{dom}(\overline{Y} \cdot \overline{S}) \rightarrow \overline{Y} \cdot \overline{S}(\zeta) \in \overline{H})$$

is true in $\overline{M}$. Since $u = S(\zeta_0)$, it follows that: $Y(u) \in H$ and:

$$X(\kappa, i_0) \in G \vee (\kappa \setminus X(u,i_0)) \in G.$$

But $G \subset F_\alpha(\kappa \setminus X(u,i_0)) \in G$ is therefore impossible, since we would then have:

$$X(\kappa, i_0) \cap (\kappa \setminus X(u,i_0)) = \emptyset \in F_\alpha.$$

Hence, $X(U,i_0) \in G$. QED (Subclaim 1)

Subclaim 2 $\sigma' : \overline{M}' \to_{\Sigma_1^{(n)}} (\overline{M}')$ mod ρ^*.

Proof. Let Q be $\Sigma_1^{(n)}(M', \rho^*)$ and $\overline{Q}$ be $\Sigma_1^{(n)}(\overline{M}')$ by the same definition. Set:

$$P(i,x) \leftrightarrow (i = 0 \wedge Q(x)),$$
$$\overline{P}(i,x) \leftrightarrow (i = 0 \wedge \overline{Q}(x)).$$

Set:

$$A(x) = \{i | P(i,x)\}, \overline{A}(x) = \{i | \overline{P}(i,x)\}.$$

Then A is the characteristic function of Q and $\overline{A}$ is the characteristic function of $\overline{Q}$. But $A(\sigma'(x)) = \overline{A}(x)$ for $x \in \overline{M}$ by Subclaim 1.

QED (Subclaim 2)

A slight reformulation of Subclaim 1 yields:

Subclaim 3 Let A be $\Sigma_1^{(n)}(M', \rho^*)$ i $p = \sigma'(\overline{p})$. Let $\overline{A}$ be $\Sigma_1^{(n)}(\overline{M}')$ in $\overline{p}$ by the same definition. Set: $\overline{H} = H_{\overline{\kappa}}^{\overline{M}}, H = H_\kappa^M$. Then $A \cap H$ is $\Sigma_1^{(n)}(M, \rho)$ in a $q = \sigma(\overline{q})$ and $\overline{A} \cap \overline{H}$ is $\Sigma_1^{(n)}(\overline{M})$ in $\overline{q}$ by the same definition.

Proof: $H = J_\kappa^E$, where $E = E^M$ and $\overline{H} = J_{\overline{\kappa}}^{\overline{E}}$ where $\overline{E} = E^{\overline{M}}$. But $\kappa, \overline{\kappa}$ are preclosed. Let $f : \kappa \overset{\text{onto}}{\longrightarrow} H$ be primitive recursive in E and let $\overline{f} : \overline{\kappa} \overset{\text{onto}}{\longrightarrow} \overline{H}$ be primitive recursive in $\overline{E}$ by the same definition. Apply subclaim 1 to

$$B = f^{-1\prime\prime} A, \overline{B} = \overline{f}^{-1\prime\prime} \overline{A}.$$

Then $B \subset \overline{\kappa}$ is $\Sigma_1^{(n)}(M, \rho)$ in a $q = \sigma(\overline{q})$ and $\overline{B} \subset \overline{\kappa}$ is $\Sigma_1^{(n)}(\overline{M})$ in $\overline{q}$. But then the same holds for $A = f'' B, \overline{A} = \overline{f}'' \overline{B}$.

QED (Subclaim 3)

For $i > n$, we know: $\rho_{\overline{M}}^i = \rho_M^i$, so we can write $\rho^i =: \rho_{\overline{M}}^i$. By the definition of ρ^*, we know: $\rho_i = \rho_i^*$ for $i > n$. We can also set:

$$\overline{H}^i = H_{\overline{M}}^i = H_{\overline{M}'}^i, H_i = H_i(M, \rho) = H_i(M', \rho^*).$$

We now prove:

Subclaim 4 Let $i > n$. Let $\overline{A}$ be $\Sigma_1^{(i)}(\overline{M}')$ in $\overline{a} \in \overline{M}'$ and let A be $\Sigma_1^{(i)}(M', \rho^*)$ in $a = \sigma'(\overline{a})$ by the same definition. Then there are $\overline{B}, B$, $\overline{q}, q$ such that

(a) $\overline{B}$ is $\Sigma_0^{(i)}(\overline{M})$ in $\overline{q} \in M$.

(b) B is $\Sigma_0^{(i)}(M, \rho)$ in $q = \sigma(\overline{q})$ by the same definition.

(c) $\overline{A} \cap \overline{H}^i = \overline{B} \cap \overline{H}^i$.

(d) $A \cap H_i = B \cap H_i$.

Proof: By induction on i. Let it hold below i. Then w.l.o.g. we can assume:

(1) $\overline{A}(x) \longleftrightarrow \langle \overline{H}^i, \overline{P} \cap \overline{H}^i \rangle \models \varphi[x]$ for $x \in \overline{H}^i$ where φ is Σ_1 and $\overline{p}$ is $\Sigma_0^{i-1}(\overline{M}')$ in $\overline{a}$.

(2) $A(x) \longleftrightarrow \langle H', P \cap H_i \rangle \models \varphi[x]$ for $x \in H_i$ where φ is the same Σ_1 formula and P is $\Sigma_0^{i-1}(M', \rho^*)$ in a by the same definition.

But then there are $\overline{Q}, Q, \overline{q}, q$ such that

(3) $\overline{P} \cap H^i = \overline{Q} \cap H^i$, where $\overline{Q}$ is $\Sigma_1^{i-1}(\overline{M})$ in $\overline{q} \in \overline{M}$.

(4) $P \cap H_i = Q \cap H_i$, where $\overline{Q}$ is $\Sigma_1^{i-1}(M, \rho)$ in $q = \sigma(q)$ by the same definition.

This is by subclaim 3 if $i = n + 1$, and otherwise by the induction hypothesis. QED (Sublemma 4)

The claim then follows easily, since σ is Σ^*–preserving mod ρ^*.
QED (Lemma 3.6.20)

We can then go on further and set:

$$\rho' = \min(M', \sigma', \rho^*).$$

It then follows that:

$$\pi``\rho_i \subset \rho'_i \leq \rho^*_i \text{ for } i < \omega.$$

To see that $\pi''\rho_i \subset \rho'_i$, we recall that $\rho'_i = \sup\{\rho'_i(n) : n < \omega\}$ where the sequence $\langle \rho'_i(n) | i < w \rangle$ is defined from ρ^*, M', σ' by a canonical recursion on n (cf. Definition 3.6.5).

But since $\rho = \min(M, \sigma, \rho)$, we have: $\rho_i = \sup_{n<w} \rho_i(n)$, where $\langle \rho_i(n) | i < w \rangle$ is defined from ρ, M, σ by the same induction on n. Since $\pi'\sigma = \pi\sigma$, it follows easily by induction on n that:

$$\pi``\rho_i(n) \subset \rho'_i(n) \text{ for } i < w.$$

The details are left to the reader.

Putting all of this together:

Theorem 3.6.21. *Let $\pi : M \to_{\Sigma^*} M'$ with critical point κ. Let $\lambda \leq \pi(\kappa)$ and let the extender F at κ, λ on M be defined by:*

$$F(X) = \pi(X) \cap \lambda.$$

Let $\sigma : \overline{M} \to_{\Sigma^*} M \min \rho$ with $\sigma(\overline{\kappa}) = \kappa$. Assume:

$$\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M, F \rangle \mod \rho$$

where $\overline{F}$ is a weakly amenable extender at $\overline{\kappa}, \overline{\lambda}$ on $\overline{M}$. Then

(a) $\overline{M}$ is $*$–extendable by $\overline{F}$, giving $\overline{\pi} : \overline{M} \to^*_{\overline{F}} \overline{M}'$.

(b) There are σ', ρ' such that

(i) $\sigma' : \overline{M}' \to_{\Sigma^*} M' \min \rho'$

(ii) σ' is defined by:

$$\sigma'(\overline{\pi}(f)(\alpha)) = (\pi\sigma)_\rho(f)(g(\alpha))$$

for $\alpha < \lambda^-, f \in \Gamma^*(\overline{\kappa}, \overline{M})$. (Hence $\sigma'\overline{\pi} = \pi\sigma$ and $\sigma' \restriction \overline{\lambda} = g$.)

(iii) $\pi''\rho_i \subset \rho'_i \leq \pi(\rho_i)$ for $i < w$ (taking $\pi(\rho_i) = \mathrm{On}_M$, if $\rho_i = \mathrm{On}_M$).

(c) The above, in fact, holds for:

$$\rho' =: \min(\rho^*) = \min(M', \sigma'\rho^*).$$

where ρ^* is defined by:

$$\rho^*_0 = \begin{cases} \sup{}''\rho_i \text{ if } \rho_{i+1} \leq \kappa_i \\ \pi(\rho_i) \text{ if } \kappa_i < \rho_{i+1} \text{ and } \rho_i < \rho^i_M \\ \rho^i_M, \text{ if } \kappa_i < \rho_{i+1} \text{ and } \rho_i = \rho^i_M. \end{cases}$$

This is the most important result on pseudo projecta.

The argumentation used in the proof of Lemma 3.6.35, Lemma 3.6.36 and Lemma 3.6.37 actually establishes a more abstract result which is useful in other contexts:

Lemma 3.6.22. *Assume that M_i, M'_i are amenable for $i < \mu$, where μ is a limit ordinal. Assume further than:*

(a) *$\pi_{i,j} : M_i \longrightarrow_{\Sigma^*} M_j$ $(i \leq j < \mu)$, where the $\pi_{i,j}$ commute.*

(b) $\pi'_{i,j} : M'_i \longrightarrow_{\Sigma^*} M'_j$ $(i \leq j < \mu)$, *where the* $\pi'_{i,j}$ *commute.*
Moreover:

$$\langle M'_i : i < \mu \rangle, \langle \pi'_{i,j} : i \leq j < \mu \rangle$$

has a transitivized direct limit $M', \langle \pi'_{i,j} : i \leq j < \mu \rangle$.

(c) $\sigma_i : M'_i \longrightarrow_{\Sigma^*} M'_j \min \rho^i$ $(i \leq j < \mu)$.

(d) $\sigma_j \pi_{i,j} = \pi'_{i,j} \sigma_i$.

(e) $\pi'_{i,j}$ "$\rho^i_n \subset \rho^i_n \leq \pi'_{i,j}(\rho^i_n)$ *for* $i \leq j < \mu, n < \omega$.

Then:

$$\langle M_i : i < \mu \rangle, \langle \pi_{i,j} : i \leq j < \mu \rangle$$

has a transitivized direct limit $M, \langle \pi_{i,j} : i < \mu \rangle$.

