- If I is of limit length μ, then it extends uniquely to an S^*-conforming iteration of length $\mu + 1$.

QED(Theorem 3.6.1)

3.7 Smooth Iterability

In this section we prove Theorem 3.7.29. This will require a deep excursion into the combinatorics of normal iteration, using methods which were manly developed by John Steel and Farmer Schluzenberg. We first answer a somewhat easier question: Let M be uniquely normally iterable and let M' be a normal iterate of M. Is M' normally iterable? Our basis tool in dealing with this is the *reiteration*: Given a normal iteration I' from M' to M'', we "reiterate" I, gradually turning it into a normal iteration I^* to an M^*. The process of reiteration mimics the iteration I'. This results in an embedding σ from M'' to M^*, thus showing that M'' is well-founded. However, σ is not necessarily Σ^*-preserving but rather Σ^*-preserving *modulo pseudoprojecta*. This means that, in order to finish the argument, we must draw on the theory of pesudoprojecta developed in §3.6. The above result is proven in §3.7.3. The path from this result to Lemma 3.7.29 is still arduous, however. It is mainly due to Schluzenberg and employs his original and surprising notion of "inflation". In order to complete the argument (in §3.7.6) we again need recourse to pseudo projecta. The remaining subsections (§3.7.1, §3.7.2, §3.7.4, §3.7.5) can be read with no knowledge of pseudoprojecta, and are of some interest in their own right.

We begin by describing a class of operations on normal iteration called *insertions*. An insertion embeds or "inserts" a normal iteration into another one.

3.7.1 Insertions

Let I be a normal iteration of M of length η. Let I' be a normal iteration of the same M having length η'. An *insertion* of I into I' is a monotone function $e : \eta \longrightarrow \eta'$ such that $E^{M_i}_{\nu_i}$ plays the same role in M_i as $E^{M'_{e(i)}}_{\nu'_{\tilde{e}(i)}}$ in $M'_{\tilde{e}(i)}$. (This is far from exact, of course, but we will shortly give a proper definition).

In one form or other, insertions have long played a role in set theory. They are implicit in the observation that iterating a single normal measure produces a sequence of indiscernibles. This situation typically arises when we have a transitive ZFC^- model M and a $\kappa \in M$ which is measurable in M with a normal ultrafilter $U \in M$. Assume that we can iterate M by U, getting:

$$M_i, \kappa_i, U_i, \pi_{i,j} : M_i \prec M_j \ \ (i \leq j < \infty),$$

where the maps $\pi_{i,j}$ are commutative and continuous at limits, $\kappa_i = \pi_{0i}(\kappa), U_i = \pi_{0i}(U)$ and:

$$\pi_{i,i+1} : M_i \longrightarrow_{U_i} M_{i+1}$$

Now let $e : \eta \longrightarrow \infty$ be any monotone function on an ordinal η. e is then an *insertion*, inducing a sequence $\langle \sigma_i : i < \eta \rangle$ of *insertion maps* such that $\sigma_i : M_i \prec M_{e(i)}$. To define there maps we first introduce an auxiliary function $\hat{e}$ defined by:

$$\hat{e}(i) =: \inf\{e(h) : h < i\}$$

Thus $\hat{e}$ is a normal function and $\hat{e}(0) = 0$.

By induction on $i < \eta$ we then define maps $\hat{\sigma}_i, \sigma_i$ as follows: We verify inductively that:

$$\hat{\sigma}_i : M_i \prec M_{\hat{e}(i)} \text{ and } \hat{\sigma}_i \bar{\pi}_{h_i} = \pi_{\hat{e}(h),\hat{e}(i)} \hat{\sigma}_h$$

Since $\hat{e}(0) = 0$, we set: $\hat{\sigma}_0 = \mathrm{id} \restriction M$. If σ_i is given, we know that $\hat{e}(i) \leq e(i)$ and hence define: $\tilde{\sigma}_i = \pi_{\hat{e}(i),e(i)} \hat{\sigma}_i$. Now let $i+1 < \eta$. Then $\hat{e}(i+1) = e(i)+1$. We know that each element of M_{i+1} has the form $\pi_{i,i+1}(f)(\kappa_i)$. Hence we can define $\hat{\sigma}_{i+1}$ by:

$$\hat{\sigma}_{i+1}(\pi_{i,i+1}(f)(\kappa_i)) = \pi_{e(i),\hat{e}(i+1)}(\sigma_i(f))(\sigma_i(\kappa_i)).$$

Finally, if $\lambda < \eta$ is a limit, then $\hat{e}(\lambda) = \mathrm{lub}\{e(i) : i < \lambda\}$, and we can define $\hat{\sigma}_\lambda$ by:

$$\hat{\sigma}_\lambda \pi_{h\lambda} = \pi_{\hat{e}(h),\hat{e}(\lambda)} \hat{\sigma}_h \text{ for } h < \lambda$$

This completes the construction. The fact that $\langle u_h : h < i \rangle$ is a sequence of indiscernibles for M_i is proven by using insertions defined on finite η.

This was a simple example, but insertions continue to play a role in the far more complex theory of mouse iterations. We define the appropriate notion of insertion as follows:

Let:

$$I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$$

be a normal iteration of M of length η. Let

$$I' = \langle \langle M'_i \rangle, \langle \nu'_i \rangle, \langle \pi'_{ij} \rangle, T' \rangle$$

be a normal iteration of the same M of length η'. Suppose that

$$e : \eta \longrightarrow \eta'$$

is monotone. Define an auxiliary function $\hat{e}$ by:

$$\hat{e}(i) =: \text{lub}\{e(h) : h < i\} \text{ for } i < \eta$$

Then $\hat{e}$ is a normal function and $\hat{e}(0) = 0$. We call e an *insertion* of I into I' iff there is a sequence $\langle \hat{\sigma}_i : i < \eta \rangle$ of *insertion maps* with the following properties:

(a) $\hat{\sigma}_i : M_i \longrightarrow_{\Sigma^*} M_{\hat{e}(i)}, \hat{\sigma}_0 = \text{id}$.

(b) $i \leq_T j \longleftrightarrow \hat{e}(i) \leq_{T'} \hat{e}(j)$. Moreover:

$$\hat{\sigma}_j \pi_{ij} = \pi'_{\hat{e}(i),\hat{e}(j)} \circ \hat{\sigma}_i, \text{ for } i \leq_T j.$$

(c) $\hat{e}(i) \leq_{T'} e(i)$ for $i < \eta'$.

Before continuing the definition, we introduce some notation. Set:

$$\pi_i = \pi'_{\hat{e}(i),e(i)},\ \sigma_i = \pi_i \hat{\sigma}_i \text{ for } i < \eta$$

We further require

(d) $\sigma_i(\nu_i) = \nu'_{e(i)}$. More precisely, one of the following holds:

- $\nu_i \in M_i \wedge \hat{\sigma}_i(\nu_i) \in \text{dom}(\hat{\pi}_i) \wedge \nu'_{e(i)} = \sigma_i(\nu_i)$
- $\nu_i \in M_i \wedge \text{dom}(\hat{\pi}_i) = M'_{\hat{e}(i)} || \hat{\sigma}_i(\nu_i) \wedge \nu'_{e(i)} = \mathsf{ON} \cap M'_{e(i)}$
- $\nu_i = \mathsf{ON} \cap M_i \wedge \text{dom}(\hat{\pi}_i) = M'_{e(i)} \wedge \nu'_{e(i)} = \mathsf{ON} \cap M'_{e(i)}$

(e) $\hat{\sigma}_i \restriction \lambda_l = \sigma_l \restriction \lambda_l$ for $l < i < \eta$.

This completes the definition.

Note. The insertion maps $\hat{\sigma}_i, \sigma_i$ are uniquely determined by e, but we have yet to prove this fact.

Note. The map $\hat{\sigma}_i$ is total on M_i, but σ_i could be partial.

Note. $e, \hat{e}$ are order preserving, and $\hat{e}$ takes $<_T$ to $<_{T'}$. On the other hand, $i <_T j$ does not imply $e_i <_T e_j$, although we have:

$$i <_T j \longrightarrow \hat{e}_i <_{T'} e_j \text{ and } e_i <_{T'} e_j \longrightarrow i <_T j.$$

Definition 3.7.1. The *identical insertion* is $\text{id} \restriction \eta$, with $\hat{\sigma}_i = \sigma_i = \text{id} \restriction M_i$ for $i < \eta$.

We shall sometimes write $e_i, \hat{e}_i$ for $e(i), \hat{e}(i)$.

Note. We use here the familiar abbreviation:

$$\kappa_i = \mathrm{crit}(E^{M_i}_{\nu_i}), \lambda_i = E^{M_i}_{\nu_i}(\kappa_i), \tau_i = \kappa_i^{+J^{E^{M_i}}_{\nu_i}}$$

for $i < \eta$. Similarly $\kappa'_i, \lambda'_i, \tau'_i$ for $i < \eta'_i$.

Note. By (e) we have:

$$h < i \leftrightarrow \hat{\sigma}_i \upharpoonright J^{E^{M_i}}_{\lambda_n} = \sigma_h \upharpoonright J^{E_{M_n}}_{\lambda_n}.$$

To see this, let:

$$J^E_\lambda = J^{E_{M_n}}_{\lambda_n} = J^{E_{M_i}}_{\lambda_n} \text{ (since } h < i).$$

Similarly let:

$$J^{E'}_{\lambda'} = J^{E_{M'_{e_n}}}_{\lambda'_{e_n}} = J^{E_{M'_{\hat{e}_i}}}_{\lambda'_{e_n}} \text{ (since } e_h < \hat{e}_i).$$

Let $x \in J^E_\lambda$. Then there is a limit ordinal $\alpha < \lambda$ and a $\beta < \alpha$ such that:

$$x = \text{ the } \beta\text{-th element of } J^E_\lambda \text{ in } <^E_\alpha,$$

where $<^E_\beta$ is the canonical well ordering of J^E_α. Let $\hat{\sigma}_i(\alpha) = \sigma_h(\alpha) = \alpha'$, $\hat{\sigma}_i(\beta) = \sigma(\beta) = \beta'$. Then:

$$\hat{\sigma}_i(x) = \sigma_h(x) = \text{ the } \beta'\text{-th element of } J^{E'}_{\alpha'} \text{ is } <^{E'}_{\alpha'}.$$

Lemma 3.7.1. *The following hold:*

(1) $\sigma_i \upharpoonright \lambda_h = \sigma_h \upharpoonright \lambda_h$ for $h \leq i \leq \eta$.
(Hence $\sigma_i \upharpoonright J^{E^{M_i}}_{\lambda_h} = \sigma_h \upharpoonright J^{E^{M_h}}_{\lambda_h}$).

Proof.

$$\mathrm{crit}(\pi_i) \geq \lambda'_{e_h} = \sigma_h(\lambda_h) \subset \sigma_h``\lambda_h$$

Hence $\sigma_h \upharpoonright \lambda_h = \pi_i \hat{\sigma}_i \upharpoonright \lambda_h = \pi_i \sigma_h \upharpoonright \lambda_h = \sigma_h \upharpoonright \lambda_h$. QED(1)

(2) *Let $\xi = T(i+1)$. Then $\kappa'_{e_i} < \lambda'_{e_\xi}$.*

Proof. $\kappa'_{e_i} = \sigma_i(\kappa_i) = \sigma_\xi(\kappa_i) < \sigma_\xi(\lambda_\xi) = \lambda'_{e_\xi}$. QED(2)

(3) *Let $\xi = T(i+1), \xi' = T'(e_i + 1)$. Then $\hat{e}_\xi \leq_{T'} \xi' \leq e_\xi$.*

Proof. $\xi' \leq e_\xi$ by (2). But $\hat{e}_\xi <_{T'} \hat{e}_{i+1} = e_i + 1$. Hence $\hat{e}_\xi \leq_{T'} \xi'$.
QED(3)

The full determination of $T'(e_i + 1)$ is as follows:

(4) *Let $\xi = T(i+1)$. Let j be the least such that*

$$\hat{e}_\xi \leq_{T'} j \leq_{T'} e_\xi \text{ and } \pi'_{j,e_\xi} \restriction \kappa'_{e_i} = \mathrm{id}\,.$$

Then $j = T'(e_i + 1)$.

Proof.

Claim. $\kappa'_{e_i} < \lambda'_j$.

Suppose not. Then $j < e_\xi$, since $\kappa'_{e_i} < \lambda'_{e_\xi}$. Set: $\kappa = \mathrm{crit}(\pi'_{j,e_\xi})$. Then $\kappa'_{e_i} < \kappa$, since otherwise:

$$\pi'_{j,e_\xi}(\kappa'_{e_i}) \geq \pi'_{j,e_\xi}(\kappa) > \kappa \geq \kappa'_{e_i} \geq \lambda'_j$$

But $\kappa < \lambda'_j$. Contradiction!

Claim. $\kappa'_{e_i} \geq \lambda_h$ for $h < j$.

If $j = \hat{e}_\xi$, then $j = T(e_i + 1)$ by (3) and Claim 1. The conclusion is then obvious. Now let $j > \hat{e}_\xi$. Then $j = \mathrm{lub}\, A$, where:

$$A = \{h : \hat{e}_\xi <_{T'} h + 1 \leq_{T'} j\}$$

Hence it suffices to show:

Claim. $\kappa'_{e_i} \geq \lambda'_h$ for $h \in A$.

Suppose not, Let $h \in A$ be the least counterexample. Let $\tau = T'(h+1)$. Then $\hat{e}_\xi \leq_{T'} \tau$. Hence

$$\mathrm{rng}(\pi'_{\hat{e}_\xi,h+1}) \subset \mathrm{rng}(\pi'_{\tau,h+1})$$

But:

$$\kappa'_{e_i} \in \mathrm{rng}(\sigma_i) \subset \mathrm{rng}(\pi'_{h+1,e_i})$$

where $\kappa'_{e_i} < \lambda'_h \leq \mathrm{crit}(\pi'_{h+1,e_i})$. Hence $\kappa'_{e_i} < \mathrm{crit}(\pi'_{h+1,e_i})$. Thus

$$\kappa'_{e_i} \in \mathrm{rng}(\pi'_{\hat{e}_i,h+1}) \subset \mathrm{rng}(\pi'_{\tau,h+1}).$$

But then $\kappa'_{e_i} \notin [\kappa'_h, \lambda'_h)$, since:

$$\mathrm{rng}(\pi'_{\tau,h+1}) \cap [\kappa'_h, \lambda'_h) = \varnothing$$

Since $\kappa'_{e_i} \leq \lambda'_h$, we conclude that $\kappa'_{e_i} < \kappa'_h$. Hence $\pi'_{\tau,e_\xi} \restriction (\kappa'_{e_i} + 1) = \mathrm{id}$. This is a contradiction, since $\tau < j < e_\xi < e_\xi$.

QED(4)

Definition 3.7.2. Let $\xi = T(i+1)$. We set:

$$e^*_i = T'(e_i + 1),\ \pi^*_i = \pi'_{\hat{e}_\xi, e^*_i},\ \sigma^*_i = \pi^*_i \hat{\sigma}_\xi$$

The following are then obvious:

(5) $M'^*_{e_i} = M'_{e^*_i}||\mu$, where μ is maximal such that τ'_{e_i} is a cardinal in $M'_{e^*_i}||\mu$.

(6) $\sigma^*_i \restriction M^*_i : M^*_i \longrightarrow_{\Sigma^*} M'^*_{e_i}$.