There is then $\sigma : M \longrightarrow M'$ defined by: $\sigma \pi_i = \pi'_i \sigma_i (i < \mu)$. Moreover:

(1) There is a unique ρ such that $\sigma : M \longrightarrow_{\Sigma^*} M' \min \rho$ and:

$$\pi' \text{“} \rho^i_n \subset \rho_n \leq \pi'_i(\rho^i_n) \text{ for } i < \mu, n < \omega.$$

(2) There is $i < \mu$ such that $\rho_n = \pi'_j(\rho^i_n)$ for $i \leq j < \mu, n < \omega$.

3.6.3 Mirrors

Let $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ be a normal iteration of length η. By a *mirror* of I we shall mean a sequence:

$$I' = \langle \langle M'_i \rangle, \langle \pi'_{ij} \rangle, \langle \sigma_i \rangle, \langle \rho^i \rangle \rangle$$

such that $\sigma_i : M_i \rightarrow_{\Sigma^*} M'_i \min \rho^i$ for $i < \eta$ and the sequence:

$$I'' = \langle \langle M'_i \rangle, \langle \nu'_i \rangle, \langle \pi'_{ij} \rangle, T \rangle$$

"mirrors" the action of I, where $\nu'_i =: \sigma_i(\nu_i)$. However, I'' will not necessarily be an iteration. If $i+1$ is not a drop point in I and $h = T(i+1)$, we will, indeed, have:

$$\pi'_{h,i+1} : M'_h \rightarrow_{\Sigma^*} M'_{i+1},$$

but M'_{i+1} is *not* necessarily an ultrapower of M'_h. None the less $\kappa'_i =: \sigma_i(\kappa_i)$ will still be the critical point and we shall have:

$$\mathbb{P}(\kappa'_i) \cap M'_h = \mathbb{P}(\kappa'_i) \cap J^{E^{M'_i}}_{\nu_i}$$

and:

$$\alpha \in E_{\nu_i}^{M_i'}(X) \leftrightarrow \alpha \in \pi'_{h,i+1}(X) \text{ for}$$
$$X \in \mathbb{P}(\kappa_i') \cap M_h' \text{ and } \alpha < \lambda_i',$$

where $\lambda_i' =: \sigma_i(\lambda_i)$.

We shall also require a measure of agreement among the maps σ_i. In particular, if $h = T(i+1)$ is as above, then:

$$\sigma_{i+1}\pi_{h,i+1} = \pi'_{h,i+1}\sigma_h;\ \sigma_i \restriction \lambda_i = \sigma_{i+1} \restriction \lambda_i.$$

Note. that this gives:

$$\langle \sigma_h, \sigma_i \restriction \lambda_i \rangle : \langle M_h, E_{\nu_i}^{M_i} \rangle \to \langle M_h', E_{\nu_i}^{M_i'} \rangle.)$$

The formal definition is:

Definition 3.6.10. Let $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ be a normal iteration of length η. By a *mirror* of I we mean a sequence:

$$I' = \langle \langle M_i' | i < \eta \rangle, \langle \pi_{ij}' | i \leq_T i \rangle, \langle \sigma_i < | i < \eta \rangle, \langle \rho^i | i < \eta \rangle \rangle$$

satisfying the following conditions:

(a) M_i' is a premouse and $\sigma_i : M_i \to_{\Sigma^*} M_i' \min \rho^i$.

(b) π_{ij}' is a partial structure preserving map from M_i' to M_j'. Moreover the π_{ij}' commute and $\pi_{ii} = \mathrm{id} \restriction M_i$. If $\lambda < \eta$ is a limit, then $M_\lambda' = \bigcup\limits_{i \top \lambda} \mathrm{rng}(\pi_{i\lambda}')$.

(c) $\sigma_i \pi_{ij} = \pi_{ij}' \sigma_i$ for $i \leq_\top j$.

(d) $\sigma_i \restriction \lambda_i = \sigma_j \restriction \lambda_i$ for $i < j < \eta$.

In order to state the further clauses we need some notation. Set:

$$\nu'_i = \sigma_i(\nu_i) =: \begin{cases} \sigma_i(\nu_i) \text{ if } \nu_i \in M_i \\ \mathrm{On} \cap M_i' \text{ if not} \end{cases}$$
$$\kappa_i' = \sigma_i(\kappa_i), \tau_i' = \sigma_i(\tau_i), \lambda_i' = \sigma_i(\lambda_i)$$

For $h = T(i+1)$ set:

$$M_i'^{*} = \begin{cases} \sigma_h(M_i^*) \text{ if } M_i^* \in M_h \\ M_h' \text{ if not.} \end{cases}$$

Noting that $\tau_i' = \sigma_h(\tau_i)$ by (d) we can easily see that:

$M_i'^* = M_h'||\mu$, where $\mu \leq \mathrm{On}_{M_h'}$ is maximal such that $\tau_o' < \mu$ and τ_i' is a cardinal in $M_h'||\mu$.

(To see that this holds for $M_i'^* = M_h'$, we note that $\tau_i' = \sigma_h(\tau_i)$ is a cardinal in $M_h'||\rho_0^h$ and ρ_0^h is cardinally absolute in M_h'.)

We now complete the definition of *mirror*:

(e) Let $h = T(i+1), i+1 \leq_T i$, and assume that there is no drop point in $(i+1, j)_T$. Then:

(i) $\pi_{h,i}' : M_i'^* \rightarrow_{\Sigma^*} M_j'$.

(ii) $\kappa_i' = \mathrm{crit}(\pi_{hj}')$.

(iii) If $X \in \mathbb{P}(\kappa_i') \cap J_{\tau_i'}^{E^{M_i}}$, then $X \in M_i'^*$ and $E_{\nu_i'}^{M_i'}(X) = \lambda_i' \cap \pi_{h,j}'(X)$.

(iv) Set:

$$\hat{\rho}^i = \begin{cases} \rho^h \text{ if } M_i'^* = M_h' \\ \min(M_i'^*, \rho_h \restriction M_i^*, \langle \rho_{M_i'^*}^n | n < w \rangle) \text{ if not.} \end{cases}$$

Then:

$$\pi_{h,j}' \text{“} \hat{\rho}_M^i \subset \rho_n^j \leq \pi_{h,j}'(\hat{\rho}_n^i) \text{ for } n < w$$

(where $\pi_{hj}'(\hat{\rho}_n^i) =: \mathrm{On}\, M_j'$ if $\hat{\rho}_n^i = \mathrm{On}_{M_i'^*}$).

(Hence, if $h \leq_T j$ and $[h,j]_T$ has no drop point, then $\pi_{h,j}' \text{“} \rho_n^h \subset \rho_n^j \leq \pi_{h,j}'(\rho_n^h)$.)

This completes the definition.

Lemma 3.6.23. $J_{\lambda_i'}^{E^{M_i'}} = J_{\lambda_i'}^{E^{M_{i+1}'}}$ *for* $i+1 < \eta_i$.

Proof: λ_i' is an inaccessible cardinal in $J_{\nu_i}^{E^{M_i}}$. Hence there are arbitrarily large primitive recursive closed ordinals $\alpha < \lambda_i'$ and it suffices to show:

Claim $J_\alpha^{E^{M_i'}} = J_\alpha^{M_{i+1}'}$ for primitive recursive closed $\alpha < \lambda_i'$.

Proof: Let $h = T(i+1)$. Since $x \in J_\alpha^E$ is J_α^E–definable from parameters $\beta_1, \ldots, \beta_n < \alpha$, it suffices to show:

Subclaim Let $\beta_1, \ldots, \beta_n < \alpha$. Let φ be a first order formula. Then:

$$J_\alpha^{E^{M_i'}} \models \varphi[\vec{\beta}] \longleftrightarrow J_\alpha^{E^{M_{i+1}'}} \models \varphi[\vec{\beta}].$$

Proof: Set: $X = \{\prec \vec{\xi}, \zeta \succ < \kappa_i' | J_\zeta^{E^{M_i'}} \mod \varphi[\vec{\xi}]\}$. Then $X \in \mathbb{P}(\kappa_i') \cap J_{\nu_i'}^{E^{M_i'}} \subset M_i'^*$ by (e) (iii). But $J_{\kappa_i'}^{E^{M_i'}} = J_{\kappa_i'}^{E^{M_i'^*}} = J_{\kappa_i'}^{E^{M_h'}}$, by (e) (i), (ii). Then:

$$\bigwedge \vec{\xi}, \zeta < \kappa_i' (\prec \vec{\xi} \succ \in X \leftrightarrow J_\zeta^E \models \varphi[\vec{\xi}]),$$

which is a first order statement in $\langle J^E_{\kappa'_i}, X\rangle$, where $E = E^{M'^*_i}$. But then the same first order statement holds in $\langle \pi'(J^E_{\kappa'_i}), \pi'(X)\rangle$, where $\pi' = \pi'_{h,i+1}$. Clearly $\pi'(J^E_{\kappa'_0}) = J^{E^{M'_{i+1}}}_{\pi'(\kappa'_i)}$. Thus:

$$\pi'(X) = \{\prec \vec{\xi}, \zeta \succ < \pi(\kappa'_i) | J^{E^{M'_{i+1}}}_{\zeta} \models \varphi[\vec{\xi}]\},$$

and we have:

$$\begin{aligned} J^{E^{M'_{i+1}}}_{\alpha} \models \varphi[\vec{\beta}] \quad &\longleftrightarrow \prec \vec{\beta}, \alpha \succ \in \pi'(X) \\ &\longleftrightarrow \prec \vec{\beta}, \alpha \succ \in E^{M'_i}_{\nu'_i}(X) \text{ by (e) (iii)} \\ &\longleftrightarrow J^{E^{M'_i}}_{\alpha} \models \varphi[\vec{\beta}]. \end{aligned}$$

QED (Lemma 3.6.23)

We know that $\lambda'_i = E^{M'_i}_{\nu'_i}(\kappa'_i) \leq \pi'(\kappa'_i)$, where $h = T(i+1)$, $\pi' = \pi_{h,i+1}$ (by (e) (iii)). Set:

$$\lambda^*_i =: \pi'_{h,i+1}(\kappa'_i) \text{ where } h = T(i+1), \text{ for } i+1 < \eta.$$

Lemma 3.6.24. *Let $i+1 < \eta$. Then $\lambda'_i \leq \lambda^*_i = \sigma_j(\lambda_i)$ for $i < j < \eta$.*

Proof: $\lambda'_i \leq \lambda^*_i$ is trivial. But then:

$$\begin{aligned} &\sigma_{i+1}(\lambda_i) = \sigma_{i+1}\pi_{h,i+1}(\kappa_i) = \pi'_{h,i+1}\sigma_h(\kappa_i) \\ &= \pi'_{h,i+1}(\kappa'_i) = \lambda^*_i. \end{aligned}$$

Hence $\sigma_j(\lambda_i) = \sigma_{i+1}(\lambda_i)$ for $j > i$, since $\lambda_i < \lambda_{i+1}$. QED (Lemma 3.6.24)

Note. The main difference between a *mirror* of I and a simple *copy* of I in our earlier sense is that we can have: $\lambda'_i < \lambda^*_i$.

Corollary 3.6.25. *$\lambda'_i < \lambda'_j$ for $i < j$, $j+1 < \eta$.*

Proof: $\lambda'_i \leq \lambda^*_i = \sigma_j(\lambda_i) < \sigma_j(\lambda_j) = \lambda'_j$. QED (Corollary 3.6.25)

Corollary 3.6.26. *If $h = T(i+1), h+1 \leq_T j$, then $\kappa'_i < \lambda'_h \leq \lambda^*_h \leq \kappa'_j$ (since $\kappa_j \geq \lambda_h$).*

Lemma 3.6.27. *$J^{E^{M'_i}}_{\lambda'_i} = J^{E^{M'_j}}_{\lambda'_i}$ for $i \leq j < \eta$.*

Proof: By induction on j

Case 1 $j = i$ trivial.

Case 2 $j = l + 1$. Then it holds at l. But $J^{E^{M_l}}_{\lambda'_l} = J^{E^{M_j}}_{\lambda'_l}$ where $\lambda'_i \leq \lambda'_l$. The conclusion is immediate.

Case 3 $j = \mu$ is a limit ordinal.

By 3.6.26 we have: $\kappa'_i < \kappa'_j$ for $i+1 \leq_T j+1 \leq_T \mu$. Moreover $\sup \kappa'_i = \sup \lambda'_i$ by 3.6.26, 3.6.25. Pick an $l+1 \leq_T \mu$ such that $\kappa'_l > \lambda'_i$. Then $J^{E^{M'_l}}_{\kappa'_l} = J^{E^{M'_\mu}}_{\kappa'_l}$ by axiom e (i), (ii) and $J^{E^{M'_i}}_{\lambda'_i} = J^{E^{M'_l}}_{\lambda'_i}$, where $\lambda'_i < \kappa'_l$.

The conclusion is immediate. QED (Lemma 3.6.27)

Lemma 3.6.28. $J^{E^{M'_{i+1}}}_{\lambda^*_i} = J^{E^{M'_j}}_{\lambda^*_i}$ *for* $i < j < \eta$.

Proof: For $j = i+1$ it is trivial. For $j > i+1$, we have $\lambda'_{i+1} = \sigma_{i+1}(\lambda_{i+1}) > \sigma_{i+1}(\lambda_i) = \lambda^*_i$ and $J^{E^{M'_{i+1}}}_{\lambda'_{i+1}} = J^{E^{M'_j}}_{\lambda'_{i+1}}$. The conclusion is immediate. QED (Lemma 3.6.28)

Lemma 3.6.29. λ^*_i *is a limit cardinal in* M'_j *for all* $j > i$.

Proof: $\lambda^*_i = \sigma_j(\lambda_i)$ is a cardinal in M'_j, since λ_i is a cardinal in M_j. (This uses that ρ^j_0 is cardinally absolute if $\rho^i_0 < \mathrm{On}_{M'_i}$.) But then λ^*_i is cardinally absolute in M'_j and:

$$J^{E^{M'_i}}_{\lambda^*_i} \models \text{there are arbitrarily large cardinals,}$$

since the same is true in $J^{E^{M_i}}_{\lambda_i}$. QED (Lemma 3.6.29)

Lemma 3.6.30. λ'_i *is cardinally absolute in* M'_j *for* $j \geq i$.

Proof: Let α be a cardinal in $J^E_{\lambda'_i} = J^{E^{M'_i}}_{\lambda'_i} = J^{E^{M'_j}}_{\lambda'_i}$. Let $h = T(i+1)$ and let:

$$X = \{\xi < \kappa'_i) J^E_{\kappa'_i} \models \xi \text{ is a cardinal}\}.$$

Then: $\alpha \in E^{M'_{i+1}}_{\nu'_i}(X) \subset \pi'_{h,i+1}(X)$. Hence:

$$J^{E^{M'_{i+1}}}_{\lambda^*_i} \models \alpha \text{ is a cardinal.}$$

But $J^{E^{M'_{i+1}}}_{\lambda^*_i} = J^{E^{M'_j}}_{\lambda^*_i}$ and λ^*_i is cardinally absolute in M'_j.

QED (Lemma 3.6.30)

But there are arbitrarily large cardinals in the sense of $J^{E^{M'_i}}_{\lambda'_i}$. Hence:

Corollary 3.6.31. *λ'_i is a limit cardinal in M'_j for $i < j$.*

Lemma 3.6.32. *Let $h = T(i+1)$. Then $J^{E^{M'_h}}_{\tau'_i} = J^{E^{M'_i}}_{\tau'_i}$.*

Proof: For $h = i$ it is trivial. Let $h < i$. Then $J^{E^{M'_h}}_{\lambda'_h} = J^{E^{M'_i}}_{\lambda'_h}$, so we need only show that $\tau'_i < \lambda'_h$. But λ'_h is a limit cardinal in M'_i and $\kappa'_i < \tau'_i$. Hence in M'_i we have: $\tau'_i \leq {\kappa'}^+_i < \lambda'_h$. QED (Lemma 3.6.32)

Corollary 3.6.33. $\mathbb{P}(\kappa'_i) \cap {M'}^*_i = \mathbb{P}(\kappa'_i) \cap J^{E^{M'_i}}_{\nu'_i}$.

Proof: Since $\tau'_i > \kappa'_i$ is a cardinal in M'^*_i, we have by acceptability:

$$\begin{aligned}\mathbb{P}(\kappa'_i) \cap M'^*_i &= \mathbb{P}(\kappa'_i) \cap J^{E^{M'_h}}_{\tau'_i} = \mathbb{P}(\kappa'_i) \cap J^{E^{M'_i}}_{\tau'_i} \\ &= \mathbb{P}(\kappa'_i) \cap J^{E^{M'_h}}_{\nu'_i}\end{aligned}$$

QED(Corollary 3.6.33)

Lemma 3.6.34. *Let $h = T(i+1), F = E^{M_i}_{\nu_i}, F' = E^{M'_i}_{\nu'_i}$. Then*

$$\langle \sigma_h \restriction M^*_i, \sigma_i \restriction \lambda_i \rangle : \langle M^*_i, F \rangle \longrightarrow \langle M'^*_i, F' \rangle.$$

Proof. Clearly $(\sigma_h \restriction M^*_i) : M^*_i \longrightarrow_{\Sigma_0} M'^*_i$. Moreover, $\mathrm{rng}(\sigma_i \restriction \lambda_i) \subset \lambda'_i$. Now let $X \subset \kappa_i, X \in M^*_i, \alpha_i, \ldots, \alpha_n < \lambda_i$. Then:

$$\begin{aligned}&\prec \vec{\alpha} \succ \in F(X) = \pi_{h,i+1}(X) \\ &\longleftrightarrow \prec \sigma_{i+1}(\vec{\alpha}) \succ \in \sigma_{i+1}\pi_{h,i+1}(X) = \pi'_{h,j+1}\sigma_h(X) \\ &\longleftrightarrow \prec \sigma_i(\vec{\alpha}) \succ \in F'(\sigma_h(X)),\end{aligned}$$

since $\sigma_i \restriction \lambda_i = \sigma_{i+1} \restriction \lambda_i$ and $F'(\sigma_h(X)) = \lambda'_i \cap \pi'_{h,i+1}(\sigma_h(X))$.

QED(Lemma 3.6.34)

We also note:

Lemma 3.6.35. *Let $\lambda < \eta$ be a limit ordinal. Then for sufficiently large $i <_T \lambda$ we have:*

$$\rho^\lambda = \pi'_{i,\lambda}(p^i_n) \text{ for } n < \omega$$

Proof. Pick $\xi < \lambda$ such that $[\xi, \lambda)_T$ has no drop points. For each $n < \omega$ and each i, j such that $\xi \leq_T i \leq_T j \leq_T \lambda$ we have:

$$\pi'_{i,j} \text{``} \rho^i_n \subset \rho^j_n \leq \pi'_{ij}(\rho^i_n).$$

(1) For each $n < \omega$ there is $i_n \in [\xi, \lambda)_T$ such that:

$$\pi'_{i,j}(\rho^i_n) = \rho^i_n \text{ for } i_n \leq_T i \leq_T j <_T \lambda.$$

Proof. Suppose not. Then there exist $i_r (r < \omega)$ such that $\xi <_T i_r <_T i_{r+1}$ and $\rho^{i_{r+1}}_n < \pi'_{i_{r+1},\lambda}(\rho^{i_{r+1}}_n) < \pi'_{i_r,\lambda}(\rho^{i_r}_n)$. Hence: $\pi'_{i_{r+1},\lambda}(\rho^{i_{r+1}}_n) < \pi'_{i_r,\lambda}(\rho^i_n)$ for $r < \omega$. Contradiction!

QED(1)

(2) $\pi'_{i,\lambda}(\rho^i_n) = \rho^\lambda_n$ for $i_n \leq_T <_T \lambda$.

Proof. Since $M, \langle \pi'_{i,\lambda} : i_n \leq_T i <_T \lambda \rangle$ is a direct limit, we have:

$$\pi'_{i,\lambda}(\rho^i_n) = \bigcup_{i_n \leq_T i <_T \lambda} \pi'_{i,\lambda} “\rho^i_n \subset \rho^\lambda_n \leq \pi'_{i,\lambda}(\rho^i_n).$$

QED(2)

(3) If $\rho^\lambda_n = \rho^n_{M_\lambda}$ then $i_n = \xi$.

Proof. If not, there is $i \in [\xi, \lambda)_T$ such that $\rho^i_n < \rho^n_{M_i}$. Hence $\rho^\lambda_n \leq \pi'_{i,\lambda}(\rho^i_n) < \rho^n_{M_\lambda}$. Contradiction!

QED(3)

But then the set $\{n : i_n > \xi\}$ is finite. Set: $i = \max\{i_n : i_n > \xi\}$. This has the desired property.

QED(Lemma 3.6.35)

Corollary 3.6.36. *Let λ be a limit ordinal. Then*

$$\pi'_{i,\lambda} : M'_i \longrightarrow_{\Sigma^*} M'_\lambda \mod (\rho^i, \rho^\lambda)$$

for sufficiently large $i \leq_T \lambda$.