Note. If $M^*_i = M_\xi$, then τ_i is a cardinal in M_ξ. Hence $\hat{\sigma}_\xi(\tau_i)$ is a cardinal in $M'_{\hat{e}_\xi}$ and $\tau'_{e_i} = \pi^*_i \hat{\sigma}_\xi(\tau_i)$ is a cardinal in $M'_{e^*_i} = M'^*_{e_i}$. If $M^*_i \in M_\xi$, then $\hat{\sigma}_\xi(M^*_i) \in M'_{\hat{e}_\xi}$ and $\pi^*_i \restriction \hat{\sigma}_\xi(M^*_i) : \hat{\sigma}_\xi(M^*_i) \longrightarrow_{\Sigma^*} M'^*_{e_i}$. (However, we cannot conclude that $M'^*_{e_i} \in M'_{e_i}$). Hence:

(7) Let $\xi = T(i+1)$. $\pi_{\xi,i+1}$ is a total function on M_ξ iff $\pi'_{\hat{e}_\xi,e_{i+1}}$ is total on $M'_{\hat{e}_\xi}$.

Hence, there is a drop point in $(\alpha,\beta]_T$ iff there is a drop point in $(\hat{e}_\alpha, e_\beta]_{T'}$.

(8) $\hat{\sigma}_{i+1}\pi_{\xi,i+1} = \pi'_{e^*_i,e_i+1}\sigma^*_i$, where $\xi = T(i+1)$.

Proof. $\hat{\sigma}_{i+1}\pi_{\xi,i+1} = \pi'_{\hat{e}_\xi,\hat{e}_{i+1}}\hat{\sigma}_\xi = \pi'_{e^*_i,e_{i+1}}\pi^*_i\hat{\sigma}_\xi = \pi_{e^*i,e_{i+1}}\sigma^*{}_i$. QED(8)

(9) $\sigma_i(X) = \sigma^*_i(X)$ *for* $X \in \mathbb{P}(\kappa_i) \cap M^*_i$.

Proof.$\sigma_i(X) = \sigma_\xi(X)$ where $\xi = T(i+1)$, since $X \in J^{E^{M_\eta}}_{\lambda_\xi}$ and $\sigma_i \restriction \lambda_\xi = \sigma_\xi \restriction \lambda_\xi$ by (1). But $\sigma_\xi(X) = \pi'_{\hat{e}_\xi,e_\xi}\hat{\sigma}_\xi(X) = \pi'_{e^*_i,e_\xi}\sigma^*_i(X)$, since $\pi'_{e^*_i e_\xi} \restriction \kappa_{e_i} + 1 = \mathrm{id}$.

QED(9)

Using notation from §3.2, then we have:

(10) $\langle \sigma^*_i \restriction M^*_i, \sigma_i \restriction \lambda_i \rangle : \langle M^*_i, F \rangle \longrightarrow \langle M'^*_{e_i}, F' \rangle$ where $F = E^{M_i}_{\nu_i}, F' = E^{M'_{e_i}}_{\nu_{e_i}}$.

Proof. $\alpha \in F(X) \longleftrightarrow \sigma_i(\alpha) \in \sigma_i(F(X)) = F'(\sigma^*_i(X))$ by (6) and (9).

QED(10)

But we are now, at last, in a position to prove:

(11) *The sequence* $\langle \hat{\sigma}_i : i < \eta \rangle$ *of insertion maps is uniquely determined by* e. *(Hence so is* $\langle \sigma_i : i < \eta \rangle$, *since* $\sigma_i = \pi'_{\hat{e}_i,e_i} \circ \hat{\sigma}_i$*).*

Proof. Suppose not. Let $\langle \hat{\sigma}'_i : i < \eta \rangle$ be a second such sequence. By induction on i we prove that $\hat{\sigma}_i = \sigma'_i$. For $i = 0$ this is immediate. Now let $\hat{\sigma}_i = \sigma'_i$. We must show that $\hat{\sigma}_{i+1}$ is unique. Let $n \leq \omega$ be maximal such that $\kappa_i < \rho^n_{M_i}$. By Lemma 3.2.19 of §3.2, we know that there is at most one σ such that

$$\sigma : M_i \longrightarrow_{\Sigma^{(n)}_0} M'_{e_i},\ \sigma\pi_{\xi,i+1} = \pi'_{e^*_i \hat{e}_{i+1}}\sigma^*_i,\ \sigma \restriction \lambda_i = \sigma_i \restriction \lambda_i$$

Hence $\hat{\sigma}_{i+1} = \sigma'_{i+1} = \sigma$ by (8).

Now let $\mu < \eta$ be a limit ordinal. Then $\hat{\sigma}_\mu = \sigma'_\mu$ is the unique $\sigma : M_\mu \longrightarrow M'_{\hat{e}_\mu}$ defined by: $\sigma\pi_{i,\mu} = \pi'_{\hat{e}_i,\hat{e}_\mu}\hat{\sigma}_i$ for $i <_{T'} \mu$.

QED(11)

We also note:

(12) *Let $\xi = T(i+1)$. Then $\pi'_{e_i^*,e_\xi} \upharpoonright (\tau'_i + 1) = \mathrm{id}$.*

(Hence $\sigma_i^* \upharpoonright (\tau_i + 1) = \sigma_\xi \upharpoonright (\tau_i + 1) = \sigma_i \upharpoonright (\tau_i + 1)$.

Proof. If $e_i^* = e_\xi$, this is immediate. Now let $e_i^* < e_\xi$. Set $\pi' = \pi'_{e_i^*,e_\xi}$. Then $\kappa'_{e_i} < \tilde{\kappa} = \mathrm{crit}(\pi')$ where $\tilde{\kappa}$ is inaccessible in M'_{e_ξ}. Hence $\tau'_{e_i+1} < \tilde{\kappa}$, since $\tau'_{e_i} = (\kappa'_{e_i})^+$ in M'_{e_ξ}. QED(12)

(13) $\hat{\sigma}_{i+1}(\nu_i) = \nu'_{e_i}$.

Proof. Let $\xi = T(i+1)$. Then:

$$\begin{aligned}\hat{\sigma}_{i+1}(\nu_i) = \hat{\sigma}_{i+1}\pi_{\xi,i+1}(\tau_i) &= \pi'_{e_i^*,e_i+1}\sigma_i^*(\tau_i)\\ &= \pi'_{e_i^*,e_i+1}(\tau'_{e_i}) = \nu'_{e_i}\end{aligned}$$

since $\tau'_{e_i^*} = \sigma_i(\tau_i) = \sigma_i^*(\tau_i)$. QED(13)

Hence:

(14) $j \geq i+1 \longrightarrow \sigma_j(\nu_i) \geq \nu'_{e_i}$.

Proof. By (13) it holds for $j = i+1$. Now let $j > i+1$. Then $\kappa_i < \lambda_{i+1}$ and

$$\hat{\sigma}_j(\nu_j) = \sigma_{i+1}(\nu_i) \geq \sigma_i(\nu_i) = \nu'_{e_i}.$$

QED(14)

We also note:

(15) $e_i <_{T'} e_j \longrightarrow i \leq_T j$.

Proof. Since $e_i < \hat{e}_j$ and $\hat{e}_j \leq_T e_j$, we conclude:

$$\hat{e}_i \leq_{T'} e_i <_{T'} \hat{e}_j; \text{ hence } i <_T j.$$

QED(15)

Extending insertion

Given an insertion e of I into I', when can we turn it into an e' which inserts an extension $\tilde{I}$ of I into an extension $\tilde{I}'$ of I'? Some things are obvious:

(16) *If e inserts I into I' and I'' extends I', then e inserts I into I''.*

(17) *If e inserts I of length $\nu+1$ into I' and $e(\nu) \leq_{T'} j$ in I', there is a unique e' inserting I into I' such that $e' \restriction \nu = e \restriction \nu$ and $e'(\nu) = j$.*

(18) *Let I be of limit length ν and let e insert I into I' of length $\nu' = \mathrm{lub}\, e``\nu$. Suppose that $b'$ is a cofinal well founded branch in $I'$ and $b = e^{-1}``b'$ is cofinal in I. Extend I' into $\tilde{I}$ of length $\eta+1$ by setting $T``\{\eta\} = b$. Extend $I'$ to $\hat{I}'$ of length $\eta'+1$ by: $T'``\{\eta\} = b'$. Then e extends uniquely to an insertion $\tilde{e}$ of $\tilde{I}$ into $\tilde{I}'$ with $\tilde{e}(\eta) = \eta'$.*

The proof is left to the reader.

These facts are obvious. The following lemma seems equally obvious, but its proof is rather arduous:

Lemma 3.7.2. *Let e insert I into I' where I is of length η and I' is of length $\eta'+1$, where $\eta' = e(\eta)$. Extend I to a potential iteration of length $\eta+2$ by appointing ν_η such that $\nu_\eta > \nu_i$ for $i < \eta$. Suppose $\sigma_\eta(\nu_\eta) > \nu'_j$ for all $j < \eta'$. Then we can extend I' to a potential iteration of length $\eta'+2$ by appointing: $\nu'_{\eta'} = \sigma_\eta(\nu_\eta)$. This determines $\xi = T(\eta+1)$, $e^*_\eta = T'(\eta'+1)$ and $M^*_i, M'^*_{e_i}$. If M'_{e_i} is $*$-extendible by $F = E^{M_i}_{\nu_i}$, then e extends uniquely to an $\tilde{e}$ inserting $\tilde{I}$ into $\tilde{I}'$, where $\tilde{I}'$ is an actual extension of I by ν_η and $\tilde{I}'$ is an actual extension of I' by $\nu'_{\eta'}$.*

Using Lemma 3.2.23 of §3.2 we can derive Lemma 3.7.2 from:

Lemma 3.7.3. *Let $e, I, I', \nu_\eta, \nu_{\tilde{e}_\eta}, M^*_\eta, M'^*_{\tilde{e}_i}, F, F'$ be as above. Then*

$$\langle \sigma^*_\eta, \sigma_\eta \restriction \lambda_\eta \rangle : \langle M^*_\eta, F \rangle \longrightarrow^* \langle M'^*_{\tilde{e}_\eta}, F' \rangle$$

We first show that Lemma 3.7.3 implies Lemma 3.7.2. Since $M'^*_{e_\eta}$ is $*$-extendible by F' we can extend I' by setting:

$$\hat{\pi}'_{e^*_\eta, e_\eta+1} : M'^*_{\sigma e_\eta} \longrightarrow^*_{F'} M'_{e_\eta+1}$$

It follows that F is close to M^*_i; hence we can set:

$$\hat{\pi}_{\xi,\eta+1} : M^*_\eta \longrightarrow^* M_{\eta+1}$$

But by Lemma 3.2.23 there us a unique

$$\sigma : M_{\eta+1} \longrightarrow_{\Sigma^*} M_{\tilde{e}_\eta+1}$$

such that $\sigma\pi_{\xi,\eta+1} = \pi'_{e^*_\eta, \tilde{e}_\eta+1}\sigma^*_\eta$ and $\sigma \restriction \lambda_\eta = \sigma_\eta \restriction \lambda_\eta$. Extend e to $\tilde{e}$ by: $\tilde{e}(\eta+1) = e_\eta + 1$. The $\tilde{e}$ satisfies the insertion axioms with $\sigma_{\eta+1} = \sigma$.

QED(Lemma 3.7.2)

We derive Lemma 3.7.3 from an even stronger lemma:

Lemma 3.7.4. *Let I, I' be as above. Let $A \subset I_\eta$ be $\Sigma_1(M_\eta||\nu_\eta)$ in a parameter p and let $A' \subset \tau'_{e_\eta}$ be $\Sigma_1(M_{e_\eta}||\nu'_{e_\eta})$ in $p' = \sigma_\eta(p)$ by the same definition. Then A is $\Sigma_1(M^*_\eta)$ in a parameter q and A' is $\Sigma_1(M'^*_{e_\eta})$ in $q' = \sigma^*_\eta(q)$ by the same definition.*

We first show that this implies Lemma 3.7.3. Repeating the proof of Lemma 3.7.1(7), we have:

$$\langle \sigma^*_\eta \upharpoonright M^*_\eta, \tilde{\sigma}_\eta \upharpoonright \lambda_\eta \rangle : \langle M^*_\eta, F \rangle \longrightarrow \langle M'^*_{e_\eta}, F' \rangle$$

where $F = E^{M_\eta}_{\nu_\eta}, F' = E^{M'_{e_\eta}}_{\nu'_{e_\eta}}$.

We can code F_α by an $\tilde{F} \subset \tau_\eta$ such that F_α is rudimentary in $\tilde{F}$ and $\tilde{F}$ is $\Sigma_i(M_\eta||\nu_\eta)$ in α, τ_η. Coding $F'_{\alpha'}$ the same way by $\tilde{F}'$, we find that $\tilde{F}'$ is $\Sigma_1(M_{e_\eta}|\nu_{e_\eta})$ in α', τ'_{e_η} by the same definition, where $\sigma_\eta(\alpha) = \alpha', \sigma_\eta(\tau_\eta) = \tau'_{e_\eta}$. Hence by Lemma 3.7.4, $\tilde{F}'$ is $\Sigma_1(M'^*_\eta)$ in a q and $\tilde{F}'$ is $\Sigma_1(M'^*_{e_\eta})$ in $q' = \sigma^*_\eta(q)$ by the same definition. Hence F_α is $\Sigma_1(M'^*_\eta)$ in q and $F'_{\alpha'}$ is $\Sigma_1(M'^*_{e_\eta})$ in $q' = \sigma^*_\eta(q)$ by the same definition.

QED(Lemma 3.7.3)

Note. We are in virtually the same situation as in §3.2, where we needed to prove the extendability of the triples we called *duplications*. Lemma 3.7.2 corresponds to the earlier Lemma 3.4.17 and Lemma 3.7.4 corresponds to Lemma 3.4.20.

We now turn to the proof of Lemma 3.7.4. Its proof will be patterned on that of Lemma 3.4.20, which, in turns, we patterned on the proof of Lemma 3.4.4.

Our proof will be rather fuller than that of Lemma 3.4.20, however, since we will face some new challengers.

Suppose Lemma 3.7.4 to be false. Let I, I' be a counterexample with $\eta = \mathrm{lh}(I)$ chosen minimally. We derive a contradiction. Let $\xi = T(\eta + 1)$.

(1) $\rho^1_{M_\eta||\nu_\eta} \leq \tau_\eta$

Proof. Suppose not. Set $\rho = \rho^1_{M_\eta||\nu_\eta}, \rho' = \rho^1_{M'_{e_\eta}||\nu'_{e_\eta}}$. Then $A \in J^{E^{M_\eta}}_\rho, A' \in J^{E^{M'_{e_\eta}}}_{\rho'}$.

Moreover, "$x = A'$" is $\Sigma^{(1)}_0(M'_\eta||\nu')$ in p, τ_η and "$x = A'$" is $\Sigma^{(1)}_0(M_\eta||\nu_\eta)$ in p', τ'_{e_η} by the same definition. Hence $\sigma_\eta(A) = A'$. Since $A \in J^{E^{M_\eta}}_{\lambda_\xi}$,

$\sigma_\eta \upharpoonright \lambda_\xi = \sigma_\xi \upharpoonright \lambda_\xi$ and $M_\xi||\lambda_\xi = M_\xi||\lambda_\xi$, we have: $\sigma_\xi(A) = \sigma_\eta(A) = A'$. But $\sigma_\eta(A) = \pi'_{e^*_\eta,e_\xi}\sigma^*_\eta(A)$ where $\pi'_{e^*_\eta,e_\eta} \upharpoonright \tau'_{e_\eta} + 1 = \mathrm{id}$ by (10). Hence $\sigma^*_\eta(A) = A'$. Hence A is $\Sigma_1(M'^*_\eta)$ in the parameter A, and A' is $\Sigma_1(M'^*_{e_\eta})$ in the parameter $A' = \sigma^*_\eta(A)$ by the same definition. Contradiction! since η was a counterexample.