Proof. Let $i_0 \leq_T i <_T \lambda$ such that $\pi'_{i,\lambda}(\rho^i_n) = \rho^\lambda_n$ for $i_0 \leq_T i < \lambda, n < \omega$. By Lemma 3.6.3 we need only show:

(1) $\rho^i_n < \rho^n_{M_i} \longrightarrow \rho^\lambda_n = \pi'_{i,\lambda}(\rho^i_n)$

(2) $\rho^i_n = \rho^n_{M_i} \longrightarrow \rho^\lambda_n = \rho^n_{M_\lambda}$

(1) is immediate. To prove (2) we note:

$$\rho^\lambda_n = \pi'_{i,\lambda}(\rho^i_n) = \pi_{i,\lambda}(\rho^n_{M_i}) \geq \rho^n_{M_\lambda} \geq \rho^\lambda_n$$

QED Corollary 3.6.36

Definition 3.6.11. By a *mirror pair* of length η we mean a pair $\langle I, I'\rangle$ such that I is a normal iteration of length η and I' is a mirror of I.

It is natural to ask whether, and in what circumstances, a mirror pair of length η can be extended to one of length $\eta+1$. For limit η the answer is fairly straightforward:

Lemma 3.6.37. *Let $\langle I, I'\rangle$ be a mirror pair of limit length. Let b be a cofinal branch in $T = T_I$. Let the sequence:*

$$\langle M'_i : i \in b\rangle,\ \langle \pi'_{ij} : i \leq j \text{ in } b\rangle$$

have a well founded direct limit. Then $\langle I, I'\rangle$ extends uniquely to a mirror pair $\langle \hat{I}, \hat{I}'\rangle$ of length $\eta+1$ with $b = \hat{T}\text{"}\{\eta\}$ (where $\hat{T} = T_{\hat{I}}$).

Proof. Let $M'_\eta, \langle \pi'_{i,\eta} : i \in b\rangle$ be the transitivized direct limit.

Note. By our convention this means that for some $j_0 \in b$, $b \setminus j_0$ is drop free and:

$$\langle M'_i : i \in b \setminus j_0\rangle, \langle \pi'_{i,j} : j_0 \leq i \leq j \text{ in } b\rangle$$

in the usual sense, and we define:

$$\pi'_{i\eta} = \pi'_{j_0,\eta} \circ \pi'_{i,j_0} \text{ for } i < j_0 \text{ in } b$$

In the same sense the sequence:

$$\langle M_i : i \in b\rangle, \langle \pi_{i,j} : i \leq j \text{ in } b\rangle$$

has a transitivized limit:

$$M, \langle M_{i\eta} : i \in b\rangle$$

The maps $\pi_{i,\eta}$, $\pi'_{i,\eta}$ are easily seen to be Σ^*-preserving for $j_0 \leq i \in b$. We extend T to $\hat{T}$ by setting $\hat{T}\text{"}\{\eta\} = b$. We define the map $\sigma_\eta : M_\eta \longrightarrow M'_\eta$ by: $\sigma_\eta \pi_{i\eta} = \pi'_{i\eta}\sigma_i$ for $i < \eta$. We must then define a good sequence $\hat{\rho} = \rho^\eta$ for M'_η. We first imitate the proof of Lemma 3.6.35 by showing that there is $i_0 \in b$ such that $b \setminus i_0$ has no drop points and for all $j \in b \setminus i_0$:

$$\pi'_{i,j}(\rho^i_n) = \rho^j_n \text{ for } n < \omega$$

Thus, setting: $\hat{\rho}_n =: \pi'_{i_0,\eta}(\rho^{i_0}_n)$, we have:

$$\hat{\rho}_n = \pi'_{j,\eta}(\rho^j_n) \text{ for } n < \omega, i_0 \leq_T j \in b$$

It is easily shown that $\hat{\rho} = \langle \hat{\rho}_n : n < \omega\rangle$ is a good sequence for M'_η. Repeating the proof of Lemma 3.6.36 we then have:

(1) $\pi'_{j\eta} : M'_j \longrightarrow_{\Sigma^*} M'_\eta \mod (\rho^i, \hat{\rho})$ for $i_0 \leq_T j \leq_T \eta$.

Using this we show:

Claim 1. $\sigma_\eta : M_\eta \longrightarrow_{\Sigma^*} M'_\eta \mod \hat{\rho}$.

Proof. Let $x_1, \ldots, x_n \in M_\eta$. Then $\vec{x} = \pi_{i\eta}(\vec{z})$ for an $i \in [i_0, \eta)$. Hence for any $\Sigma_0^{(n)}$ formula:

$$\begin{aligned} M_\eta \models \varphi[\vec{x}] &\longleftrightarrow M_i \models \varphi[\vec{z}] \\ &\longleftrightarrow M'_i \models \varphi[\sigma_i(\vec{z})] \mod \rho^i \\ &\longleftrightarrow M'_i \models \varphi[\pi'_{i,\eta}\sigma_i(\vec{z})] \mod \hat{\rho} \end{aligned}$$

where $\pi'_{i,\eta}\sigma_i(\vec{z}) = \sigma_\eta \pi_{i,\eta}(\vec{z}) = \sigma_\eta(\vec{x})$.

QED(Claim 1)

We must also show:

Claim 2. $\sigma_\eta : M_\eta \longrightarrow_{\Sigma^*} M'_\eta \min \hat{\rho}$.

Proof. We must show:

$$\hat{\rho} = \min(M_\eta, \sigma_\eta, \tilde{\rho})$$

Let $\langle \hat{\rho}_l(n) : l < \omega \rangle$ be defined by induction on $n < \omega$ as in Definition 3.6.5. We must show: $\hat{\rho}_l = \bigcup_{n<\omega} \hat{\rho}_l(n)$. Let $\xi < \hat{\rho}_l$. Then $\xi = \pi'_{i,\eta}(\bar{\xi})$ where $i_0 \leq_T <_T \eta$ and $\bar{\eta} < \rho^i_l$. But $\rho^i_l = \bigcup_{n<\omega} \rho^i_l(n)$. Thus $\bar{\xi} < \rho^i_l(n)$ for some n. Using (1) and Definition 3.6.5, we easily get:

$$\pi'_{i,\eta}{}'' \rho^i_l(n) \subset \hat{\rho}_l(n) \text{ by induction on } n$$

But then $\xi = \pi'_{i,\eta}(\bar{\xi}) \in \hat{\rho}_l(n)$.

QED(Claim 2)

Using these facts it is easy to see that the extension $\langle \hat{I}, \hat{I}' \rangle$ we have defined satisfies the axiom (a)-(e) and is, therefore a mirror pair of length $\eta + 1$. (We leave the detail to the reader). The uniqueness of the maps $\pi_{i,\eta}, \pi'_{i,\eta}, \sigma_\eta$ is immediate from our construction. Finally, we must show that $\hat{\rho} = \rho^\eta$ is unique. This is because $\hat{\rho}_n = \pi'_{i_0,\lambda}(\rho^{i_0}_n)$ where $\pi'_{i_0,\lambda}$ is unique.

QED(Lemma 3.6.37)

We now ask how we can extend a mirror pair of length $\eta+1$ to one of length $\eta + 2$. This will turn out to be more complex.

If $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ is a normal iteration of length $\eta+1$, we can turn it into a *potential iteration* of length $\eta+2$ simply by appointing a ν_η such that $E^{M_\eta}_{\nu_\eta} \neq \varnothing$ and $\nu_\eta > \nu_i$ for $i < \eta$. This then determines $h = T(\eta+1)$ and M^*_η. (The notion of potential iteration was introduced in §3.4, where we gave a more formal definition). If $\langle I, I'\rangle$ is a mirror pair of length $\eta+1$, we can then form a *potential mirror pair* of length $\eta+2$ by appointing $\nu'_\eta =: \sigma_\eta(\nu_\eta)$. This determines M'^*_η. Our main lemma on "1-step extension" of mirror pair reads:

Lemma 3.6.38. *Let $\langle I, I'\rangle$ be a mirror pair of length $\eta+1$. Form a potential pair of length $\eta+2$ by appointing ν_η and $\nu'_\eta = \sigma_\eta(\nu_\eta)$. Let:*

$$\pi' : M'^*_\eta \longrightarrow_{\Sigma^*} M' \text{ such that } \kappa'_\eta = \mathrm{crit}(\pi')$$

and

$$E^{M'_\eta}_{\nu_\eta}(X) = \lambda'_\eta \cap \pi'(X) \text{ for } X \in \mathbb{P}(\kappa'_\eta) \cap J^{E^{M'_\eta}}_{\nu'_\eta}$$

Our potential pair then extends to a full mirror pair with:

$$M' = M'_{\eta+1},\ \pi' = \pi'_{h,\eta+1} \text{ where } h = T(\eta+1)$$

In order to prove this, we must first form a $*$-ultrapower:

$$\pi : M^*_\eta \longrightarrow^*_F M \text{ where } F = E^{M_\eta}_{\nu_\eta}$$

We must then define σ, ρ such that:

$$\pi'``\hat{\rho}_n \subset \rho_n \leq \pi'(\hat{\rho}_n) \text{ for } n < \omega$$

where $\hat{\rho}$ is defined as in axiom (e)(iv). If we then set:

$$M_{\eta+1} =: M, M'_{\eta+1} =: M', \pi_{h,\eta+1} =: \pi, \pi'_{h,\eta+1} =: \pi', \sigma_{\eta+1} = \sigma, \rho^{\eta+1} = \rho$$

we will have defined the desired extension. (We leave it to the reader to verify the axioms (a)-(e)). By the proof of Lemma 3.6.34 we have:

$$\langle \sigma_h \restriction M^*_\eta, \sigma_\eta \restriction \lambda_\eta\rangle : \langle M^*_i, F\rangle \longrightarrow \langle M^*_i, F'\rangle$$

where $F = E^{M_\eta}_{\nu_\eta}$, $F' = E^{M'_\eta}_{\nu'_\eta}$.

Lemma 3.6.19 then points us in the right direction. In order to get the full result, however, we must use Theorem 3.6.21 together with:

Lemma 3.6.39. *Let $\langle I, I'\rangle, \nu_\eta, \nu'_\eta, \pi'$ be as in Lemma 3.6.38. Set: $\xi = T(\eta+1), F = E^{M_\eta}_{\nu_\eta}$, $F' = E^{M'_\eta}_{\nu'_\eta}$. Set:*

$$\hat{\rho} = \begin{cases} \rho^\xi & \text{if } M'^*_\eta = M'_\xi \\ \min(M'^*_\eta, \sigma_h \restriction M'^*_\eta, \langle \rho^n_{M'^*_\eta} : n < \omega\rangle) & \text{if not} \end{cases}$$

Then:

$$\sigma_h \restriction M^*_h, \sigma_\eta \restriction \lambda_\eta : \langle M^*_\eta, F\rangle \longrightarrow^{**} \langle M'^*_\eta, F'\rangle \mod \hat{\rho}$$

We leave it to the reader to see that Theorem 3.6.21 and Lemma 3.6.39 give the desired result.