(2) $\xi < \eta$.

Proof. Suppose not. Then A is $\Sigma_1(M_\eta||\nu_\eta)$ in p and A' is $\Sigma_1(M'_{e_\eta}||\nu'_{e_\eta})$ in $p' = \sigma_\eta(p)$ by the same definition. But $\sigma_\eta = \pi'_{e^*_\eta,e_\eta}\sigma^*_\eta$, since $\xi = \eta$ and:

$$\pi'_{e^*_\eta,e_\eta} \upharpoonright \tau'_{e_\eta} + 1 = \mathrm{id}$$

Hence A' is $\Sigma_1(M_{e^*_\eta}||\nu^*)$ in $\sigma^*_\eta(p)$ by the same definition, where $\nu^* = \sigma^*_\eta(\nu_\eta)$. But $M_\eta||\nu_\eta = M^*_\eta$ since $\rho^1_{M_\eta||\nu_\eta} \leq \tau_\eta$. But $\rho^1_{M'_{e^*_\eta}||\nu^*} \leq \tau'_{e_\eta}$, since $\sigma^*_\eta \upharpoonright M^*_\eta$ takes M^*_η in a Σ^* way to $M'_{e^*_\eta}||\nu^* \bigwedge x^1(x^1 \neq \tau_\eta)$ hold in M^*_η. But then $M'^*_{e_\eta} = M'_{e^*_\eta}||\nu^*$. Hence A is $\Sigma_1(M^*_\eta)$ in p and A' is $\Sigma_1(M'_{e_\eta})$ in $\sigma^*_\eta(p)$ by the same definition. Contradiction! QED(2)

Since $\xi < \eta$ and $\tau'_{e_\eta} = \sigma_\xi(\tau_\eta)$, we have:

$$\tau'_{e_\eta} = \sigma_\eta(\tau_\eta) = \pi_\eta\hat{\sigma}_\eta(\tau_\eta) = \pi_\eta\sigma_\xi(\tau_\eta) = \pi_\eta(\tau'_{e_\eta})$$

Hence $\mathrm{crit}(\pi_\eta) > \tau'_{e_\eta}$ if $\hat{e}_\eta \neq e_{\eta'}$. Hence A' is $\Sigma_1(M_\eta||\nu_\eta)$ in p and A' is $\Sigma_1(M'_{\hat{e}_\eta}||\nu'_{e_\eta})$ in $\hat{\sigma}_\eta(p)$ by the same definition. But then we can set $I'' = I'|e_\eta + 1$ and define e' inserting I into I'' by:

$$e_h = \begin{cases} e_h & \text{if } h < \eta \\ \hat{e}_\eta & \text{if } h = \eta \end{cases}$$

$\langle e', \eta, I, I''\rangle$ is obviously still a counterexample to Lemma 3.7.2. Thus we may henceforth assume:

(3) $e_\eta = \hat{e}_\eta$

(4) $\nu_\eta = \mathsf{ON}_{M_\eta}$.

Proof. $\tau_\eta < \lambda_\xi$, where λ_ξ is inaccessible in M_η. Hence, if $\nu_\eta \in M_\eta$, we would have: $\rho^1_{M_\eta||\nu_\eta} \geq \lambda_\xi > \tau_\eta$, contradicting (1). QED(4)

(5) η is not a limit ordinal.

Proof. Suppose not. Let A, A', p, p' be as above. By (2), $\xi < \eta$ where $\xi = T(\eta + 1)$. By (4) $M_\eta = M_\eta||\nu_\eta$ is an active premouse. But $\sigma_\eta : M_\eta \longrightarrow_{\Sigma^*} M'_{e_\eta}$ and $\sigma_\eta(\nu_\eta) = \nu'_{e_\eta}$. Pick $l <_T \eta$ such that:

- $\mathrm{crit}(\pi_{l,\eta}) > \lambda_\xi$,
- $\pi_{l,\eta}$ is a total map on M_l,

- $p \in \mathrm{rng}(\pi_{l,\eta})$.

Set $\bar{p} = \pi_{l,\eta}^{-1}(p)$. Then A is $\Sigma_1(M_l)$ in $\bar{p}$ and A is $\Sigma_1(M_\eta)$ in p by the same definition. Define a potential iteration $\bar{I}$ of length $l+2$ extending $I|l+1$ by appointing: $\bar{\nu}_l =: \pi_{l,\eta}^{-1}(\nu_\eta)$. Then $\bar{M}_l = M_l||\bar{\nu}_l$. Since $\pi_{l,\eta}(\kappa_\eta) = \kappa_\eta$ it follows that $\bar{\kappa}_l = \kappa_\eta$ and $\bar{M}_l^* = M_\eta^*$. Define $\bar{e} : l+1 \longrightarrow \eta'$ by: $\bar{e} \upharpoonright l+1 = e \upharpoonright l+1, \bar{e}_{l+1} = e_\eta + 1$ (hence $\bar{\hat{e}}_l = e_\eta$). Then $\bar{e}$ inserts $\bar{I}$ into I', giving the insertion maps:

$$\bar{\sigma}_i = \sigma_i \text{ for } i < l, \bar{\sigma}_l = \sigma_\eta \pi_{l,\eta}$$

Then $\bar{\kappa}_l = \kappa_\eta$. It follows easily that $\bar{M}_l^* = M_\eta^*$ and $\bar{\sigma}_l^* = \sigma_\eta^*$. But $l < \eta$, so by the minimality of η there is a q such that A is $\Sigma_1(M_\eta^*)$ in q and A' is $\Sigma_1(M'^*_{e_\eta})$ in $\sigma_\eta^*(q)$ by the same definition. Contradiction! QED(5)

Now let $\eta = j+1, h = T(\eta)$. Then $e_\eta = \hat{e}_\eta = e_j + 1$. We know

$$\pi_{h,\eta} \upharpoonright M_j^* : M_j^* \longrightarrow_{\Sigma^*} M_\eta = \langle J_{\nu_\eta}^E, E_{\nu_\eta} \rangle$$

Hence M_j^* has the form:

(6) $M_j^* = \langle J_\nu^E, E_\nu \rangle$ where $E_\nu \neq \varnothing$.

(7) $\tau_\eta < \kappa_j$.

Proof. $\tau_\xi \leq \kappa_j$ since $\xi < \eta = j+1$. Hence $\tau_\eta < \lambda_\eta \leq \lambda_j$. But $\tau_\eta \in \mathrm{rng}(\pi_{h,\eta})$, where:

$$[\kappa_j, \lambda_j) \cap \mathrm{rng}(\pi_{h,\eta}) = \varnothing$$

QED(7)

(8) $\rho^1_{M_j^*} \leq \tau_\eta$.

Proof. Suppose not. Then $\tau_\eta = \pi_{h,\eta}(\tau_\eta) < \pi``_{h,\eta}\rho^1_{M_j^*} \subset \rho^1_{M'_\eta}$, contradicting (1). QED(8)

Thus:

(9) $\pi_{h,\eta} : M_j^* \longrightarrow_{E_{\nu_i}} M_\eta$ is a Σ_0 ultrapower.

(10) $\sigma_j^*(\tau_\eta) = \tau'_{e_\eta}$.

Proof. $\tau_\eta < \kappa_j < \lambda_h$ by (7). Hence:

$$\tau'_{e_\eta} = \hat{\sigma}_\eta(\tau_\eta) = \sigma_h(\tau_\eta) = \pi'_{e_j^*,e_h}\sigma_j^*(\tau_\eta) = \sigma_j^*(\tau'_\eta),$$

since $\sigma_j^*(\tau_\eta) < \sigma_j^*(\kappa_j) = \kappa'_{e_j}$ and $\pi'_{e_j^*,e_h} \upharpoonright \kappa'_{e_j} = \mathrm{id}$.

QED(10)

(11) $\rho^1_{M^*_{e_j}} = \tau'_{e_\eta}$.

Proof. $\bigwedge x^1(x^1 \neq \tau_\eta)$ holds in M^*_j by (8). But:

$$\sigma^*_j \restriction M^*_j : M_j \longrightarrow_{\Sigma^*} M'^*_{e_j}$$

Hence $\bigwedge x^1(x^1 \neq \sigma^*_j(\tau_\eta))$ holds in $M'^*_{e_j}$, where $\sigma^*_j(\tau_\eta) = \tau'_{e_j}$. QED(11)

But then:

(12) $\pi'_{e^*_j, e_\eta} : M'^*_{e_j} \longrightarrow_{E_{\nu_{e_j}}} M_{e\eta}$ is a Σ_0-ultrapower.

We can now prove:

(13) A is $\Sigma_1(M^*_j)$ in an r and A' is $\Sigma_1(M'^*_{e_j})$ in $r' = \sigma^*_j(r)$ by the same definition.

Proof. Let $p = \pi_{h,\eta}(f)(\alpha)$, where $f \in M^*_j$, $\alpha < \lambda_i$. Then $p' = \pi'_{e^*_j, e_\eta}(f')(\alpha')$, where: $f' = \sigma^*_j(f), \alpha' = \tilde{\sigma}_j(\alpha)$. Let $F =: E^{M_j}_{\nu_j}, F' = E^{M'_{e_j}}_{\nu_{e_j}}$. F_α can of course be coded by an $\tilde{F} \subset \tau_j$ which is $\Sigma_1 < (M_j||\nu_j)$ in α, τ_j and F'_α is coded by an $\tilde{F}' \subset \tau'_{e_j}$ which is $\Sigma_1(M'_{e_j})$ in α', τ'_{e_j} by the same definition. By the minimality of η we can conclude: F_α is $\Sigma_1(M^*_j)$ in a parameter a and $F'_{\alpha'}$ is $\Sigma_1(M'^*_{e_j})$ in the parameter $a' = \sigma^*_j(a)$ by the same definition. Now suppose:

$$A(\mu) \longleftrightarrow \bigvee yB(\mu, y, p) \text{ and}$$
$$A'(\mu) \longleftrightarrow \bigvee yB'(\mu, y, p')$$

where B is $\Sigma_0(M_\eta)$ and B' is $\Sigma_0(M'_{e_j})$ by the same definition. Let B^* be $\Sigma_0(M^*_j)$ and B'^* be $\Sigma_0(M'^*_{e_j})$ by the same definition. Since the map $\pi = \pi_{h,\eta}$ takes M^*_j cofinally to M_η, we have:

$$\begin{aligned} A(\mu) &\longleftrightarrow \bigvee u \in M^*_j \bigvee y \in \pi(u) B(\mu, y, \pi(f)(\alpha)) \\ &\longleftrightarrow \bigvee u \in M^*_j \{\gamma < \kappa_j : \bigvee y \in uB^*(\mu, y, f(\gamma))\} \in F_\alpha \end{aligned}$$

Hence A is $\Sigma_1(M^*_j)$ in $r = \langle a, f \rangle$. By the same argument, however, A' is $\Sigma_1(M'^*_{e_j})$ in $r' = \langle a', f' \rangle$ by the same definition. QED(13)

Now extend $I|h+1$ to a potential iteration I^+ of length $h+2$ by appointing: $\nu^+_h = \pi^{-1}_{h,\eta}(\nu_\eta)$. (Hence $M^*_j = M_h||\nu^+_h$). Set: $h' = e^*_j$. Extend $I'|h'+1$ to I'^+ of length $h'+2$ by appointing: $\nu'^+_{h'} = \pi'_{h', e_\eta}(\nu'_\eta)$. (Hence $M'^*_{e_j} = M'_{h'}||\nu'^+_{h'}$). Obviously, $\sigma^*(\nu^+_h) = \nu'^+_{h'}$. Now extend $e \restriction h$ to $e^+ : h+1 \longrightarrow h'+1$ by:

$$e^+_i = \begin{cases} e_i & \text{if } i < h \\ e^*_j & \text{if } i = h \end{cases}$$

Then e^+ is easily seen to insert I^+ into I'^+, giving the insertion maps:

$$\sigma_i^+ = \begin{cases} \sigma_i & \text{for } i < h \\ \sigma_j^* = \pi'_{\hat{e}_h, h'} \circ \hat{\sigma}_j & \text{for } i = h \end{cases}$$

Then $\sigma_h^+(\nu_h^+) = \nu'^+_{h'}$. We note that $\tau_h^+ = \tau_\eta, \tau'^+_{h'} = \tau'_{e_\eta}$. It follows easily that $(M_h^+)^* = M_\eta^*, (M'^+_{h'}) = M'^*_{e_\eta}$ and $(\sigma_h^+) = \sigma_\eta^*$. By the minimality of η we conclude that A is $\Sigma_1(M_\eta^*)$ and $(\sigma_h^+)^* = \sigma_\eta^*$. By the minimality of η we conclude that A is $\Sigma_1(M_\eta^*)$ in a q and A' is $\Sigma_1(M'^*_{e_\eta})$ in $\sigma_\eta^*(q)$ by the same definition. Contradiction! QED(Lemma 3.7.4)

Composing insertions

Lemma 3.7.5. *Let e insert I into I', with insertion maps $\hat{\sigma}_i^e, \sigma_i^e$. Let f insert I' into I'' with insertion maps $\hat{\sigma}_i^f, \sigma_i^f$. Then*

(i) *fe inserts I into I''*

(ii) $\widehat{f \circ e} = \hat{f} \circ \hat{e}$.

(iii) $\sigma_i^{fe} = \sigma_{e_i}^f \circ e_i^e$

(iv) $\hat{\sigma}_i^{fe} = \hat{\sigma}_{\hat{e}_i}^f \circ \hat{\sigma}_i^e$.

Proof. We show that $f \circ e$ satisfies the insertion axioms (a)-(e) with $\hat{\sigma}_i^{fe} = \hat{\sigma}_{e_i}^f \circ \hat{\sigma}_i^e$. In the process we shall also verify (ii), (iii). We first note:

$$\widehat{fe}(i) = \text{lub}(fe)\text{"}i = \text{lub}\, f\text{"}(\text{lub}\, e\text{"}i) = \hat{f}\hat{e}(i)$$

Axioms (a), (b), (c) then follow trivially. By definition we then have:

$$\begin{aligned} \sigma_i^{fe} &= \pi''_{\hat{f}\hat{e}(i), fe(i)} \hat{\sigma}_i^{ef} \\ &= \pi''_{\hat{f}e(i), fe(i)} \circ \pi''_{\hat{f}\hat{e}(i), \hat{f}e(i)} \circ \hat{\sigma}^f_{\hat{e}(i)} \circ \hat{\sigma}_i^e \\ &= (\pi''_{\hat{f}e(i), fe(i)} \circ \hat{\sigma}^f_{e(i)}) \circ (\pi'_{\hat{e}(i), e(i)} \circ \hat{\sigma}_i^e) \\ &= \sigma^f_{e(i)} \circ \sigma_i^e \end{aligned}$$

Axioms (d), (e) then follow easily. QED(Lemma 3.7.5)

We now consider "towers" of insertions. Let I^ξ be an iterate of M for $\xi < \Gamma$, where $e^{\xi,\mu}$ inserts I^ξ into I^μ for $\xi \le \mu < \Gamma$. (We take $e^{\xi,\xi}$ as the identical insertion).

Definition 3.7.3. We call:

$$\langle\langle I^\xi : \xi < \Gamma\rangle, \langle e^{\xi,\mu} : \xi < \mu < \Gamma\rangle\rangle$$

a *commutative insertion system* iff $e^{\zeta,\mu} \circ e^{\xi,\zeta} = e^{\xi,\mu}$ for $\xi \le \zeta \le \mu < \Gamma$.

Now suppose that Γ is a limit ordinal. Is there a reasonable sense in which we could form the *limit* of the above system? We define:

Definition 3.7.4. $I, \langle e^\xi : \xi < \Gamma\rangle$ is a good limit of the above system iff:

- I is an iterate of M and e^ξ inserts I^ξ into I.
- $e^\mu \circ e^{\xi,\mu} = e^\xi$ for $\xi \le \mu < \Gamma$.
- If $i < \mathrm{lh}(I)$, then $i = e^\xi(h)$ for some $\xi < \Gamma$, $h < \mathrm{lh}(I^\xi)$.

Note. Let $\eta_i = \mathrm{ht}(I^i)$ for $i < \Gamma$. It is a necessary but not sufficient condition for the existence of a good limit that:

$$\langle \eta_i : i < \Gamma\rangle, \langle e^{ij} : i \le j < \Gamma\rangle$$

have a well founded limit.

If η, $\langle \tilde{e}^i : i < \Gamma\rangle$ is the transitivised direct limit of the above system, then any good limit must have the form $\langle I, \langle e^i : i < \Gamma\rangle\rangle$.

Fact. Let $\eta, \langle e^i : i < \Gamma\rangle$ be as above. Let $\xi < \eta$ and let $\hat{e}^i(\xi_i) = \xi$ for an $i < \Gamma$. For $i \le j < \Gamma$ set:

$$\xi_j =: \hat{e}^{i,j}(\xi_i) = (\hat{e}^j)^{-1}(\xi)$$

Then $e^j(\xi_j) = \hat{e}^j(\xi_j) = \xi$ for sufficiently large $j < \Gamma$.

Proof. Suppose not. Then there is a monotone sequence $\langle j_n : n < \omega\rangle$ in $[i, \Gamma)$ such that $e^{j_n, j_{n+1}}(\xi_{j_n}) > \xi_{j_{n+1}}$.

Hence $e^{j_{n+1}}(\xi_{j_{n+1}}) < e^{j_n}(\xi_{j_n})$ for $n < \omega$. Contradiction! QED

We then get:

Lemma 3.7.6. *Let $\langle I^\xi\rangle, \langle e^\xi, \mu\rangle$ be a commutative system of insertions of limit length θ. Then there is at most one good limit $I, \langle e^\xi\rangle$. Moreover, if $i < \mathrm{lh}(I)$, then $|M_i| = \bigcup\{\mathrm{rng}(\tilde{\sigma}^\xi_h) : e^\xi(h) = i\}$.*

Proof. Let $\langle I\langle e^\xi\rangle\rangle, \langle I'\langle e'^\xi\rangle\rangle$ be two distinct good limits. We derive a contradiction. Set $\eta_\xi = \text{lh}(I^\xi)$ for $\xi < \Gamma$. Then $\langle\eta_\xi\rangle, \langle\tilde{e}^\xi, \mu\rangle$ has a transitive direct limit $\eta, \langle f^\xi\rangle$. Moreover $\eta = \text{lh}(I)$ and $e^\xi = e'^\xi = f^\xi$ for $\xi < \Gamma$. Hence $\hat{e}^\xi = \hat{e}'^\xi = \text{lub}\{f^h : h < \xi\}$ for $\xi < \Gamma$. By induction on $i < \xi$ we prove:

(a) $M_i = M_i'$

(b) $\sigma_h^\xi = \sigma_h'^\xi$ for $e^\xi(h) = i$.

(c) $|M_i| = \bigcup\{\text{rng}\,\sigma_h^\xi : e^\xi(h) = i\}$.

For $i = 0$ this is trivial. Now let $i = j + 1$. Then:

$$\nu_j = \nu_j' = \sigma_h^\xi(\nu_h^\xi) \text{ whenever } e^\xi(h) = j$$

This fixes $\mu =: T(j+1) = T'(j+1)$. But then we have: $M_j^* = M_j'^*$. Thus $M_i = M_i'$ and $\pi_{\mu+i} = \pi'_{\mu_i}$ are determined by:

$$\pi_{\mu+i} : M_i^* \longrightarrow_F M_i, \text{ where } F = E_{\nu_j}^{M_j} = E_{\nu_j'}^{M_j'}$$

We must still show:

Claim. If $x \in M_i$, then $x = \sigma_l^\xi(\bar{x})$ for a $\xi < \theta$ such that $e^\xi(l) = i$.

Proof. Let $n \leq \omega$ be maximal such that $\kappa_i < \rho_{M_i}^n$. Then $x = \pi_{1i}(f)(\alpha)$ for an $f \in \Gamma^n(\kappa_j, M_i^*)$. Let either $f = p \in M_i^*$ or else $f(\xi) \cong G(\xi, p)$ where $p \in M_i^*$ and G is a good $\Sigma_1^{(m)}(M_i^*)$ function for a $m < n$. Pick $\xi < \theta$ such that there are μ_ξ, j_ξ, i_ξ with:

$$e^\xi(\mu_\xi) = \mu,\ e^\xi(i_\xi) = i,\ e^\xi(j_\xi) = j$$

Assume furthermore that $\sigma_{\bar{\mu}}(\bar{p}) = p$ and $\sigma_{j_\xi}^\xi(\bar{\alpha}) = \alpha$. Since $\sigma_{j_\xi}(\nu_{j_\xi}^\xi) = \nu_j$, it follows easily that $\mu_\xi = T^\xi(i_\xi)$ and:

$$\sigma_{\bar{\mu}}^\xi \restriction M_{i_\xi}^{\xi *} : M_{i_\xi}^{\xi *} \longrightarrow_{\Sigma^*} M_i^*$$

Let $\bar{f}$ be defined from $\bar{p}$ over $M_{i_\xi}^\xi$ as f was defined from p over M_i. Let $\bar{x} = \pi_{\bar{\mu}, i_\xi}^\xi(\bar{f})(\bar{\alpha})$. Then $\sigma_{i_\xi}(\bar{x}) = x$ by Lemma 3.7.1(5). QED(Claim)

Now let $\lambda < \theta$ be a limit ordinal. We first prove:

Claim. $i <_T \lambda$ iff whenever $e(i_\xi) = i$ and $e^\xi(\lambda_\xi) = \lambda$, then $i_\xi <_{T^\xi} \lambda_\xi$.

Proof. ($\longrightarrow$) is immediate by Lemma 3.7.1(10). We prove ($\longleftarrow$). Suppose not. Let A be the set of $\xi < \theta$ such that there are i_ξ, λ_ξ with $e^\xi(i_\xi) = i$, $e^\xi(\lambda_\xi) = \lambda$. Then $i \not<_T \lambda$ but $i_\xi <_{T^\xi} \lambda_\xi$ for $\xi \in A$. Then:

$$\hat{e}^\xi(i_\xi) <_T \hat{e}^\xi(\lambda_\xi) \leq_T e^\xi(\lambda_\xi) = \lambda.$$

Set: $j = \sup\{\hat{e}^\xi(i_\xi) : \xi \in A\}$. Then $j <_T \lambda$ by the fact that $T``\{\lambda\}$ is club in λ. Hence $j < i$. Let $\xi \in A$ such that $e^\xi(j_\xi) = j$. Then $j_\xi < i_\xi$, since e^ξ is order preserving. Hence:

$$j = e^\xi(j_\xi) < \hat{e}^\xi(i_\xi) \leq j.$$

Contradiction! QED(Claim)

But then $T``\{\lambda\} = T'``\{\lambda\}$. Hence $M_\lambda = M'_\lambda, \pi_{i,\lambda} = \pi'_{i,\lambda}$ are given as the transitivized limit of:

$$\langle M_i : i <_T \lambda \rangle,\ \langle \pi_{i,j} : i \leq_T j < \lambda \rangle.$$

Finally, we show that each $x \in M_\lambda$ has the form $\sigma^\xi_{\lambda_\xi}(\bar{x})$ for an $\xi \in A$. We know that $x = \pi_{i,\lambda}(x')$ for an $i <_T \lambda$. Pick $\xi < \theta$ such that $e^\xi(i_\xi) = i$, $e^\xi(\lambda_\xi) = \lambda$ and $x' = \sigma^\xi_{i_\xi}(\bar{x}')$. Set: $\bar{x} = \pi^\xi_{i_\xi,\lambda_\xi}(\bar{x}')$. Then $\sigma^\xi_\lambda(\bar{x}) = x$ by Lemma 3.7.1(10).

QED(Lemma 3.7.6)

In the following we take a more local approach for forming a good limit and ask if and when the proven can be break down. It is of course a necessary condition that the limit be indexed in a well founded way, so we assume that.

In the following let $\mathbb{C} = \langle \langle I^\xi \rangle, \langle e^{\xi,\mu} \rangle \rangle$ be a commutative insertion system of limit length θ. Let $\eta_\xi = \text{length}(I^\xi)$ for $\xi < \theta$. Suppose that

$$\langle \eta_\xi : \xi < \theta \rangle, \langle e^{\xi,\mu} : \zeta \leq \mu < \theta \rangle$$

has the transitivized direct limit:

$$\eta,\ \langle e^\xi : \xi < \theta \rangle$$

(Thus if $\mathbb{C}$ had a good limit, it would have the form $\langle I, \langle e^\xi : \xi < \theta \rangle \rangle$).

Definition 3.7.5. Let $\mathbb{C}, \eta$, etc. be as above. Let $i < \eta$. Let I be a normal iteration of M of length $i + 1$. I is a *good limit of* $\mathbb{C}$ *at* i iff whenever $\gamma < \theta$ and $e^\gamma(h) = i$, then $e^\gamma \upharpoonright h + 1$ inserts $I^\gamma | h + 1$ into I.

Note. By Lemma 3.7.6 it follows that there is at most one good limit of $\mathbb{C}$ at i. To see this, let $\gamma < e$ such that $e^\gamma(h) = i$ and apply Lemma 3.7.6 to the structure:

$$\mathbb{C}' = \langle \langle \tilde{I}^\xi : \gamma \leq \xi < \theta \rangle, \dots \rangle \text{ where } \tilde{I}^\xi = I | e^{\gamma,\xi}(h) + 1.$$

Moreover, if I is a good limit of $\mathbb{C}$ at i and $h < i$, thus $I|h+1$ is the good limit of $\mathbb{C}$ at h. Thus we can unambiguously denote the good limit of $\mathbb{C}$ at i, if it exists, by: $I|i+1$. By uniqueness we then have:

$$(I|i+1)|h+1 = I|h+1 \text{ for } h < i$$

It is clear that I is the unique good limit of $\mathbb{C}$ iff $I|i+1$ exists for all $i < \eta$, and $I = \bigcup_{i<\eta} I|i+1$. We also note that $I|1 = \langle\langle M\rangle, \varnothing, \langle \text{id} \rangle, \varnothing\rangle$ is trivially the good limit at 0.

Recall that we call a premouse M *uniquely iterable* iff it is normally iterable and has the *unique branch property* -i.e. whenever I is a normal iteration of M of limit length, then it has at most one cofinal well founded branch. (Similarly for uniquely α-iterable). In the later subsection of §3.7 we shall always assume unique iterability of M and make use of the following two lemmas:

Lemma 3.7.7. *Let $\mathbb{C}, \eta$ be as above and let M be uniquely η-iterable. Let $i+1 < \eta$. If $I|i+1$ exists, then so does $I|i+2$.*

Proof. Let $I = I|i+1$. Pick $\mu < \theta$ such that $e^\mu(i_\mu) = i$ and $e^\mu(i_\mu+1) = i+1$. Set: $\nu_i = \sigma^\mu_{i_\mu}(\nu^\mu_{i_\mu})$. For $\mu \leq \delta < \theta$, we have $\nu_\delta = \sigma^\delta_{i_\delta}(\nu^\delta_{i_\delta})$ and $\nu^\delta_{i_\delta} \geq \nu^\delta_j$ for $j < i_\delta$.

It follows easily that $\nu_i > \nu_j$ in I whenever $j < i$. Thus ν_i determines a potential extension of $I|i+1$, giving: $\xi = T'(i+1), M^*_i$. Let $F = E^{M_i}_{\nu_i}$ in I.

Set:

$$\pi'_{\eta,i+1} : M^*_i \longrightarrow^*_F M'_{i+1}$$

This gives us an iteration I' of length $i+2$ extending I, it follows by Lemma 3.7.2 that $e^\mu|i_\mu+2$ inserts $I^\mu|i_\mu+2$ into I'. But this holds for sufficiently large $\mu < \theta$. Now let $\overline{\mu} < \theta$ with $e^{\overline{\kappa}} = i+1$. Let $\mu \geq \overline{\mu}$ be as above. Then $e^{\overline{\mu},\mu}(h) = i_\mu + 1$, and $e^{\overline{\mu},\mu} \upharpoonright h+1$ inserts $I^{\overline{\mu}}|h+1$ into $I^\mu|i_\mu+2$. Hence $e^{\overline{\mu}} = e^\mu \circ e^{\overline{\mu},\mu}$ inserts $I^{\overline{\mu}}|h+1$ into I'.

QED(Lemma 3.7.7)

Now let $\delta < \eta$ be a limit ordinal and let $I|i+1$ be defined for all $i < \delta$. If $I|\delta+1$ defined? Not necessarily. Set: $I = \bigcup_{i<\delta} I|i+1$. Then I is a normal iteration of length δ. Hence it has a unique cofinal well founded branch b. We can then extend I to I' of length $\delta+1$, taking $T'``\{\delta\} = b$. However I' will only be a good limit of $\mathbb{C}$ at δ if a certain condition on b is fulfilled:

Lemma 3.7.8. *Let $\mathbb{C}, I, b, I'$, etc. be as above. Assume that there are arbitrarily large $\gamma < \theta$ such that:*

(*) *$e^\gamma(\overline{\delta}) = \delta$ for some $\overline{\delta}$. Moreover, either $\hat{e}^\gamma(\overline{\delta}) \in b$ or $\hat{e}^\gamma(\overline{\delta}) = \delta$ and $\hat{e}^\gamma(i) \in b$ whenever $i <_{T^\gamma} \overline{\delta}$.*

Then I' is a good limit of $\mathbb{C}$ at δ.

Proof. Let $\gamma, \overline{\delta}$ as in (*). We show that $e^\gamma \upharpoonright \overline{\delta}+1$ inserts $I^\gamma|\overline{\delta}+1$ into $I'|\delta+1$. We consider two cases:

Case 1: $\hat{e}^\gamma(\overline{\delta}) \in b$.

Let $\xi = \hat{e}^\gamma(\overline{\delta})$. Then $\xi \leq_{T'} \delta$. It is easily verified that $e^\gamma \upharpoonright \overline{\delta}+1$ inserts $I^\gamma|\overline{\delta}+1$ into I' with $\hat{\sigma} = \hat{\sigma}^\gamma_{\overline{\delta}}$, $\sigma = \sigma^\gamma_{\overline{\gamma}}$ defined as follows:

By the above Fact there is $\gamma' > \gamma$ such that $e^{\gamma'}(\delta') = \xi$, where $\delta' = \hat{e}^{\gamma,\gamma'}(\overline{\delta})$. Thus $e^{\gamma'} \upharpoonright \delta'+1$ inserts $I_{\delta'}|\delta+1$ into $I|\xi+1$. Set:

$$\hat{\sigma} =: \hat{\sigma}^\gamma_{\delta'} \circ \hat{\sigma}^{\gamma,\gamma'}_{\overline{\delta}}, \sigma =: \pi'_{\xi,\delta} \circ \hat{\sigma}$$

QED(Case 1)

Case 2: $e^\gamma(\overline{\delta}) = \delta$.

Then e^γ takes $\overline{\delta}$ cofinally to δ. Thus $e^\gamma \upharpoonright \overline{\delta}+1$ inserts $I^\gamma|\overline{\delta}+1$ into $I|\delta+1$, where $\sigma = \sigma^\gamma_{\overline{\delta}} = \hat{\sigma}^\gamma_{\overline{\delta}}$ is defined by:

$$\sigma \pi^\gamma_{i,\overline{\delta}} = \pi_{e^\gamma_{\overline{\delta}}(i),\delta} \circ \hat{\sigma}^\gamma_i$$

The verification is again straightforward.