Note. It is clear that $\pi_{h,\eta+1}, \pi'_{h,\eta+1}, \sigma_{\eta+1}$ are uniquely determined by the choice of $\nu_\eta, \nu'_\eta, \pi'$. If we wished, we could use clause (c) of Theorem 3.6.21 to make $\rho^{\eta+1}$ unique.

We are actually in familiar territory here. The notion of mirror is clearly analogous to that of *copy* developed in §3.4.2. The analogue of mirror pair was there called a *duplication*. The role of Lemma 3.4.16 is now played by Lemma 3.6.38 and that of Theorem 3.4.16 by Lemma 3.6.39, which verifies the weaker principle $\longrightarrow^{**}$ in place of $\longrightarrow^{*}$ (which was, in turn, patterned on the proof of Theorem 3.4.3), which said that, if I is a potential normal iteration of length $\eta+2$, then $E_\eta^{M_\eta}$ is close to M_η^*).

We now turn to the proof of lemma 3.6.39. Just as in §3.4.2 we derive it from a stronger lemma. In order to formulate this properly we define:

Definition 3.6.12. Let M be acceptable. Let $\kappa \in M$ be inaccessible in M such that $\mathbb{P}(\kappa)\cap M \in M$. $A \subset \mathbb{P}(\kappa)\cap M$ is *strongly* $\Sigma_1(M)$ *in the parameter* p iff there is $B \subset M$ such that B is $\Sigma_0(M)$ and:

- $x \in A \longleftrightarrow \bigvee z B(z,x,p)$
- If $u \in M$ such that $u \subset \mathbb{P}(\kappa)$ and $\overline{\overline{u}}^M \leq \kappa$, then:

$$\bigvee v \in M \bigwedge X \in u \bigvee z \in v (B(z,X,p) \vee B(z,\kappa\setminus X,p))$$

We shall derive:

Lemma 3.6.40. *Let $\langle I, I'\rangle, \eta, \xi, \nu_\eta, \nu'_\eta, \pi'$ be as in Lemma 3.6.39. Let $A \subset \mathbb{P}(\kappa_\eta)$ be strongly $\Sigma_1(M_\eta||\nu_\eta)$ in p. Let $A' \subset \mathbb{P}(\kappa'_\eta)$ be $\Sigma_1(M'_\eta||\nu'_\eta)$ in $p' = \sigma_\eta(p)$ by the same definition. Then there is $q \in M^*_\eta$ such that*

- *A is strongly $\Sigma_1(M^*_\eta)$ in q.*
- *Let A'' be $\Sigma_1(M'^*_\eta)$ in $q' = \sigma_\xi(q)$ by the same definition. Then $A'' \subset A'$.*

Before proving this, we show that it implies Lemma 3.6.39:

Lemma 3.6.41. *Assume Lemma 3.6.40. Let ρ^* be good for M'^* and let:*

$$\sigma_\xi \restriction M^*_\eta : M^*_\eta \longrightarrow_{\Sigma^*} M'^*_\eta \quad \mathrm{mod}\ \rho^*.$$

Then:

$$\langle \sigma_\xi \restriction M^*_\eta, \sigma_\eta \restriction \lambda_\eta\rangle : \langle M^*_\eta, F\rangle \longrightarrow^{**} \langle M'^*_\eta, F'\rangle \quad \mathrm{mod}\ \rho^*.$$

Proof. Let $\alpha < \lambda_\eta, \alpha' = \sigma_\eta(\alpha)$. Then F_α is $\Sigma_1(J_{\nu_\eta}^{E^{M_\eta}})$ in α, since:

$$X \in F_\alpha \longleftrightarrow \bigvee Y(Y = F(X) \wedge \alpha \in Y)$$

We know, however, that if $u \in J_{\nu_\eta}^{E^{M_\eta}}, u \subset \mathbb{P}(\kappa)$, and $\overline{\overline{u}} \leq \kappa$ in $J_{\nu_\eta}^{E^{M_\eta}}$, then:

$$\bigvee v \in J_{\nu_\eta}^{E^{M_\eta}} \wedge X \in u \bigvee Y \in v(Y = F(X) \wedge (\alpha \in Y \vee \alpha \in (\kappa \setminus Y)))$$

Hence F_α is strongly $\Sigma_1(J_{\nu_\eta}^{E^{M_\eta}})$ in α. Obviously $F_{\alpha'}^{\alpha'}$ is $\Sigma_1(J_{\nu'_\eta}^{E^{M'_\eta}})$ in $\alpha' = \sigma_\eta(\alpha)$ by the same definition. Hence $\overline{G} = F_\alpha$ is strongly $\Sigma_1(M_\eta^*)$ in a parameter q. Moreover, if G' in $\Sigma_1({M'}_\eta^*)$ in $\sigma_\xi(q)$ by the same definition, then $G' \subset F'_{\alpha'}$. Now let G be $\Sigma_1({M'}_\eta^*, \rho^*)$ in $\sigma_\xi(q)$ by the same definition. Then $G \subset G' \subset F'_{\alpha'}$. Now let:

$$X \in \overline{G} \longleftrightarrow \bigvee z\overline{B}(z, X, q)$$

be the strongly $\Sigma_1(M_\eta^*)$-definition of G in q. Then:

$$X \in G \longleftrightarrow \bigvee zB(z, X, q')$$

where $q' = \sigma_\eta(q)$ and B is $\Sigma_0(M_\eta^*, \rho^*)$ by the same definition. (In other words, B is $\Sigma_0({M'}_\eta^*|\rho_0^*)$ by the same definition). Now let $\overline{H}$ be the set of $f \in M_\eta^* \cap {}^\kappa\mathbb{P}(\kappa)$ such that

$$\bigvee z \bigwedge i < \kappa(\overline{B}(z, f(i), q) \vee \overline{B}(z, \kappa \setminus f(i), q))$$

Then $\overline{H} = M_\eta^* \cap {}^\kappa\mathbb{P}(\kappa)$ by the strongness of our definition. But if H has the same $\Sigma_1(M_\eta^*, \rho^*)$ definition in q', then we obviously have:

$$f \in H \longrightarrow \bigwedge i < \kappa'(f(i) \in G \vee \kappa \setminus f(i) \in G)$$

QED(Lemma 3.6.41)

(In the application we, of course, take $\rho^* = \hat{p}$, where $\hat{p}$ is defined as in Lemma 3.6.39).

We now turn to the proof of Lemma 3.6.40. Suppose not. Let η be the least counterexample. We again have fixed ν_η and $\nu'_\eta = \sigma_\eta(\nu_\eta)$, which gives us $\kappa_\eta, \kappa'_\eta\tau_\eta, \tau'_\eta, \lambda_\eta, \lambda'_\eta, \xi = T(\eta + 1), M_\eta^*, {M'}_\eta^*$ and ρ^*.

(1) $\xi < \eta$.

Proof. Suppose not. Let $A \subset \mathbb{P}(\kappa)$ be strongly $\Sigma_1(M_\eta||\nu_\eta)$ in p and let $A' \subset \mathbb{P}(\kappa'_\eta)$ be $\Sigma_1(M'_\eta||\nu'_\eta)$ in $p' = \sigma_\eta(p)$ by the same definition.

Clearly τ_η is a cardinal in $M_\eta||\nu_1$, so $M_\eta^* = M_\eta||\mu$ for a $\mu \geq \nu_\eta$. Similarly $M_\eta'^* = M_\eta'||\mu'$ where:

$$\mu' = \begin{cases} \sigma_\eta(\mu) & \text{if } \mu \in M_\eta \\ \mathsf{ON} \cap M_\eta & \text{if not} \end{cases}$$

Now suppose $\nu_\eta \in M_\eta^*$ (i.e. $\mu > \nu_\eta$). Then $A \in M_\eta^*$ and $A' \in M_\eta'^*$ where $\sigma_\eta(A) = A'$. Then A is trivially strongly $\Sigma_1(M_\eta^*)$ in the parameter A and A' is $\Sigma_1(M_\eta^{*'})$ in $A' = \sigma_\eta(A)$ by the same definition, where $A' \subset A'$. Contradiction!

Now let $M_\eta^* = M_\eta||\nu_\eta$. Then $M_\eta'^* = M_\eta'||\nu_\eta'$ and A' is $\Sigma_1(M_\eta'^*)$ definable in $p' = \sigma_\eta(p)$ by the same definition. But A is strongly $\Sigma_1(M_\eta^*)$ in p, since $M_\eta^* = M_\eta|\nu_\eta$. Contradiction!

QED(1)

(2) $\nu_\eta = \mathsf{ON} \cap M_\eta$.

Proof. Suppose not. Then $\lambda_\xi > \tau_\eta$ is inaccessible in M_η. Hence $A \in J_{\lambda_\xi}^{E^{M_\eta}} = J_{\lambda_\xi}^{E^{M_\xi}} \subset M_\eta^*$. Similarly $A' \in J_{\lambda_\xi'}^{E^{M_\eta'}} = J_{\lambda_\xi'}^{E^{M_\xi'}} \subset M'^*_\eta|\rho_0^*$. Then A is strongly $\Sigma_1(M_\eta^*)$ in $A' = \sigma_\xi(A)$ by the same definition. Contradiction!

QED(2)

(3) $\tau_\eta \geq \rho_{M_\eta}^1$.

Proof. Suppose not. Then $\tau_\eta < \rho_{M_\eta}^1$. Hence $A \in J_{\rho_{M_\eta}^1}^{E^{M_\eta}}$ since $A \subset J_{\tau_\eta}^{E^{M_\eta}}$. Hence $A \in J_{\lambda_\xi}^{E^{M_\eta}} = J_{\lambda_\xi}^{E^{M_\xi}} \subset M_\eta^*$. Hence A is strongly $\Sigma_1(M_\eta^*)$ in the parameter A_r. Now let A'' be $\Sigma_1(M_\eta'|\rho_0^\eta)$ in $p' = \sigma_\eta(p)$ by the same definition. Then $A'' \subset A'$. But since

$$\sigma_\eta : M_\eta \longrightarrow_{\Sigma^*} M_\eta' \min(\rho^\eta),$$

we have: $A'' = \sigma_\eta(A)$. But λ_ξ'' is inaccessible in M_η'; hence $A'' \in J_{\lambda_\xi'}^{E^{M_\eta}} = J_{\lambda_\xi'}^{E^{M_\xi}} \subset M_\eta'^*$. Hence $A'' = \sigma_\xi(A)$ is $\Sigma_1(M_\eta'^*)$ in $A'' = \sigma_\xi(A)$ by the same definition. Contradiction!

QED(3)

(4) η is not a limit ordinal.