QED(Case 2)

Now let $\mu < \theta$ be arbitrary such that $e^\mu(\delta') = \delta$. Let $\gamma > \mu$ satisfy (*) with $e^\gamma(\overline{\delta}) = \delta$. Then $e^{\mu,\gamma}$ inserts $I^\mu|\delta'+1$ into $I^\gamma|\overline{\delta}+1$ and e^γ inserts $I^\gamma|\overline{\delta}+1$ into $I'|\delta+1$. Hence $e^\mu = e^\gamma \cdot e^{\mu,\gamma}$ inserts $I^\mu|\delta'+1$ into $I'|\delta+1$.

QED(Lemma 3.7.8)

Remark. It follows that every $\gamma < \theta$ such that $\delta \in \mathrm{rng}(e^\gamma)$ satisfies (*).

Building on what we have just proven, we show that we can disperse with the iterability assumption if the length of the commutative system has cofinality greater than ω.

Lemma 3.7.9. *Let $\mathbb{C}$ be a commutative insertion system of length θ. If $\mathrm{cf}(\theta) > \omega$, then $\mathbb{C}$ has a good limit.*

Proof.

Claim. $\langle \eta_i : i < \theta \rangle, \langle e^{\xi,\mu} : \xi \leq \mu < \theta \rangle$ has a transitivized direct limit:

$$\eta, \langle e^{\xi} : \xi < \theta \rangle$$

Proof. Suppose not. Let $\langle u, <^* \rangle, \langle e^{\xi} : \xi < \theta \rangle$ be a direct limit, where $<^*$ is a linear ordering of u. Then there are x_n $(n < \omega)$ such that $x_{n+1} <^* x_n$ for $n < \omega$. Since $\mathrm{cf}(\theta) > \omega$, there must be $\gamma < \theta$ such that $x_n \in \mathrm{rng}(e^{\gamma})$ for $n < \omega$. Let $e^{\gamma}(\alpha_n) = x_n$ $(n < \omega)$. Then $\alpha_{n+1} < \alpha_n$ in η_δ for $n < \omega$. Contradiction!

QED(Claim)

We now prove by induction on $i < \eta$ that $\mathbb{C}$ has a good limit $I|i$ at i.

Case 1. $i = 0$. The 1-step iteration of M: $\langle \langle M \rangle, \varnothing, \langle \mathrm{id} \rangle, \varnothing \rangle$ is the good limit at 0 (with $e_0^0 = \hat{e}_0^0 = \mathrm{id} \restriction \{0\}$).

Case 2. $i = h + 1$.

Let $\nu_i, \xi = T'(i+1), M_i^*, F = E_{\nu_i}^{M_i}$ be as in the proof of Lemma 3.7.7. The proof of Lemma 3.7.7 goes through exactly as before if we can show:

Claim. M_i^* is extendible by F.

Proof. Suppose not. Then there are $f_n \in \Gamma^*(\kappa_i, M_i^*), \alpha_n \in \lambda_i$ $(n < \omega)$ such that

$$\{\langle \mu, \tau \rangle : f_{n+1}(\mu) \in f_n(\tau)\} \in F_{\langle \alpha_{n+1}, \alpha_n \rangle} \quad \text{for } n < \omega$$

Let $p_n \in M_i^*$ such that either $p_n = f_n$ or f_n is defined by: $f_n(\beta) \cong G(p_n, \beta)$, where G is good over M_ξ^*. Since $\mathrm{cf}(\theta) > \omega$, we can pick $\gamma < \theta$ such that

- $e^{\gamma}(i_\gamma) = i, e^{\gamma}(\xi_\gamma) = \xi$
- $\sigma^{\gamma}_{\xi_\gamma}(\overline{p}_n) = p_n$ $(n < \omega)$
- $\sigma^{\gamma}_{i_\gamma}(\overline{\alpha}_n) = \alpha_n$ $(n < \omega)$
- $[\overline{e}^{\gamma}(\xi_\gamma), e^{\gamma}(\xi_\gamma)]_T$ has no drop point in I. (Hence $\sigma^{\gamma *}_{\xi_\gamma}, M^{\gamma}_{\xi_\gamma} \longrightarrow_{\Sigma^*} M_\xi$, since $\sigma^{\gamma}_{\xi_\gamma} = \pi_{\xi_\gamma} \hat{\sigma}^{\gamma}_{\xi_\gamma}$).

We note that $\xi_\gamma = T^{\gamma}(i_\gamma + 1)$. (Suppose not. Let $t = T^{\gamma}(i_\gamma + 1)$. Then $\xi \in [\hat{e}^{\gamma}(t), e^{\gamma}(t)]$ by Lemma 3.7.1 (3). But thus $t < \xi$ and $\xi < t$ are both impossible. Contradiction!) It follows that:

$$\sigma^{\gamma}_{\xi} \upharpoonright M^{\gamma *}_{i_\gamma} \longrightarrow_{\Sigma^*} M^*_i$$

If $\overline{f}_n$ is defined from $\overline{p}_n$ as f_n was defined from p_n, we then have:

$$\{\langle \mu, \tau \rangle : \overline{f}_{n+1}(\mu) \in \overline{f}_n(\tau)\} \in \overline{F}_{\langle \overline{\alpha}_{n+1}, \overline{\alpha}_n \rangle}$$

where $\overline{F} = E^{M^{\gamma}_{i_\gamma}}_{\nu_{i_\gamma}}$. But:

$$\pi^{\gamma}_{\xi_\gamma, i_\gamma} : M^{\gamma *}_{i_\gamma} \longrightarrow^{*}_{\overline{F}} M^{\gamma}_{i_\gamma + 1}$$

Hence $M^{\gamma}_{i_\gamma+1}$ would be ill founded. Contradiction!

QED(Case 2)

Case 3: $i = \mu$ is a limit ordinal.

Let b' be the set of $j < \mu$ such that for some $\gamma < \theta$ and $\overline{\mu} < \eta_\gamma$ we have $e^{\gamma}(\overline{\mu}) = \mu$ and $j = \hat{e}^{\gamma}(i)$ for an $i \leq_{T^\gamma} \overline{\mu}$. Let b be the closure of b' under limit points below μ. Then b is a cofinal branch in I. Moreover, b satisfies $(*)$.

τ_{i_n} is not a cardinal in Lemma 3.7.8. Hence we can simply repeat the proof of Lemma 3.7.8 if we can show:

Claim. b is a well founded branch in I.

Proof. We must first show:

Subclaim. b has at most finitely many drop points.

Proof. Suppose not. Let $\langle i_n : n < \omega \rangle$ be monotone such that $i_n + 1$ is a drop point in b. Since $i_n + 1$ is not a limit point in b, we have $i_n + 1 \in b'$. Hence for each n there is a $\gamma < \theta$ and a $\overline{\mu}$ such that $e^{\gamma}(\overline{\mu}) = \mu, \hat{e}^{\gamma}(h_n + 1) = i_n + 1, h_n + 1 <_{T^\gamma} \overline{\mu}$. If γ has this property, so will every larger $\gamma' < \theta$. Since $\mathrm{cf}(\theta) > \omega$, we know that sufficiently large $\gamma < \theta$ will have the property for all n. We can also suppose without lose of generality that $e^{\gamma}(t_n) = t_n$, where $t_n = T(i_n + 1)$ in I. Just as in Case 2 we then have $I_n = T^{\gamma}(h_n + 1)$. As in Case 2 we can assume γ chosen big enough that $[\hat{e}^{\gamma}(\overline{t}_n), e^{\gamma}(\overline{t}_n))_T$ has no drop point in I. (Hence the map $\sigma^{\gamma}_{\overline{t}_n}$ is Σ^*-preserving). Then τ_{i_n} is not a cardinal in M_{t_n} and $\tau_{i_n} = \sigma^{\gamma}_{h_n}(\tau_{h_n}) = \sigma^{\gamma}_{\overline{t}_n}(\tau_{h_n})$. Hence τ_{h_n} is not a cardinal in $M^{\gamma}_{h_n}$. Hence $h_n + 1$ is a drop point in I^{γ}. Hence $T^{\gamma}``\{\overline{\mu}\}$ has infinitely many drop points. Contradiction!

QED(Subclaim)

We now prove the claim. Suppose not, Let $b'' =: b' \setminus \beta$, where $\beta < \bar{\mu}$ is big enough that no $i \in b''$ is a drop point. Then there is a monotone sequence $\langle i_n : n < \omega \rangle$ such that $i_n \in b'', x_n \in M_{i_n}$ and

$$x_{n+1} \in \pi_{i_n, i_{n+1}}(x_n) \text{ for } n < \omega$$

Pick $\gamma < \theta$ big enough that $e^\gamma(\bar{\mu}) = \mu$ and $\hat{e}^\gamma(h_n) = i_n$, where $h_n <_{T^\gamma} \bar{\mu}$. We can also pick it big enough that $x_n = \hat{\sigma}_{i_n}(\bar{x}_n)$ for $n < \omega$. Hence

$$\bar{x}_{n+1} \in \pi^\gamma_{h_n, h_{n+1}}(\bar{x}_n) \text{ for } n < \omega$$

Hence $M^\gamma_{\bar{\mu}}$ is ill founded. Contradiction!

QED(Lemma 3.7.9)

3.7.2 Reiterations

From now on assume that M is a uniquely normally iterable mouse (i.e. every normal iteration of limit length has exactly one cofinal well founded branch). (Our results will go through mutatis mutandis if we assume unique normal α-iterability for a regular cardinal $\alpha > \omega$).

Interpolating extenders

Let $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ be a normal iteration of M of length $\eta + 1$. A "reiteration" of I occurs when we "interpolate" new extender which were not on the sequence $\langle \nu_i : i < \eta \rangle$. This rounds very vague, or course, but we can make it more explicit by considering the case of a single extender $F = E^{M_\eta}_\nu$ which we had neglected to place on the sequence. Set: $\tau = \tau^{+M_\eta || \nu}, \kappa = \text{crit}(F), \lambda = \lambda(F) =: F(u)$. For the moment let us assumer that τ is a cardinal in M_η. The interpolation gives rise to a new iteration I'. I' coincides with I up to the point at which F should have been applied. At that point we apply F and thereafter simply copy what we did in I. The point s at which F should have been applied is defined as follows:

$$s = \text{ the least point such that } s = \eta \text{ or } s < \eta \text{ and } \nu < \nu_s$$

We want $I|s+1 = I'|s+1$, but at stage s we apply F instead of $E^{M_s}_{\nu_s}$. Thus we set: $\nu_s = \nu$. This determines $t = T'(s+1)$ and M'^*_s. We then form:

$$\pi'_{t,s+1} : M'^*_r \longrightarrow^*_F M'_{s+1}$$

There is then an obvious insertion f of $I|t+1$ into $I'|s+2$ defined by:

$$f \upharpoonright t = \text{id}, \ f(t) = s+1$$

f induces the new insertion embeddings:

$$\hat{\sigma}_t = \mathrm{id} \upharpoonright M_t,\ \pi_t = \pi'_{t,s+1},\ \sigma_t = \pi_t \hat{\sigma}_t$$

If $t = \eta$ (hence $s = \eta$), then $I' = I'|s+2$ is fully defined. Now let $t < \eta$.

Then $M'^*_s = M_t||\mu$, where $\mu \leq \mathsf{ON}_{M_t}$ is maximal with: τ is a cardinal in $M_t||\mu$. But then $\tau \in J^{E^{M_\eta}}_{\nu_t} \subset J^{E^{M_\eta}}_{\nu}$, so τ is a cardinal in $J^{E^{M_\eta}}_{\nu_t}$. Hence $\mu \geq \nu_t$ and $\sigma_t(\nu_t)$ is defined. Set: $\nu'_{s+1} = \sigma_t(\nu_t)$. This defines a potential extension of $I'|s+2$, since

$$\nu'_s = \pi_t(\tau) < \pi_t(\nu_t) = \nu'_{s+1}$$

where $\pi_t = \pi'_{t,s+1}$.

Now define e on η by:

$$e \upharpoonright t = \mathrm{id}, e(t+i) = s+1+i \text{ for } t+i \leq \eta$$

Then $e \upharpoonright t+1 = f$. It is easily seen that $\hat{e}(t) = t$ and $e(t) = s+1$. But for $i \neq t$ we have $\hat{e}(i) = e(i)$. We prove:

Claim. e inserts I into a unique I' of length $e(\eta)+1$.

To show this we prove the following subclaim by induction on i:

Subclaim. If $t+1+i \leq \eta$, then $e \upharpoonright (t+1+i+1)$ inserts $I|(t+1+i+1)$ into a unique $I'' = I'|(s+2+i+1)$ of length $s+2+i+1$.

Proof. Case 1: $i = 0$.

We have seen that $\sigma_t(\nu_t)$ exists and that $\sigma_t(\nu_t) > \nu'_t$. Hence we can appoint $\nu'_{t+1} = \sigma(\nu_t)$, which determines $\xi = T'(s+2)$ and M'^*_{s+1}. M'^*_{s+1} is $*$-extendible by $F = E^{M'_{s+1}}_{\nu'_{s+1}}$ by the fact that M is uniquely iterable. By Lemma 3.7.2 we conclude that $e|t+2$ inserts $I|t+2$ into a unique $I'|s+3$ extending $I'|s+2$.

QED(Case 1)

Case 2: $i = j+1$.

Then $I'|s+2+i$ is given. Set: $h = t+1+j$. Then $e(h) = \hat{e}(h) = s+2+j$. We are given: $\sigma_h(\nu_h) = \hat{\sigma}_h(\nu_h)$. Set $\nu'_{e(h)} =: \sigma_h(\nu_h)$. This determines a potential extension of $I'|e(h)+1$, since:

$$\nu'_{e(h)} > \sigma_h(\nu_l) \geq \nu'_{e(l)} \text{ for } t \leq l < h$$

But M'^*_h is $*$-extendible by $E^{M'_{e(h)}}_{\nu_{e(h)}}$ by unique iterability. Hence by Lemma 3.7.2, $e|h+2$ inserts $I|h+2$ into a unique $I'|e(h)+2$ extends $I'|e(h)+1$ by Lemma 3.7.2.

QED(Case 2)

Case 3: $i = \lambda$ is a limit ordinal.

We first observe that the componentwise union $I' = \bigcup_{i<\lambda} I'|e(i)$ is the unique iteration of length $e(\lambda)$ into which $e|\lambda$ inserts $I|\lambda$. Now let b' be the unique cofinal well founded branch in $I'|e(\lambda)$. Then $b = \{i : e(i) \in b'\}$is the unique cofinal well founded branch in $I|\lambda$. Hence $b = T``\{\lambda\}$. By Lemma 3.7.1 (18), $e|\lambda + 1$ inserts $I|\lambda + 1$ into a unique $I'|e(\lambda) + 1$ extending $I'|e(\lambda)$.

QED(Case 3)

QED(Claim)

We must still consider the case that τ is not a cardinal in M_η. If $t < \eta$, then τ is not a cardinal in $J^{E^{M_t}}_{\lambda_t}$ since $J^{E^{M_t}}_{\lambda_t} = J^{E^{M_\eta}}_{\lambda_t}$ and λ_t is a cardinal in M_η. M'^*_s thus has the form: $M_t||\mu = M_\eta||\mu$. (Hence we truncate to the same place that we would if we applied F directly to M_η). Clearly $\mu < \lambda_t < \nu_t$ if $t < \eta$. Hence the "copying" process we performed in the previous case is impossible. (Note, too, that $t = s$, since if $t < s$, then λ_t would be inaccessible in $J^{E^{M_s}}_{\nu}$ and $\tau < \lambda_t$ would be a cardinal in $J^{E^{M_s}}_{\lambda_t} = J^{E^{M_t}}_{\lambda_t}$. Contradiction!). We set:

$$I^\nu = I|t + 1$$

We can extend I^* to I' by setting $\nu'_t = \nu$. Set $e\upharpoonright t = \mathrm{id}, e(t) = s + 1 = t + 1$. Then e inserts I^* into I'.

The I' which we have described above is called a *simple reiteration* of I. If I' is obtained by a chain of simple reiterations, we also call it a simple reiteration. However, we must still show that an infinite chain of simple reiterations has a well founded limit. This will require considerable effort. Before doing that we develop the notion of *normal reiteration*, which is easier to deal with.

Now let $\langle I^i : i < \omega\rangle$ be a chain of simple reiterations with

$$I^0 = \langle\langle M^i_h\rangle, \langle\nu^i_h\rangle, \langle\pi^i_h\rangle, T^i\rangle \text{ of length } \eta_i.$$

Let I^{i+1} be obtained from I^i by interpolating $F_i = E^{M^i_{\xi_i}}_{\nu_i}$ into I^i, giving rise to the insertion e^i of I^{i*} into I^{i+1}. In an effort to tame the complexity of these structures, we could impose the normality condition: $\nu_i < \nu_j$ for $i < j < \omega$. It turns out that we can impose a far more powerful normality condition by requiring that F_i be interpolated in the *earliest possible* I^h with $h \le i$, rather than necessarily into I_i itself. This gives the concept of *normal reiteration*, which is clearly analogous to that of normal iteration. First, however, we must redo our definitions in order to make this notion precise. To say that I^h is a *possible* candidate for interpolation of F_i means simply that $h \leq i$ and $I^h|t+1 = I^i|t+1$, where t is defined from as before from ν_i, I^i. In a normal reiteration it will then turn out that either $t = \eta_h$ or $\nu_t^i \leq \nu_t^h$ (ν_t^i will exits if $h < i$). In a normal reiteration we will then have: $I^j|t+1 = I^i|j+1$ for $h \leq j \leq i$.

We now give a precise definition of the operation we perform when we apply F_i to I^h.

Definition 3.7.6. Let $I = \langle\langle M_h^i\rangle, \langle\nu_h^i\rangle, \langle\pi_h^i\rangle, T\rangle$ be a normal iteration of M of length η. Let
$$I' = \langle\langle M_h'^i\rangle, \langle\nu_h'^i\rangle, \langle\pi_h'^i\rangle, T'\rangle$$
be a normal iteration of M of length η'. Let $F = E_\nu^{M'_\eta} \neq \varnothing$. Set:
$$\kappa =: \mathrm{crit}(F), \lambda = \lambda(F) =: F(\kappa), \tau = \kappa^{+M||\nu}.$$
Let s be least such that
$$s = \eta' \ \vee \ (s < \eta' \wedge \nu' < \nu_s)$$
Let t be least such that:
$$t = \eta^i \ \vee \ (t = \eta^i \wedge \kappa' < \lambda'_t)$$
(Hence $t \leq s$).

Assume that $I|t+1 = I'|t+1$ and $\nu'_t \leq \nu_t$. We define an operation:
$$W(I, I', \nu) = \langle I^*, I'', e\rangle$$
by cases as follows:

Case 1: $t = \eta$ and τ is a cardinal in M_η.

Extend I to I'' by appointing $\nu''_\eta = \nu$. Then $\pi''_{\eta,\eta+1} : M \longrightarrow_F^* M_{\eta+1}$. e is then the insertion of I into I'' defined by $e \upharpoonright \eta = \mathrm{id}, e(\eta) = \eta + 1$. (Hence $\pi_\eta = \pi'_{\eta,\eta+1}$ and $\sigma_\eta = \mathrm{id} \upharpoonright M_\eta, \tilde{\sigma}_\eta = \tilde{\pi}_\eta \sigma_\eta$). We set: $I^* = I$.

Case 2: $t < \eta$ and τ is a cardinal in M_η. We set $I''|s+1 = I'|s+1$. We then appoint $\nu''_s = \nu$. Thus $t = T''(s+1)$ and $M_s''^* = M_t||\mu$, where $\mu \leq \mathsf{ON}_{M_t}$ is maximal such that τ is a cardinal in $M_t||\mu$. But τ is a cardinal in $J_{\nu_t}^{E^{M_t}} = J_{\nu_t}^{E^{M_\eta}}$. Hence $\mu \geq \nu_t$. Let f be the insertion of $I|t+1$ into $I''|s+2$ defined by
$$f \upharpoonright t = \mathrm{id}, f(t) = s+1.$$
Then:
$$\hat{\sigma}_t = \mathrm{id} \upharpoonright M_t, \pi_t = \pi_{t,s+1}, \sigma_t = \pi_t \circ \sigma_t$$

(Hence $\sigma_t(\nu_t) > \nu''_t$ as before).

Now define e on $\eta + 1$ by

$$e \upharpoonright t = \mathrm{id}, e(t+i) = s+1+i.$$

Set $\eta'' =: e(\eta)$. I'' is then the unique iteration of length $\eta''+1$ extending $I'|s+2$ such that e inserts I into I''. We set: $I^* =: I$.

The existence and uniqueness proofs are exactly as before.

Case 3: τ is not a cardinal in M_η. If $t < \eta$, then τ is not a cardinal in $J^{E^{M_t}}_{\nu_t}$. Hence $M''^*_s = M_t||\mu$, where $\mu < \nu_t$. Set: $I^* =: I|t+1$. Set: $\nu''_s =: \nu$. This gives:

$$\pi''_{t,s+1} : M''^*_s \longrightarrow^*_F M''_{s+1}$$

which defines $I'' = I''|s+2$. e is thus the insertion of I^* into I'' defined by: $e \upharpoonright t = \mathrm{id}, e(t) = s+1$.

Note that $e \upharpoonright t = \mathrm{id}$ (hence $\hat{e} \upharpoonright t+1 = \mathrm{id}$ in all three cases.)

This completes the definition. We are now in a position to define the notion of *normal reiteration*. First, however, we prove a particularly useful lemma:

Lemma 3.7.10. *If $j \in (t,s]$ and $s < \mu$, then $j \not<_{T''} \mu$.*

Proof. We proceed by induction on μ.

Case 1: $\mu = s+1$. Then $t = T''(\mu)$ and $j \not<_{T''} t$, since $t < j$. Hence $j \not<_{T''} \mu$.

Case 2: $\mu > s+1$ is a successor. Let $\mu = \gamma + 1$. Then $\gamma \geq s+1$ and $\gamma = e(\overline{\gamma})$ where $\bar{\gamma} \geq t$. Let $\xi = T''(\gamma+1)$. Let $j \in (t,s]$ such that $j <_{T''} \mu$, then $j \leq_{T''} \xi$. We derive a contradiction. Let $\overline{\xi} = T(\overline{\gamma}+1)$. Then:

$$\hat{e}(\overline{\xi}) \leq_{T''} \xi \leq_{T''} e(\overline{\xi}).$$

If $\overline{\xi} = t$, then $t \leq_{T'} \xi \leq_{T''} s+1$. Hence $\xi \notin (t,s]$ by Case 1. Hence either $\xi = t < j$ or $\xi = s+1 >_{T'} j$, contradicting the induction hypothesis. If $\overline{\xi} < t$ then $\xi = \hat{e}(\overline{\xi}) = e(\overline{\xi}) = \overline{\xi} < j$. Contradiction! If $\overline{\xi} > t$, then $\xi = \hat{e}(\overline{\xi}) = e(\xi) \geq s+1$. Hence $j <_{T'} \xi < \mu$, contradicting the induction hypothesis.

QED(Case 2)

Case 3: μ is a limit ordinal.

Pick $i <_{T''} \mu$ such that $i > s$. Then $j \not<_{T''} i$ by the induction hypothesis. Hence $j \not<_{T''} \mu$.

QED(Lemma 3.7.10)

As we have seen, if e is an insertion of I to I' and $h = T(i+1)$, then the determination of $e^*(i) = T'(e(i)+1)$ is important. In the case of the e defined above, this determination is as follows:

Lemma 3.7.11. *Let $h = T(i+1)$. If $\kappa_i < \kappa$, then $\hat{e}(h) = h = T''(e(i)+1)$. If $\kappa_i \geq \kappa$, then $e(h) = T''(e(i)+1)$, where $e(h) > s+1$.*

Proof. Let $h' = T''(e(i)+1)$. We know:

$$\hat{e}(h) \leq_{T'} h' \leq_{T'} e(h).$$

The cases: $h < t$ and $h > t$ are straightforward. Now let $h = t$. As in Case 2 of the above proof we conclude: $h' = t$ or $h' = s+1$. But $\kappa''_{e(i)} = \pi(\kappa_i)$, where $\pi = \pi''_{t,s+1}$. Hence, if $\kappa_i < \kappa = \text{crit}(\pi)$ we have: $\pi(\kappa_i) = \kappa_i < \lambda_t$. Hence $h' = t$. If $\kappa \leq \kappa_i$, then: $\pi(\kappa_i) \geq \pi(\kappa) = \lambda \geq \lambda_i$. Hence $h' = s+1$.

QED(Lemma 3.7.11)

We now turn to the definition of a *normal reiteration.*

$R = \langle\langle I^i : i < \eta\rangle, \langle\nu_i : i+1 < \eta\rangle, \langle e^{i,j} : i \leq_T j\rangle, T\rangle$ is a *normal reiteration on M* iff the following hold:

(a) $\eta \geq 1$ and each $I^i = \langle\langle M^i_h\rangle, \langle\nu^i_h\rangle, \langle\pi^i_h\rangle, \tau^i\rangle$ is a normal iteration of M of length $\eta_i + 1$.

(b) T is a tree on η such that $iTj \longrightarrow i < j$.

(c) $F_i =: E^{M^i_{\eta_i}}_{\nu_i} \neq \varnothing$. Moreover, $\nu_i < \nu_j$ for $i < j$.

Set: $\kappa_i =: \text{crit}(F_i), \lambda_i = \lambda(F_i) =: F_i(\kappa_i)$, $\tau_i = \tau(F_i) =: \kappa^{+J^E_{\nu_i}}$, where $E = E^{M^i_{\eta_i}}$.

(d) $e^{i,j}$ inserts a segment $I^i|\mu$ into I^j. Moreover, $e^{h,i} = e^{ij} \circ e^{hi}$ for $h \leq_T i \leq_T j$. e^{ii} is the identical insertion on I^i.

(e) Set: $s = s_i =:$ the least s such that $s = \eta_i$ or $s < \eta_i$ and $\nu_i < \nu^i_s$. Then: $I^i|s+1 = I^j|s+1$ and $\nu^j_s = \nu_i$ for $i < j \leq \eta$.

(f) Let $i+1 < \eta$. Let h be least such that $h = i$ or $h < i$ and $\kappa_i < \lambda_h$. Then h is the immediate predecessor of $i+1$ in T. (In symbols: $h = T(i+1)$). Before continuing with the definition, we note some consequences:

Set:

$$t = t_i =: \text{ the least } t \text{ such that } t = \eta_i \text{ or } t < \eta_i \wedge \kappa < \lambda^i_t$$

(Hence $t_i \leq s_i$). In the following assume: $h = T(i+1), t = t_i$. Then:

(1) $I^i|t+1 = I^h|t+1$. Moreover $\nu^h_t \geq \nu^i_t$ if $t < \eta_h$.
Proof. If $h = i$ this is trivial. Now let $h < i$. Then

$$\kappa < \lambda_h = \lambda^i_{s_h} \text{ by (e).}$$

Hence $t \leq s_h$. Clearly by (e) we have:

$$I^h|s_h + 1 = I^i|s_h + 1 \text{ and } \nu^i_{s_h} = \nu_h \tag{*}$$

Hence $I^h|t+1 = I^i|t+1$. If $t = s_h$, we then have: $\nu^h_t > \nu_h = \nu^i_t$ if $t < \eta_h$. If $t < s_h$, then: $\nu^h_t = \nu^i_t$ by (*).

QED(1)

(2) h is least such that $I^i|t = I^h|t$.
Proof. Let $l < t$. Then $\lambda^i_{s_l} = \lambda l \leq \kappa < \lambda^i_t$. Hence $s_l < t$. But $\nu^h_{s_l} = \nu_l < \nu^l_{s_l}$ if $s_l < \eta_l$. Hence $I^l|t \neq I^h|t$.

QED (2)

By (1), the conditions for forming $W(I^h, I^i, \nu_i)$ are given. Our next axiom reads:

(g) Let $h = T(i+1)$. Then $e^{h,i+1}$ inserts I^i_* into I^{i+1} where:

$$\langle I^i_*, I^{i+1}, e^{h,i+1}\rangle = W(I^h, I^i, \nu_i)$$

We define:

Definition 3.7.7. $i+1$ is a *drop point* (or *truncation point*) in R iff τ_i is not a cardinal in $M^h_{\eta_h}$ where $h = T(i+1)$. (This is the only case in which $I^i_* \neq I^h$ is possible).

Our final axioms read:

(h) If $\lambda < \eta$ is a limit ordinal, then $T``\{\lambda\}$ is club in $\lambda$. Moreover, $T``\{\lambda\}$ contain at most finitely many drop points.

(i) If λ is as above and $(h, \lambda)_T$ has no drop points, then $e^{i,\lambda}$ inserts I^h into I^λ and:

$$I^\lambda, \langle e^{i,\lambda} : h \leq_T i \leq_T \lambda\rangle$$

is the good limit of:

$$\langle I^i : h \leq_T i <_T \lambda\rangle, \langle e^{i,j} : h \leq_T i \leq_T j < \lambda\rangle$$

Note. As usual, we will then refer to $I^\lambda, \langle e^{i,\lambda} : i <_T \lambda\rangle$ as the direct limit of:

$$\langle I^i : i \leq_T \lambda\rangle, \langle e^{i,j} : i \leq_T j < \lambda\rangle,$$

since the missing points are supplied by: $e^{l,\lambda} = e^{h,\lambda} \circ e^{l,h}$ for $l \leq h$.

Definition 3.7.8. If $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle$ is a reiteration of length η and $o < \mu \leq \eta$, we let $R|\mu$ denote:

$$\langle\langle I^i : i < \mu\rangle, \langle\nu_i : i+1 < \mu\rangle, \langle e^{i,j} : i \leq_T j < \mu\rangle, T \cap \mu^2\rangle$$

Lemma 3.7.12. *If R is a reiteration and $0 < i \leq \mathrm{lh}(R)$. Then $R|i$ is a reiteration.*

Lemma 3.7.13. *Let $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{ij}\rangle, T\rangle$ be reiteration of length $\gamma+1$, where I^i have length $\eta_i + 1$ for $i \leq \gamma$. Let $E_\nu^{M^\gamma_{\eta\gamma}} \neq \varnothing$, where $\nu > \nu_i$ for $i < \gamma$. Then there is a unique extension of B to a reiteration R' of length $\gamma+2$ such that $R'|\gamma+1 = R$ and $\nu'_\gamma = \nu$.*

Proof. Let $i = T'(\gamma+1)$. Then $W(I^i, I^\gamma, \nu)$ is defined.

A much deeper result is:

Lemma 3.7.14. *Let R be a reiteration of limit length η. There is a unique extension R' such that $R'|\eta = R$ and $\mathrm{lh}(R') = \eta+1$.*

The proof of this theorem will be the main task of this subsection. It will require a long train of lemmas.