Proof. Suppose not. Pick $\overline{\eta} <_T \eta$ such that $\overline{\eta} = \mu + 1$. $\pi_{\overline{\eta}\eta}$ is total on $M_{\overline{\eta}}, \kappa = \mathrm{crit}(\pi_{\overline{\eta},\eta}) > \lambda_\eta$ and $p \in \mathrm{rng}(\pi_{\overline{\eta},\eta})$. Then $\pi'_{\overline{\eta},\eta}$ is total in $M'_{\overline{\eta}}$, $\kappa' = \mathrm{crit}(\pi'_{\overline{\eta},\eta}) > \lambda'_\eta$ and $p' \in \mathrm{rng}(\pi'_{\overline{\eta},\eta})$, where $p' = \sigma_\eta(p)$. Set $\overline{p} = \pi^{-1}_{\overline{\eta},\eta}(p), \overline{p}' = \pi^{-1}_{\overline{\eta},\eta}(p')$. Then $\sigma_{\overline{\eta}}(\overline{p}) = p$. Then $M_{\overline{\eta}} =$

$\langle J^{E^{M_{\overline{\eta}}}}_{\overline{\nu}}, \overline{F}\rangle, M'_{\overline{\eta}} = \langle J^{E^{M'_{\overline{\eta}}}}_{\overline{\nu}'}, \overline{F}\rangle$. Extend the mirror $\langle I|\overline{\eta}+1, I'|\overline{\eta}+1\rangle$ to a potential mirror $\langle \overline{I}, \overline{I}'\rangle$ of length $\overline{\eta}+2$, by setting: $\overline{\nu}_{\overline{\eta}} = \overline{\nu}, \overline{\nu}'_{\overline{\eta}} = \overline{\eta}'$. Then $\overline{M}^*_{\overline{\eta}} = M^*_\eta, \overline{M}'^*_{\overline{\eta}} = M'^*_{\overline{\eta}} = M'^*_\eta, \xi = \overline{T}(\overline{\eta}+1) = T(\eta+1)$ and $\sigma_\xi \upharpoonright M^*_{\overline{\eta}} : \overline{M}^*_{\overline{\eta}} \longrightarrow_{\Sigma^*} \overline{M}'^*_{\overline{\eta}} \min \rho^*$. It is easily seen that A is $\Sigma_1(M_{\overline{\eta}})$ in $\overline{p}'$ by the same definition. By the minimality of η we conclude that there is $q \in M^*_\eta = \overline{M}^*_{\overline{\eta}}$ such that A is strongly $\Sigma_1(M^*_\eta)$ in q and A is $\Sigma_1(M'^*_\eta)$ in $q' = \sigma_\xi(q)$ by the same definition. Contradiction!

QED(4)

Now let $\eta = \mu+1$. Let $\zeta = T(\mu+1)$. Then $\pi_{\zeta,\eta} : M^*_\mu \longrightarrow_{\Sigma^*} M_\eta$ and $\kappa_\mu = \text{crit}(\pi_{\zeta,\eta})$. Hence M^*_μ has the form $\overline{M} = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{F}\rangle$ where $\overline{F} \neq \varnothing$. Set: $\overline{\kappa} = \text{crit}(\overline{F}), \overline{\tau} = \tau(\overline{F}) =: \overline{\kappa}^{+\overline{M}}, \overline{\lambda} = \lambda(\overline{F}) =: \overline{F}(\overline{\kappa})$. Similarly M'^*_μ has the form $\overline{M}' = \langle J^{\overline{E}'}_{\overline{\nu}'}, \overline{F}'\rangle$ and we define $\overline{\kappa}', \overline{\tau}', \overline{\lambda}'$ accordingly.

Set: $\pi = \pi_{\zeta,\eta}, \pi' = \pi'_{\zeta,\eta}$.

(5) $\kappa_\mu > \overline{\kappa}$,

since otherwise $\kappa_\eta = \pi(\overline{\kappa}) \geq \pi(\kappa_\mu) = \lambda_\mu \geq \lambda_\xi > \kappa_\eta$. Contradiction!

QED(5)

But then $\kappa_\mu > \overline{\tau}$ and hence $\overline{\tau} = \tau_\eta, \overline{\kappa} = \kappa_\eta$. Similarly $\kappa'_\mu > \overline{\tau}'$ and $\overline{\tau}' = \tau'_\eta, \overline{\kappa}' = \kappa'_\eta$. But then:

(6) $\kappa_\mu > \rho^1_{\overline{M}}$,

since otherwise $\rho^1_{M_\eta} \geq \pi(\kappa_\mu) = \lambda_\mu > \tau_\eta$. Contradiction! by (3).

QED(6)

Hence, since $\pi : \overline{M} \longrightarrow^*_{E_{\nu_\mu}} M_\eta$, we have:

(7) $\pi : \overline{M} \longrightarrow_{E_{\nu_\mu}} : M_\eta$ is a Σ_0 ultraproduct and $\rho^1_{\overline{M}} = \rho^1_{M_\eta}$.

Recall that A is strongly $\Sigma_1(M_\eta)$ in p and A' is $\Sigma_1(M'_\eta)$ in $p' = \sigma_\eta(p)$ by the same definition. By (7) we know:

(8) $p = \pi(f)(\alpha)$ where $\alpha < \lambda_\mu, f \in \overline{M}$ and $f : \kappa_\mu \longrightarrow \overline{M}$. Hence

(9) $p' = \pi'(f')(\alpha')$ where $f' = \sigma_f(f), \alpha' = \sigma_\mu(\alpha)$.

Proof. $p' = \sigma_\eta(\pi(f)(\alpha)) = (\sigma_\eta\pi(f))(\sigma_\eta(\alpha)) = (\pi'\sigma_\zeta(f))(\sigma_\mu(\alpha))$.

QED(9)

Note. $\sigma_\mu \upharpoonright \lambda_\mu = \sigma_\eta \upharpoonright \lambda_\mu$ since $\mu < \eta$.

Let A be strongly $\Sigma_1(M_\eta)$ in p as witnessed by $\bigvee zB(z, X, p)$, where B is $\Sigma_0(M_\eta)$. Set:

$$B_0(u, X, p) \longleftrightarrow: \bigvee z \in uB(z, X, p).$$

Then A is strongly $\Sigma_1(M_\eta)$ in p as witnessed by $\bigvee uB_0(u,X,p)$. Note that for all u,u':

(10) $(B_0(u,X,p)\wedge u\subset u')\longrightarrow B_0(u',X,p)$.

Let B_1 be $\Sigma_0(\overline{M})$ by the same definition as B_0 over M_η. Set $\tilde{F}=:E^{M_\mu}_{\nu_\mu},\tilde{F}'=E^{M'_\mu}_{\nu'_\mu}$. By the cofinality of the map $\overline{p}:\overline{M}\longrightarrow M_\eta$ and (10) we have:

(11)
$$\begin{aligned} AX &\longleftrightarrow \bigvee u\in\overline{M}B_0(\pi(u),X,p)\\ &\longleftrightarrow \bigvee u\in\overline{M}\{\gamma<\kappa_\mu : B_\gamma(u,X,f(\gamma))\}\in\tilde{F}_\alpha. \end{aligned}$$

But $\tilde{F}_\alpha$ is strongly $\Sigma_1(M_\mu||\nu_\mu)$ in α and $\tilde{F}'_{\alpha'}$ is $\Sigma_1(M'_\mu||\nu'_\mu)$ in α' by the same definition.

Hence by the minimality of η we conclude:

(12) There is $q\in\overline{M}$ such that the following hold:

(a) $G=\tilde{F}_\alpha$ is strongly $\Sigma_1(\overline{M})$ in q.

(b) Let G' be $\Sigma_1(\overline{M}')$ in $q'=\sigma_\gamma(q)$ by the same definition. Then $G'\subset\tilde{F}'_{\alpha'}$, where $\alpha'=\sigma_\mu(\alpha)$.

Let: $\bigvee zG_0(z,X,q)$ witness the fact that G is strongly $\Sigma_1(\overline{M})$ in q. Then:

$$\begin{aligned} AX &\longleftrightarrow \bigvee u\in\overline{M}B_0(\pi(u),X,\pi(f)(\alpha))\\ &\longleftrightarrow \bigvee u\in\overline{M}\{\gamma<\kappa_\mu : B_1(u,X,f(\gamma))\}\in G\\ &\longleftrightarrow \bigvee v\in\overline{M}\bigvee u\in v\bigvee\in v\bigvee z\in v\\ &\quad (Y=\{\gamma<\kappa_\mu : B_1(u,X,f(\gamma))\}\wedge G_0(z,Y,q)) \end{aligned}$$

This has the form:

(13) $AX\longleftrightarrow\bigvee vB_2(v,X,r)$, where $r=\langle q,f\rangle$ and B_2 is $\Sigma_0(\overline{M})$.

For this B_2 we claim:

(14) A is strongly $\Sigma_1(\overline{M})$ in r are witnessed by $\bigvee B_2(v,X,r)$.

Proof. Let $w\subset\mathbb{P}(\overline{\kappa})\cap\overline{M},\overline{\overline{w}}<\overline{\kappa}$ in $\overline{M}$.

Claim. There is $v\in\overline{M}$ such that

$$\bigwedge X\in w(B_2(v,X,r)\wedge B_2(v,\overline{\kappa}\setminus X,r))$$

For the sake of simplicity we can assume without lose of generality that $X \in w \longleftrightarrow (\overline{\kappa} \setminus M) \in \omega$. Fix $u \in \overline{M}$ such that

$$\bigwedge X \in w(B_0(\pi(u), X, p) \wedge B_0(\pi(u), (\overline{\kappa} \setminus X), p))$$

For $X \in w$ set:

$$\theta(X) =: \{\gamma < \kappa_\mu : B_1(u, X, f(\gamma))\}$$

Then:

$$\bigwedge x \in w(\theta(X) \in G \vee \theta(\overline{\kappa} \setminus X) \in G)$$

By rudimentary closure, $\langle \theta(X) : X \in w \rangle \in \overline{M}$. Hence θ“$w \in \overline{M}$ and $\mathrm{card}(\theta$“$w) \leq \overline{\kappa} < \kappa_\mu$ in $\overline{M}$. Thus there is $z \in \overline{M}$ such that:

$$\bigwedge X \in w(G_0(z, \theta(X), q) \vee G_0(z, \kappa_\mu \setminus \theta(X), q))$$

Claim. $\bigwedge X \in w(G_0(z, \theta(X), q) \vee G_0(z, \theta(\overline{\kappa} \setminus X), q))$.

Proof. Suppose not. Then there is $X \in w$ such that:

$$\kappa_\mu \setminus \theta(X),\ \kappa_\mu \setminus \theta(\overline{\kappa} \setminus X) \in G = \tilde{F}_\alpha.$$

Hence $\neg B_0(\pi(u), X, p)$ and $\neg B_0(\pi(u), \overline{\kappa} \setminus X, p)$. Contradiction!