For now on let:

$$R = \langle\langle I^\xi\rangle, \langle\nu_\xi\rangle, \langle e^{\xi,\mu}\rangle, T\rangle$$

be a reiteration of limit length η. Let:

$$I^\xi = \langle\langle M_i^\xi\rangle, \langle\nu_i^\xi\rangle, \langle\pi_{ij}^\xi\rangle, T^\xi\rangle$$

be of length $\eta_\xi + 1$ for $\xi < \eta$.

Lemma 3.7.15. *Let $\xi < \mu < \eta$. Then:*

(a) $s_\xi < s_\mu$

(b) $\nu_\xi = \nu^\mu_{s_\xi}$

Proof. (b) holds by (e) in Definition **??**. We prove (a). Suppose not. $\eta_\mu > s_\xi$ since $\nu^\mu_{s_\xi}$ exists. Hence $s_\mu < \eta_\mu$. Hence $\nu_\mu < \nu^\mu_{s_\mu} \leq \nu^\mu_{s_\xi} = \nu_\xi$. Contradiction!

QED(Lemma 3.7.15)

Lemma 3.7.16. *Let $\xi + 1 \leq_T \mu$. Then $e^{\xi+1,\mu} \upharpoonright s_\xi + 1 = \mathrm{id}$.*

We proved by induction on μ. For $\mu = \xi+1$ it is trivial. Now let $\xi+1 <_T \mu+1$ and let it hold at $\gamma = T(\mu + 1)$. Then $\xi < \gamma$ and hence: $\kappa_\mu \geq \lambda_\xi = \lambda^\mu_{s_\xi}$. Hence $t_\mu \geq s_\xi + 1$ and:

$$e^{\gamma,\mu+1} \upharpoonright t_\mu = \mathrm{id}$$

by (g). Hence:

$$e^{\xi+1,\mu+1}(\alpha) = e^{\gamma,\mu+1} e^{\xi+1,\gamma}(\alpha) = \alpha \text{ for } \alpha \leq s_\eta.$$

Now let μ be a limit ordinal and let the induction hypothesis hold at γ for all γ with: $\xi + 1 \leq_T \gamma <_T \mu$. For $i \leq_T j <_T \mu$ we then have: $e^{i\mu}(\alpha) = e^{j\mu} e^{ij}(\alpha) = e^{j\mu}(\alpha)$.

Let $\alpha \leq s_\xi$ be least such that $\alpha < e^{\xi+1,\mu}(\alpha)$. Let $\xi + 1 \leq_T \delta <_T \mu$ such that $e^{\delta,\mu}(\overline{\alpha}) = \alpha$. Then $\overline{\alpha} < \alpha = e^{\xi+1,\delta}(\alpha)$. Hence $e^{\delta,\mu}(\overline{\alpha}) = \overline{\alpha} < \alpha$. Contradiction!

QED(Lemma 3.7.16)

Definition 3.7.9. $\hat{s}_\gamma =: \mathrm{lub}\{s_\xi : \xi < \gamma\}$.

Lemma 3.7.17. *Let $\gamma = T(\xi + 1)$. Then $\hat{s}_\gamma \leq t_\xi \leq s_\gamma$.*

Proof.

(1) $\hat{s}_\gamma \leq t_\eta$, since if $i < \gamma$, then $\lambda_i = \lambda^\gamma_{s_i} \leq \kappa_\xi$.

(2) $t_\xi \leq s_\gamma$.

This is trivial for $\gamma = \xi$. Now let $\gamma < \xi$. Then $\kappa_\eta < \lambda_\gamma = \lambda^\xi_{s_\gamma}$. Hence $t_\xi \leq s_\gamma$.

QED(Lemma 3.7.17)

Definition 3.7.10. X is *in limbo* at μ iff $X \subset \hat{s}_\mu$ and there is no pair $\langle i, j \rangle$, such that $i \in X$, $j \geq \hat{s}_\mu$ and $i <_{T^\mu} j$.

Lemma 3.7.18. *If $\xi + 1 \leq_T \mu$, then $(t_\xi, s_\xi]$ is in limbo at μ.*

Proof. By induction on μ.

Case 1: $\mu = \xi + 1$ by Lemma 3.7.10.

Case 2: $\mu = \delta + 1 >_T \xi + 1$.

Let $\gamma = T(\delta+1)$. Then it holds at γ. Moreover, $\hat{s}_\gamma \leq t_\gamma \leq s_\gamma$. Let $i \in (t_\xi, s_\xi]$ and $i <_{T^\mu} j$, where $j \geq \hat{s}_\mu = s_\delta + 1$. We derive a contradiction.

$j \geq \hat{s}_\mu = s_\delta + 1$. Hence $j = s_\delta + 1 + l$. Hence $e^{\gamma,\mu}(k) = j$, where $k = t_\delta + l$. Since $e^{\gamma,\mu}(i) = i$, we conclude: $i <_{T^\gamma} k$, where $\hat{s}_\gamma \leq t_\delta \leq k$. Contradiction!

QED(Case 2)

Case 3: μ is a limit ordinal.

Suppose $i \in (t_\xi, s_\xi]$ with $i \leq_{T^\mu} h$, $h \geq \hat{s}_\mu$. Then $h = e^{\gamma+1,\mu}(\overline{h})$ for a γ such that

$$\xi + 1 <_{T^\mu} \gamma + 1 <_{T^\mu} \mu$$

But $e^{\gamma+1,\mu} \restriction s_\gamma + 1 = \mathrm{id}$ by Lemma 3.7.16. Hence $\overline{h} > s_\gamma$. Hence $\overline{h} \geq \hat{s}_\gamma = s_\gamma + 1$. Hence $i \not<_{T^{\gamma+1}} \overline{h}$ by the induction hypothesis. Hence $i \not<_{T^\mu} h$.

QED(Lemma 3.7.18)

By Lemma 3.7.16, $I^\xi | s_\xi + 1 = I^\gamma | s_\xi + 1$ for $\xi \leq \gamma < \eta$. The componentwise union:

$$\tilde{I} = \bigcup_{\xi<\eta} I^\xi | s_\xi$$

is then a normal iteration of length

$$\tilde{\eta} = \mathrm{lub}\{s_\xi : \xi < \eta\}$$

For $\xi < \tilde{\eta}$ set:

Definition 3.7.11. $\gamma(i) =:$ the least γ such that $i \leq s_\gamma$.

(Hence $\hat{s}_\gamma \leq i \leq s_\gamma$). The following lemma establishes an important connection between the normal iteration $\tilde{I}$ and the reiteration R.

Lemma 3.7.19. *Let $i \leq_{\tilde{T}} j$. Then $\gamma(i) \leq_T \gamma(i)$.*

Proof. Suppose not. Let i, j be a counterexample. Then $\gamma(i) \not\leq_T \gamma(j)$. Hence $i < j$ and $\gamma(i) < \gamma(j)$. Set: $\gamma = \gamma(j)$. There is $\mu + 1 \leq_T \gamma$ such that $T(\mu+1) < \gamma(i) < \mu + 1$. Set $\tau = T(\mu+1)$. Then $s_\tau < i$, since $\tau < \gamma(i)$. Hence $t_\mu \leq s_\tau < i$ by Lemma 3.7.17. But $i \leq s_{\gamma(i)} \leq s_\mu$, since $\gamma(i) \leq \mu$. Hence $i \not<_{T^\gamma} j$ by Lemma 3.7.18, since $j \geq \hat{s}_\gamma$. Hence $i \not<_{\tilde{T}} j$, since $I^\gamma | s_\gamma + 1 = \tilde{I} | s_\gamma + 1$. Contradiction!

QED(Lemma 3.7.19)

Lemma 3.7.20. *Let $\tau = T(\xi+1) \leq_T \mu$. Then:*

$$\mathrm{crit}(e^{\tau,\mu}) = t_\xi \text{ and } e^{\tau,\mu}(t_\xi) \leq \hat{s}_\mu$$

Proof. By induction on μ.

Case 1. $\mu = \xi + 1$. $e^{\tau,\xi+1}(t_\xi) = s_\xi + 1 = \hat{s}_{\xi+1} > t_\eta$, but

$$e^{t,\xi+1}(i) = \hat{e}^{\tau,\xi+1}(i) = i \text{ for } i < t_\xi$$

Case 2. $\mu = \delta + 1$ is a successor.

Let $\gamma = T(\delta + 1)$. Then:

$$\begin{aligned} e^{\tau,\mu}(t_\xi) &= e^{\gamma,\mu} \circ e^{\tau,\mu}(\hat{s}_\gamma) \\ &\leq e^{\gamma,\mu}(t_\delta) = s_\delta + 1 = \hat{s}_\mu \end{aligned}$$

By the induction hypothesis we have:

$$e^{\tau,\mu}(t_\xi) = e^{\gamma,\mu} \circ e^{\tau,\gamma}(e_\xi) \geq e^{\tau,\gamma}(t_\xi) > t_\eta$$

For $i < t_\xi$ we have:

$$e^{\tau,\mu}(i) = e^{\gamma,\mu} e^{\tau,\gamma}(i) = e^{\gamma,\mu}(i) = i$$

(since $i < t_\gamma$).

QED(Case 2)

Case 3. μ is a limit cardinal. Then $e^{\tau,\mu} \restriction t_\xi = \text{id}$, since $e^{\tau,\gamma} \restriction t_\xi = \text{id}$ for $t \leq_T \gamma <_T \mu$ (cf. the proof of Lemma 3.7.16). Moreover $e^{\tau\mu}(t_\xi) \geq e^{\tau\gamma}(t_\xi) > t_\xi$.

Claim. $e^{\tau,\mu}(t_\xi) \leq \hat{s}_\mu$.

Proof. Let $h < e^{\tau,\mu}(t_\xi)$. Then $h = e^{\gamma,\tau}(\overline{h})$ where $\xi \leq_T \gamma <_T \mu$. Assume w.l.o.g. that $\gamma = T(\delta + 1)$, where $\delta + 1 <_T \mu$. Then:

$$\overline{h} < e^{\tau,\gamma}(t_\xi) \leq \hat{s}_\gamma \leq t_\delta.$$

But $e^{\gamma,\mu} \restriction t_\delta = \text{id}$ by the induction hypothesis.

Hence:

$$h = e^{\gamma,\mu}(\overline{h}) = \overline{h} < \hat{s}_\gamma \leq \hat{s}_\mu$$

QED(Lemma 3.7.20)

In order to prove Theorem 3.7.14 we must find a cofinal branch b in T such that

$$\langle I^i : i \in b \rangle, \langle e^{i,j} : i < j \text{ in } b \rangle$$

has a good limit. An obvious necessary condition is that

$$\langle \eta_i : i \in b \rangle, \langle e^{i,j} : i < j \text{ in } b \rangle$$

have a transitivized direct limit:

$$\eta, \langle e^i : i \in b \rangle.$$

Note. This does not say that e^i inserts I^i into a good limit I. It simply gives us a system of indices which, with luck, might be used to construct a good limit.

We obtain a rather surprising result:

Lemma 3.7.21. *Let b be any cofinal branch in T. Then the commutative system:*

$$\langle \eta_i : i \in b \rangle, \langle e^{i,j} : i \leq j \textit{ in } b \rangle$$

has a well founded limit.

Note. This is surprising since, as we shall see, there is only one branch which yields a good limit, whereas these could be many cofinal branches.

We now turn to the proof of Lemma 3.7.21. Let $i_0 \in b$ such that there is no drop point in $b \setminus i_0$. Hence $e^{i,j}(\eta_i) = \eta_i$ for $i \leq j$, $i, j \in b$. Let $\hat{\eta} + 1$, $\langle e^i : i \in b \setminus i_0 \rangle$ be the direct limit of

$$\langle \eta_i + 1 : i \in b \setminus i_0 \rangle, \langle e^{i,j} : i \leq j \text{ in } b \setminus i_0 \rangle$$

We claim that $\hat{\eta}$ is well founded.

Set: $\tilde{\kappa}_\tau =: t_\xi$ for $\tau, \xi + 1 \in b \setminus i_0, \tau = T(\xi + 1)$. Using Lemma 3.7.20 it is straightforward to see that:

(a) $\hat{e}^{\tau,\mu} \upharpoonright \tilde{\kappa}_\tau = \text{id}$ for $\tau \leq \mu$ in $b \setminus i_0$.

(b) $\tilde{\kappa}_\tau < e^{\tau,\xi+1}(\tilde{\kappa}_\tau) \leq \tilde{\kappa}_{\xi+1}$.

(c) $e^{\tau,\xi+1}(\tilde{\kappa}_\tau + j) = e^{\tau,\xi+1}(\tilde{\kappa}_\tau) + j$.

(d) If τ is a limit ordinal, then:

$$\eta_\tau = \bigcup \{\text{rng}\, e^{i,\tau} : i_0 < i < \tau \text{ in } b\}.$$

Given this, the conclusion follows from a sublemma, which -in an effort to simplify notation- we formulate abstractly:

Sublemma. *Let η be a limit ordinal. Let $\langle\delta_i : i < \eta\rangle$ be a sequence of ordinals and $e_{ij} : \delta_i \longrightarrow \delta_j$ $(i \leq j < \eta)$ be a commutative system of order preserving maps. Let*

$$\Delta, \langle e_i : i < \eta\rangle$$

be the direct limit of

$$\langle\delta_i : i < \eta\rangle, \langle e_{i,j} : i \leq j < \eta\rangle$$

Let $<_\Delta$ be the induced order on Δ. Assume that $\kappa_i < \delta_i$ for $i < \eta$ such that the following hold:

(a) $e_{i,j} \upharpoonright \kappa_i = \mathrm{id}$

(b) $\kappa_i < e_{i,i+1}(\kappa_i) \leq \kappa_{i+1}$

(c) $e_{i,i+1}(\kappa_i + j) = e_{i,i+1}(\kappa_i) + j$

(d) $\delta_\lambda = \bigcup_{i<\lambda} \mathrm{rng}(e_{i,\lambda})$ *for limit* $\lambda < \eta$.

Then $<_\Delta$ is well founded.

Proof. Set $\tilde{\Delta} = \mathrm{wfc}(\langle\Delta, <_\Delta\rangle)$. Assume w.l.o.g. that $\tilde{\Delta}$ is transitive and $<_\Delta \cap \tilde{\Delta}^2 = \in \cap \tilde{\Delta}^2$. Thus, our assertion amounts to: $\tilde{\Delta} = \Delta$.

(1) $\kappa_j \geq \kappa_i$ for $j > i$.

Proof. Otherwise $e_{i,j+1}(\kappa_i) > \kappa_j$ where $\kappa_j < \kappa_i$, contradicting (a).

(2) $\kappa_j > \kappa_i$ for $j > i$.

Proof. $\kappa_j \geq \kappa_{j-1} > \kappa_i$ by (b).

(3) Let $e_i(h) \in \tilde{\Delta}$. Let $\mu \leq \delta_i$ and:

$$e_{i,j}(h + l) = e_{i,j}(h) + l \text{ for } i \geq j \text{ and } h + l < \mu.$$

Then $e_i(h + l) = e_i(h) + l$ for $h + l \leq \mu$.

Proof. Suppose not. Let l be the least counterexample. Then $l > 0$. Let $e_j(\alpha) = e_i(h) + l$ for a $j \geq i$. Then $e_{ij}(h) < \alpha < e_{ij}(h) + l$, since

$$e_j e_{ij}(h) < e_j(k) < e_j(e_{ij}(h) + l)$$

Hence $\alpha = e_{ij}(h) + k$ for a $k < l$. Hence:

$$e_j(\alpha) = e_j(e_{ij}(h) + k) = e_i(h) + k < e_j(h) + l = e_j(\alpha).$$

Contradiction!