QED(Claim)

Pick $V \in \overline{M}$ such that $u \in v, z \in v$ and θ”$w \subset v$. Then:

$$\bigwedge X \in w(B_2(v, X, r) \vee B_2(v, \overline{\kappa} \setminus X), r)$$

QED(14)

(15) Let A'' be $\Sigma(\overline{M})$ in $r' = \sigma_\zeta(r)$ by the same definition. Then $A'' \subset A'$.

Proof. Let B_0' be $\Sigma_0(M')$ by the same definition as B_0 over M. Let B_1' be $\Sigma_0(\overline{M})$ by the same definition. $A''X$ says that there is $u \in \overline{M}$ with:

$$\{\gamma < \kappa_\mu' : B_1'(u, X, f'(\gamma))\} \in G'$$

where $f' = \sigma_\zeta(f)$. But $G' \subset \tilde{F}_{\alpha'}$. Hence $B_0'(\pi(u), X, \pi'(f')(\alpha'))$, where $p' = \pi'(f')(\alpha')$. Hence $A'X$.

QED(15)

Now extend $\langle I|\zeta{+}1, I'(\zeta{+}1)\rangle$ to a potential mirror pair $\langle \hat{I}, \hat{I}'\rangle$ of length $\zeta + 2$ by setting: $\nu_\zeta = \overline{\nu}, \nu_\zeta' = \overline{\nu}'$. Since $\overline{\kappa} = \kappa_\eta$, $\overline{\tau} = \tau_\eta$, we have:

$$\xi = \hat{T}(\zeta + 1), \hat{M}_\zeta^* = M_\eta^*, \hat{M}_\zeta'^* = M_\eta'^*$$

But $\zeta \leq \mu < \eta$. By the minimality of η and by (14), (15), we conclude that there is a parameter $s \in M_\eta^*$ such that:

- A is strongly $\Sigma_1(M^*_\eta)$ in s.
- If A''' has the same $\Sigma_1(M'^*_\eta)$ definition in $s'(\sigma_\xi(s))$, then $A''' \subset A''$ (hence $A''' \subset A'$).

This contradicts the fact that η was a counterexample.

QED(Lemma 3.6.40)

The argumentation used in the proof of Lemma 3.6.35, Lemma 3.6.36 and Lemma 3.6.37 actually establishes a more abstract result which is useful in other contexts:

Lemma 3.6.42. *Assume that M_i, M'_i are amenable for $i < \mu$, where μ is a limit ordinal. Assume further that:*

(a) $\pi_{i,j} : M_i \longrightarrow_{\Sigma^*} M_j$ $(i \leq j < \mu)$, *where the* $\pi_{i,j}$ *commute.*

(b) $\pi'_{i,j} : M'_i \longrightarrow_{\Sigma^*} M'_j$ $(i \leq j < \mu)$, *where the* $\pi'_{i,j}$ *commute. Moreover:*

$$\langle M'_i : i < \mu \rangle, \langle \pi'_{i,j} : i \leq j < \mu \rangle$$

has a transitivized direct limit $M', \langle \pi'_i : i < \mu \rangle$.

(c) $\sigma_i : M'_i \longrightarrow_{\Sigma^*} M'_j \min \rho^i$ $(i \leq j < \mu)$.

(d) $\pi'_{i,j}``\rho^i_n \subset \rho^j_n \leq \pi'_{i,j}(\rho^i_n)$ *for* $i \leq j < \mu, n < \omega$. *Then*

$$\langle M_i : i < \mu \rangle, \langle \pi_{i,j} : i \leq j < \mu \rangle$$

has a transitivized direct limit $M, \langle \pi_i : i < \mu \rangle$. *There is then* $\sigma : M \longrightarrow M'$ *defined by:* $\sigma\pi_i = \pi'_i\sigma_i$ $(i < \mu)$. *Moreover:*

(1) *There is a unique* ρ *such that* $\sigma : M \longrightarrow_{\Sigma^*} M' \min \rho$ *and:*

$$\pi'_i``\rho^i_n \subset \rho_n \leq \pi'_i(\rho^i_n) \text{ for } i < \mu, n < \omega.$$

(2) *There is* $i < \mu$ *such that* $\rho_n = \pi'_j(\rho^j_n)$ *for* $i \leq j < \mu$, $n < \omega$.

3.6.4 The conclusion

In this section we show that every smoothly iterable premouse is fully iterable. We first define some auxiliary concepts:

Definition 3.6.13. Let $\langle I, I'\rangle$ be a mirror pair of length η with:

$$I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle \text{ and } I' = \langle\langle M_i'\rangle, \langle\pi_{ij}'\rangle, \langle\sigma_i\rangle, \langle\rho^i\rangle\rangle$$

Let N be a premouse such that $M_0' = N||\mu$ for some $\mu \leq \mathsf{ON}_N$. As usual set: $\nu_i' = \sigma_i(\nu_i)$. Let:

$$I'' = \langle\langle N_i\rangle, \langle\nu_i''\rangle, \langle\pi_{ij}''\rangle, T\rangle$$

be an iteration on N of length η. (T being the same as in I). Set:

$$\mu_i = \begin{cases} \pi_{0j}''(\mu) & \text{if } \mu \in \mathrm{dom}(\pi_{0j}'') \\ \mathsf{ON}_{N_i} & \text{if not.} \end{cases}$$

We say that the mirror pair $\langle I, I'\rangle$ is *backed* by I'' (or *M-backed by* I'') iff:

$$M_i' = N_i||\mu_i, \nu_i' = \nu_i'', \pi_{ij}' = \pi_{ij}'' \restriction M_i' \text{ for } i \leq_T j < \eta.$$

Now suppose that $\langle I, I'\rangle$ is a mirror pair of length $\eta + 1$ backed by I''. Extend I to a potential iteration I^+ of length $\eta + 2$ by appointing ν_η such that $E_{\nu_\eta}^{M_\eta} \neq \varnothing$ and $\nu_\eta > \nu_i$ for $i < \eta$. This determines $\zeta = T(\eta + 1)$ and M_η^*. If we then set: $\nu_\eta' = \sigma_\eta(\nu_\eta)$, we have determined $M_\eta^{'*}$ and turned $\langle I, I'\rangle$ into a potential mirror pair $\langle I^+, I^{'+}\rangle$. But ν_η' also extends I'' to a potential iteration $I^{''+}$ of length $\eta + 2$, determining N_η^*. We then say that $I^{''+}$ *potentially backs* $\langle I^+, I^{'+}\rangle$.

Note that if $M_\eta^* \in M_\xi$, then:

$$M_\eta^{'*} = \sigma_\xi(M_\eta^*) = N_\eta^*.$$

If, however, $M_\eta^* = M_\xi$, then we have $M_\eta^{'*} = M_\xi'$, but if is still possible that $M_\eta^{'*} \in N_\eta^*$ and even that $N_\eta^* \in N_\xi$. This can happen if $M_\xi' = N_\xi||\mu_\xi$ and $\mu_\xi \in N_\xi$. There might then be $\gamma > \mu_\xi$ such that τ_η' is a cardinal in $N_\xi||\gamma$. Hence $M_\eta^{'*} = M_\xi' \in N_\xi'||\gamma \subset N_\eta^*$. But if the largest such γ is an element of N_ξ, we then have $N_\eta^* \in N_\xi$.

Note. If $I^+, I^{'+}, I^{''+}$ are as above, we certainly have: $E_{\nu_\eta'}^{M_\eta'} = E_{\nu_\eta'}^{N_\eta}$.

Using Lemma 3.6.38 we can then prove:

Lemma 3.6.43. *Let $I^+, I^{'+}, I^{''+}$ be as above. Suppose that N_η^* is $*$-extendible by $F' = E_{\nu_\eta'}^{N_\eta}$. Then $\langle I^+, I^{'+}\rangle$ extends to an actual mirror pair $\langle \hat{I}, \hat{I}'\rangle$ with $\hat{\nu}_\eta = \nu_\eta$ and $I^{''+}$ extends to an iteration $\hat{I}''$ which backs $\langle \hat{I}, \hat{I}'\rangle$.*

Proof. Set $\pi'' : N_\eta^* \longrightarrow_{F'}^* N'$. Then I''^{+} extends uniquely to $\hat{I}''$ with: $N_{\eta+1} = N', \pi''_{\xi,\eta+1} = \pi''$.

Set: $\pi' =: \pi'' \upharpoonright M'^{+}_\eta$. Then:

$$\pi' : M'^{*}_\eta \longrightarrow_{\Sigma^*} M'$$

where:

$$M' = \begin{cases} \pi''(M'^{*}_\eta) & \text{if } M'^{*}_\eta \in N_\eta^* \\ M' & \text{if not} \end{cases}$$

Then $\mathrm{crit}(\pi') = \kappa'_\nu$ and $F' = E^{M'_\eta}_{\nu'_\eta}$. Hence by Lemma 3.6.38, $\langle I, I' \rangle$ extends to a mirror $\langle \hat{I}, \hat{I}' \rangle$ of length $\eta + 2$ with: $M' = M'_{\eta+2}$. Obviously, $\hat{I}''$ backs $\langle \hat{I}, \hat{I}' \rangle$.

QED(Lemma 3.6.43)

Note. If $M'^{*}_\eta \in N_\eta^*$, then $\langle \pi', M' \rangle$ is not necessarily an ultraproduct of $\langle M'^{*}_\eta, F' \rangle$.

Using Lemma 3.6.37 we also get:

Lemma 3.6.44. *Let $\langle I, I' \rangle$ be a mirror pair of limit length η which is backed by I''. Let b be a well founded cofinal branch in I''. Then $\langle I, I' \rangle$ extend uniquely to $\langle \hat{I}, \hat{I}' \rangle$ of length $\eta + 1$ such that $b = \hat{T}``\{\eta\}$. Moreover I'' extends uniquely to $\hat{I}''$ which backs $\langle \hat{I}, \hat{I}' \rangle$.*

The proof is straightforward and is left to the reader.

But by the same lemmata we get:

Lemma 3.6.45. *Suppose that N is normally iterable. Let $M = N||\mu$. Then M is normally α-iterable.*

Proof. Fix a successful iteration strategy S for N. We must define a strategy S^* for M. Let:

$$I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$$

be an iteration of M of length η. We first note:

Claim. There is at most one pair $\langle I', I'' \rangle$ such that $\langle I, I' \rangle$ is a mirror pair backed by I'' and I'' is S-conforming.

Proof. By induction on $\mathrm{lh}(I)$. We leave this to the reader.

We now define an iteration strategy S^* for M. Let I be a normal iteration of M of limit length η. If there is no pair $\langle I', I''\rangle$ satisfying the above claim, then $S^*(I)$ is undefined. If not, we set:

$$S^*(I) =: S(I'')$$

$b = S^*(I)$ is then a cofinal well founded branch is I. (Clearly, if we extend each of I, I', I'' by the branch b, we obtain $\langle \tilde{I}, \tilde{I}', \tilde{I}''\rangle$ satisfying the Claim). It is then obvious that if I is of length $\eta + 1$ and we pick $\nu > \nu_i (i < \eta)$ such that $E_\nu^{M_\eta} \neq \varnothing$, then I extends to an S^*-conforming iteration of length $\eta+1$. Hence S^* is successful.

QED(Lemma 3.6.45)

This is fairly weak result which could have been obtained more cheaply. We now show, however, that our methods establish Theorem 3.6.1. We begin by defining the notion of a *full mirror* I' of a full iteration I.

Definition 3.6.14. Let $I = \langle I^i : i < \mu\rangle$ be a full iteration of M, inducing M_i, π_{ij} $(i \leq j < \mu)$. Let:

$$I^i = \langle\langle M_h^i\rangle, \langle \nu_h^i\rangle, \langle \pi_{hj}\rangle, T^i\rangle$$

By a *full mirror* of I we mean $I' = \langle I'^i : i < \mu\rangle$ such that

$$I'^i = \langle\langle M_h'^i\rangle, \langle \pi_{hj}'^i\rangle, \langle \sigma_h^i\rangle, \langle \rho^{i,h}\rangle\rangle$$

is a mirror of I^i for $i < \mu$, and I' induces $\langle M_i' : i < \mu\rangle, \langle \pi_{ij}' : i \leq j < \mu\rangle, \langle \sigma_i : i < \mu\rangle, \langle \rho^i : i < \mu\rangle$ such that:

(a) $\sigma_i : M_i \longrightarrow_{\Sigma^*} M_i' \min \rho^i$

(b) π_{ij}' is a partial structure preserving map from M_i' to M_j'. Moreover, they commute and $\pi_{i,i}' = \mathrm{id} \restriction M_i'$. If $\alpha < \mu$ is a limit ordinal, then $M_\alpha' = \bigcup_{i<\alpha} \mathrm{rng}(\pi_{i,\alpha}')$.

(c) $\sigma_j \pi_{ij} = \pi_{ij}' \sigma_i$ for $i \leq j < \mu$.

(d) If $i \leq j < \mu$ and $[i, j)$ has no drop point in I, then:

$$\pi_{ij}' : M_i' \longrightarrow_{\Sigma^*} M_j' \text{ and } \pi_{ij}' \text{“}\rho^i \subset \rho^i \leq \pi_{ij}'(\rho^i)$$

(e) $M_0' = M_0 = M; \sigma_0 = \mathrm{id} \restriction M$, and

$$\rho^0 = \langle \rho_M^n : n < \omega\rangle$$

(f) $M_{i+1}' = M_{l_i}'^i$ where I^i has length $l_i + 1$. Moreover, $\sigma_{i+1} = \sigma_{l_i}^i$ and $\rho^{i+1} = \rho^{i,l_i}$ and $\pi_{i,i+1} = \pi_{i,l_i}^i$.

We leave it to a reader to see that $\langle M_i : i < \mu\rangle, \langle\pi'_{ij} : i \leq j < \mu\rangle, \langle\sigma_i : i < \mu\rangle$ are uniquely characterized by (a)-(f), given the triple $\langle M, I, I'\rangle$. In particular if $\alpha < \mu$ is a limit ordinal, then:

$$M'_\alpha, \langle\pi'_{i\alpha} : i < \alpha\rangle$$

is the transitivized direct limit of

$$\langle M'_i : i < \alpha\rangle, \langle\pi'_{ij} : i \leq j < \alpha\rangle.$$

(This makes sense by (d), since I has only finitely drop points $i < \alpha$). σ_α is then defined by: $\sigma_\alpha\pi_{i\alpha} = \pi'_{i\alpha}\sigma_i$. By the method of §3.6.2 it follows that there is only one ρ^α satisfying our conditions and that, in fact, for sufficiently large $i < \alpha$ we have:

$$\rho^\alpha_n = \pi'_{i\alpha}(\rho^i_n) \text{ for } i < \omega.$$

$\langle I, I'\rangle$ is then called a *full mirror pair.*

We leave to the reader to verify:

Lemma 3.6.46. *Let $\langle I, I'\rangle$ be a full mirror pair of limit length μ. Suppose further, that, if $[i_o, \mu)$ has no drop point, then:*

$$\langle M'_i : i_0 \leq i < \mu\rangle, \langle\pi'_{ij} : i_0 \leq i \leq j < \mu\rangle$$

has a well founded limit. Then $\langle I, I'\rangle$ extends uniquely to a mirror pair of length $\mu + 1$.

We recall that a full iteration $I = \langle I^i : i < \mu\rangle$ is called *smooth* iff $M_i = M^i_0$ for all $i < \mu$. We define:

Definition 3.6.15. Let $I = \langle I^i : i < \mu\rangle$ be a full iteration of M. Let $\langle I, I'\rangle$ be a full mirror pair. Let:

$$I'' = \langle I''^i : i < \mu\rangle$$

be a smooth iteration of M inducing

$$\langle M''_i : i < \mu\rangle, \langle\pi''0_{ij} : i \leq j < \mu\rangle$$

such that $M'^i_0 \lhd M'_i \lhd M''_i$ and I''^i backs $\langle I^i, I'^i\rangle$ for $i < \mu$.

We then say that I'' *backs* $\langle M, I, I'\rangle$.

It is obvious that, if I'' backs $\langle M, I, I'\rangle$ then I'' is uniquely determined by $\langle M, I, I'\rangle$. Building on the last lemma we get:

Lemma 3.6.47. *Let $\langle I, I' \rangle$ be a full mirror pair of limit length μ. Let I'' be a smooth iteration of M of length $\mu + 1$, such that $I''|\mu$ backs $\langle M, I, I' \rangle$. Then $\langle I, I' \rangle$ extends uniquely to a pair of length $\mu + 1$ which is backed by I''.*

Proof. (Sketch). The extension is easily defined using Lemma 3.6.46 if we can show:

Claim. I has finitely many drop points.

We first note that if I^i has a truncation on the main branch, then so do I'^i and I''^i. Hence there are only finitely many such I^i. Now suppose that $M_0^i \neq M_i$ for infinitely many i. Let $\langle i_n : n < \omega \rangle$ be a monotone sequence of such i such that $[i_n, i_{n+1})$ has no drop. Then, letting $M_i' = M''_{i_n}||\mu_n$ for $n < \omega$, we have: $\mu_{n+1} < \pi''_{i_n, i_{n+1}}(\mu_n)$.

Hence $\pi''_{i_{n+1},\mu}(\mu_{n+1}) < \pi''_{i_n,\mu}(\mu_n)$. Contradiction!

QED(Lemma 3.6.47)

Now let S be a successful smooth iteration strategy for M. (Thus S is defined only on smooth iterations $I = \langle I^i : i \leq \eta \rangle$ such that I^η is a normal iteration of limit length. $S(I)$, if defined, is then a well founded cofinal branch b in I^η. We call S *successful* for M iff every S-conforming smooth iteration I of M can be extended in an M-conforming manner. (This is defined precisely in §3.5.2).).

Claim. Let I be a full iteration of M. There is at most one pair $\langle I', I'' \rangle$ such that $\langle I, I' \rangle$ is a full mirror pair, I'' backs $\langle I, I' \rangle$ and is S-conforming.

Proof. By induction on $\text{lh}(I)$ and for $\text{lh}(I) = i + 1$ by induction on $\text{lh}(I^i)$. The details are left to the reader.

We now define a full iteration of length $i + 1$ where I^i is of limit length. If there exist $\langle I', I'' \rangle$ as in the above claim, we set $S^*(I) = S(I'')$. If not, then $S^*(I)$ is undefined. It follows as before that an S^*-conforming full iteration of M can be properly extended in any permissible way to an S^*-conforming iteration. More precisely:

- If I is of length $i + 1$ and I^i is of limit length, then $S^*(I)$ exists.
- If I is of length $i + 1$ and I^i is of successor length $j + 1$ and $\nu > \nu_h^i$ for $h < j$, where $E_\nu^{M_\nu^i} \neq \varnothing$, then I extends to and S^*-conforming $\hat{I}$, $\hat{I}_i$ extends I^i and $\nu_j = \nu$ in $\hat{I}^i$.
- If I, i, j are as before and $\tilde{M} \lhd M_j^i$, then I extends to an S^*-conforming $\hat{I}$ of length $i + 1$ such that $\tilde{M} = M_0^{i+1}$.

- If I is of limit length μ, then it extends uniquely to an S^*-conforming iteration of length $\mu + 1$.

QED(Theorem 3.6.1)

3.7 Smooth Iterability

In this section we prove Theorem 3.7.29. This will require a deep excursion into the combinatorics of normal iteration, using methods which were manly developed by John Steel and Farmer Schluzenberg. We first answer a somewhat easier question: Let M be uniquely normally iterable and let M' be a normal iterate of M. Is M' normally iterable? Our basis tool in dealing with this is the *reiteration*: Given a normal iteration I' from M' to M'', we "reiterate" I, gradually turning it into a normal iteration I^* to an M^*. The process of reiteration mimics the iteration I'. This results in an embedding σ from M'' to M^*, thus showing that M'' is well-founded. However, σ is not necessarily Σ^*-preserving but rather Σ^*-preserving *modulo pseudoprojecta*. This means that, in order to finish the argument, we must draw on the theory of pesudoprojecta developed in §3.6. The above result is proven in §3.7.3. The path from this result to Lemma 3.7.29 is still arduous, however. It is mainly due to Schluzenberg and employs his original and surprising notion of "inflation". In order to complete the argument (in §3.7.6) we again need recourse to pseudo projecta. The remaining subsections (§3.7.1, §3.7.2, §3.7.4, §3.7.5) can be read with no knowledge of pseudoprojecta, and are of some interest in their own right.

We begin by describing a class of operations on normal iteration called *insertions*. An insertion embeds or "inserts" a normal iteration into another one.

3.7.1 Insertions

Let I be a normal iteration of M of length η. Let I' be a normal iteration of the same M having length η'. An *insertion* of I into I' is a monotone function $e : \eta \longrightarrow \eta'$ such that $E^{M_i}_{\nu_i}$ plays the same role in M_i as $E^{M'_{e(i)}}_{\nu'_{\tilde{e}(i)}}$ in $M'_{\tilde{e}(i)}$. (This is far from exact, of course, but we will shortly give a proper definition).

In one form or other, insertions have long played a role in set theory. They are implicit in the observation that iterating a single normal measure produces