QED(3)

Taking $h = 0$, we have $e_{ij}(l) = i$ for $l < \kappa_i$. Hence:

(4) $\kappa_i \subset \tilde{\Delta}$ and $e_i \upharpoonright \kappa_i = \mathrm{id}$.

(5) Let $e_{ij}(h) \geq \kappa_j$. Then $e_{ij}(h+l) = e_{ij}(h) + l$ for all $h + l < \delta_i$.

Proof. By induction on $j \geq i$. The case $i = j$ is trivial. Now let $j = k+1$, where it holds at k. Then $e_{i,k}(h) \geq \kappa_k$, since otherwise:

$$e_{ij}(h) = e_{k,k+1}e_{ik}(h) = e_{i,k}(h) < \kappa_h < \kappa_j.$$

Hence:

$$\begin{aligned} e_{i,k}(h+l) &= e_{kj}e_{ik}(h+l) = e_{kj}(e_{ik}(h)+l) \\ &= e_{kj}(h) + l \end{aligned}$$

since if $e_{ik}(h) = \kappa_k + a$, then:

$$\begin{aligned} e_{k,k+1}(h+l) &= e_{k,k+1}(\kappa_k + a + l) = e_{k,k+1}(\kappa_k) + a + l \\ &= e_{h,k+1}(\kappa_k + a) + l = e_{k,k+1}(h) + l \end{aligned}$$

Now let j be a limit ordinal. Then:

$$\delta_j, \langle e_{ij} : i < j \rangle$$

is the limit of

$$\langle \delta_i : i < j \rangle,\ \langle e_{h,i} : h \leq i < j \rangle$$

and we apply (3).

QED(5)

We now prove $\Delta \subset \tilde{\Delta}$ by cases as follows:

Case 1: For all $i < \eta, h < \delta_i$ there is $j > i$ such that $e_{ij}(h) < \kappa_j$.

Then $e_i(h) = e_j e_{i,j}(h) \subset \kappa_j$, since $e_j \upharpoonright \kappa_j = \mathrm{id}$. Thus $\Delta = \bigcup_i \mathrm{rng}(e_i) \subset \bigcup_i \kappa_i \subset \tilde{\Delta}$.

Case 2: Case 1 fails.

Then there is i such that for some $h < \delta_{i_0}$, we have: $e_{ij}(h) \geq \kappa_i$ for all $j \geq i$. Since $e_{jk}e_{ik}(h) \geq e_{ik}(h) \geq \kappa_k$ for $i_0 \leq j \leq k$, there is for each $j \geq i_0$ a least h_j such that $e_{jl}(h_j) \geq \kappa_l$ for all $l \geq j$.

Claim. $e_{ij}(h_j) = h_j$ for $i_0 \leq i \leq j$.

Proof. Suppose not. Let j be the least counterexample. Using (3) it follows that $j = l+1$ is a successor. Then $h_j < e_{l,j}(h_l)$. But $h_j \geq \kappa_j \geq e_{lj}(\kappa_l)$.

Hence $h_j = e_{lj}(\kappa_l) + a = e_l(\kappa_l + a)$, where $\kappa_l + a < h_l$. But for $j' \geq j$ we have:

$$h_{l,j}(\kappa_l + a) = h_{j,j'}(e_{l,j}(\kappa_l) + a) \geq \kappa_{j'}.$$

Hence $h_l \leq \kappa_l + a < h_l$. Contradiction!

QED(Claim)

But then $e_i(h_i) = e_j(h_j)$ for $i_0 \leq i \leq j < \eta$. Now let $\tilde{h} = e_i(h_i)$ for $i_0 \leq i < \eta$. Then:

Claim. $\tilde{h} = \bigcup\{h_i : i_0 \leq i < \eta\}$.

Proof. $\tilde{h} = \bigcup_i e_i``h_i$. But if $a < h_i$, then $e_{ij}(a) < \kappa_j$ for some $j \geq i$ by the minimality of h_i. Hence $e_i(a) = e_j(e_{i,j}(a)) = e_{i,j}(a) < h_j$, since $e_j \upharpoonright \kappa_j = \mathrm{id}$.

QED(Claim)

Hence $\tilde{h} \in \tilde{\Delta}$ and:

$$e_j(h_j + l) = \tilde{h} + l \text{ for } h_j + l < \delta_j,$$

by (3), (5). Hence $\mathrm{rng}(e_j) \subset \tilde{\Delta}$ and $\Delta = \tilde{\Delta}$. This proves the sublemma and with it Lemma 3.7.21.

QED(Lemma 3.7.21)

Note that $\eta_0 \geq \tilde{\kappa}_i$ for $i \in b \setminus i_0$ where $e^i(\eta_i) = \hat{\eta}$. Hence as a corollare of the proof we have:

Corollary 3.7.22. *Set $\tilde{\eta}_i =$ the least h such that $e^{i,j}(h) \geq \tilde{\kappa}_j$ for all $j \geq i$. Then $\tilde{\eta}_i$ is defined for sufficiently large i and $e^i(\tilde{\eta}_i) = \tilde{\eta}$. Moreover $\tilde{\eta} =$* $\mathrm{lub}\{\tilde{\eta}_i : i < \eta\}$.

However, in order to prove Theorem 3.7.14 we must find the "right" cofinal branch in T. Lemma 3.7.19 suggests an obvious strategy: Let $\tilde{b}$ be the unique well founded cofinal branch in $\tilde{I}$. Set:

$$\hat{b} = \{\gamma(i) : i \in \tilde{b}\}, b = \{\tau : \bigvee \gamma \in \hat{b},\ \tau \leq_T \gamma\}$$

Then b is a cofinal branch in T. We show that this branch works, thus establishing the existence assertion of Theorem 3.7.14.

By Lemma 3.7.21, the commutative system

$$\langle \eta_i + 1 : i \in b \rangle, \langle e^{i,j} : i \leq j \text{ in } b \rangle$$

has a transitivized direct limit:

$$\hat{\eta}+1, \langle e^i : i \in b\rangle$$

This gives us a system of indices with which to work.

We must show that the commutative insertion system:

$$\langle I^h : h \in b\rangle, \langle e^{h,j} : h \le j \text{ in } b\rangle$$

has a good limit I. By induction on $i < \hat{\eta}$ we, in fact, show:

Lemma 3.7.23. *Let $i < \hat{\eta}$. Then the above commutative system has a good limit $I|i+1$ with respect to i in the sense of Definition 3.7.5 at the end of §3.7.1. In other words, $I|i+1$ has length $i+1$ and $e^\xi \restriction h+1$ inserts $I^\xi|h+1_i$ into $I|i+1$ whenever $e^\xi(h)=i$.*

Remark on notation. In §3.7.1 we showed that there can be at most one good limit below i. We denote this, if it exists, by $I|i+1$. But then $(I|i+1)|h+1 = I|h+1$ by uniqueness.

We recall that we defined: $\tilde{\kappa}_\tau = t_\xi$ where $\tau = T(\xi+1), \xi+1 \in b$, and that $\tilde{\kappa}_\tau = \text{crit}(e^{\tau,j}) = \text{crit}(e^\tau)$ for $\tau < j$ in b.

But then $\tilde{I} = \bigcup_{\tau\in b} I^\tau|\tilde{\kappa}_\tau$, since if $\tau = T(\xi+1), \xi+1 \in b$, then:

$$I^\tau|\tilde{\kappa}_\tau = (I^\xi|s_{\eta+1})|\tilde{\kappa}_\tau = \tilde{I}|\tilde{\kappa}_\tau.$$

But $\bigcup_{\tau\in b}\tilde{\kappa}_\tau = \bigcup_{i<\eta} s_i+1,$ since if $\tau=\delta+1$, then:

$$\hat{s}_\tau = s_\delta + 1 \le t_\xi = \tilde{\kappa}_\tau.$$

We prove Lemma 3.7.23 by induction on $i \le \hat{\eta}$.

Case 1. $i < \tilde{\eta} = \text{lh}(\tilde{I})$.

Let $e^\xi(h) = i$. Let $\xi <_T \tau \in b$, where $i+1 < \tilde{\kappa}_\tau$. Then $e^\xi|h+1 = (e^\tau|i+1)(e^{\xi,\tau}|h+1)$ where $e^\tau|i+1 = \text{id}$. Hence:

$$e^\xi|h+1 = e^{\xi,\tau}|h+1 \text{ inserts } I^\xi|h+1 \text{ into } I^\tau|h+1 = I|h+1$$

QED(Case 1)

Case 2. $i = \tilde{\eta}$.

Let $\tilde{b}$ be the unique cofinal well founded branch in $\tilde{I}$. Let $M_{\tilde{\eta}}, \langle \tilde{\pi}_{i,\tilde{\eta}} : i \in \tilde{b} \rangle$ be the transitivized direct limit of: $\langle M_i : i \in b \rangle$, $\langle \tilde{\pi}_{ij} : i \leq_T j \in \tilde{b} \rangle$. This gives us $I|\tilde{\eta}+1$. We must prove that whenever $e^{\xi}(\overline{\eta}) = \tilde{\eta}, \xi \in b$, then e^{ξ} inserts $I^{\xi}|\overline{\eta}+1$ into $I|\tilde{\eta}+1$. By Lemma 3.7.8 it suffices to show that for arbitrarily large $\xi \in b$:

(∗) $e^{\xi}(\overline{\eta}) = \tilde{\eta}$, where either $\hat{e}^{\xi}(\overline{\eta}) \in \tilde{b}$ or else $\hat{e}^{\xi}(\overline{\eta}) = \tilde{\eta}$ and $\hat{e}^{\xi}(i) \in \tilde{b}$ for all $i <_{T^{\xi}} \overline{\eta}$.

We know: $\tilde{\kappa}_\tau = \mathrm{crit}(e^{\tau,\xi+1}) = t_\xi$ for $\tau = T(\xi+1)$, $\xi+1 \in b$. Set:

$$\tilde{\lambda}_\tau =: e^{\tau,\xi+1}(\tilde{\kappa}_\tau) = s_\xi + 1 \text{ for } \tau = t(\xi+1),\ \xi+1 \in b.$$

Then:

(1) $\tilde{b} \cap \bigcup_{\tau \in b} (\tilde{\kappa}_\tau, \tilde{\lambda}_\tau) = \varnothing$.

Proof. Suppose not. Let $i \in \tilde{b} \cap (\tilde{\kappa}_\tau, \tilde{\lambda}_\tau)$ where $\tau \in b$. Let $\mu > \tau$ such that:

$$\mu \in \hat{b} = \{\gamma(i) : i \in \tilde{b}\}.$$

Let $\mu = \gamma(j), j \in \tilde{b}$. Then $\hat{s}_\mu \leq j \leq s_\mu$. Then $i < j$ in $\tilde{b}$, since:

$$i \leq s_\xi < \hat{s}_\mu \leq j, \text{ where } \tau = T(\xi+1),\ \xi+1 \in b.$$

But $\tilde{T}|s = T^{\mu}|s_\mu + 1$. Hence $i <_{T^\mu} j$ in I^μ. But:

$$(\tilde{\kappa}_\tau, \tilde{\lambda}_\tau) = (t_\xi, s_\xi].$$

Hence $(\tilde{\kappa}_\tau, \tilde{\lambda}_\tau)$ is in limbo at μ, since $\xi+1 \leq_T \mu$. Hence $i \not<_{T^\mu} j$. Contradiction!

QED(1)

Set:

$$A = \{\tau \in b : \hat{s}_\tau < \tilde{\kappa}_\tau\}.$$

The set A strongly determines what happens at $\tilde{\eta}$. We first consider the case:

Case 2.1. A is cofinal in b.

There is then a $\tau_0 \in b$ such that $\hat{s}_\tau = \tilde{\kappa}_\tau$ for all $\tau \in b \setminus \tau_0$. (Recall that, if $T = T(\xi+1)$ and $\xi+1 \in b$, then $\tilde{\kappa}_\tau = t_\xi$ and $\hat{s}_\tau \leq t_\xi \leq s_\tau < \tilde{\lambda}_\tau$ by Lemma 3.7.17.) By (1) we have:

$$\tilde{b} \setminus \tau_0 \subset B =: \{\hat{s}_i : \tau_0 \leq i \text{ in } b\} = \{\tilde{\kappa}_i : \tau_0 \leq i \text{ in } b\}.$$

(2) $\tilde{b} \setminus \tau_0 = B$.

Proof. Suppose not. Let $i \in B \setminus \tilde{b}_0$ be the least counterexample. Then $i > \tau_0$. Moreover, i is not a limit ordinal, since otherwise $i = \text{lub}\{\hat{s}_j : j \in B \cap i\}$, where $B \cap i \subset \tilde{b}$ and $\tilde{b}$ is closed in $\tilde{\eta}$. Hence:

$$i = \hat{s}_{\xi+1} = s_\xi + 1, \text{ where } \xi + 1 \in b \setminus (\tau_0 + 1).$$

Let $\tau = T(\xi + 1)$. Then $\tau \geq \tau_0$ in b and

$$\hat{s}_\tau = \tilde{\kappa}_\tau = t_\xi, \ s_\xi + 1 = \tilde{\lambda}_\xi.$$

Hence $\hat{s}_\tau = \tilde{T}(s_\xi + 1)$, where $\hat{s}_\tau \in B$. Clearly $\hat{s}_\tau \in \tilde{b}$, by the minimality of i. Now let $j+1 \in \tilde{b}$ such that $\hat{s}_\tau = \tilde{T}(j+1)$. Then $j+1 \geq \tilde{\lambda}_\tau = s_\xi + 1$, since $j + 1 > \tilde{\kappa}_\tau$ and $(\tilde{\kappa}_\tau, \tilde{\lambda}_\tau) \cap \tilde{b} = \varnothing$. Let $\gamma = \gamma(j + 1)$. Then $j + 1 = \hat{s}_\gamma = \tilde{\kappa}_\gamma$ is a successor ordinal. Hence $\hat{s}_\gamma = s_\delta + 1$, where $\gamma = \delta + 1$. Let $\mu = T(\delta + 1)$. Then $\hat{s}_\mu = \hat{\kappa}_\mu = \hat{T}(s_\delta + 1)$. Hence $\hat{s}_\mu = \hat{s}_t$. Hence $\mu = \tau, \delta = \xi$ and $i = \hat{s}_\xi + 1 = j + 1 \in \tilde{b}$. Contradiction!

QED(2)

But then every $\tau \in b \setminus \tau_0$ satisfies (*), since:

(3) Let $\tau_0 \leq \tau \in b$. Then $e^\tau(\tilde{\kappa}_\tau) = \tilde{\eta}$ and $e^\tau \restriction \tilde{\kappa}_\tau = \text{id}$. (Hence $\hat{e}^\xi(\tilde{\kappa}_\tau) = \tilde{\kappa}_\tau \in \tilde{b}$).

Proof. We know that if $\tau = T(\xi + 1), \xi + 1 \in b$, then:

$$e^{\tau,\xi+1} \restriction \tilde{\kappa}_\tau = \text{id}, \ e^{\tau,\xi+1}(\tilde{\kappa}_\tau) = \tilde{\lambda}_\tau = s_\xi + 1 = \tilde{\kappa}_{\xi+1}$$

Using this we prove by induction on $\xi \in b \setminus \tau_0$ that if $\tau_0 \leq \tau < \xi$, $\tau \in b$, then:

$$e^{\tau,\xi} \restriction \tilde{\kappa}_\tau = \text{id}, e^{\tau,\xi}(\tilde{\kappa}_\tau) = \tilde{\kappa}_\xi.$$

At limit ξ we use the fact that:

$$e^{\tau,\xi}(i) = \bigcup_{\tau \leq \tau' \in b} e^{\tau',\xi} \text{''} e^{\tau,\tau'}(i).$$

But then the same proof shows:

$$e^\tau \restriction \tilde{\kappa}_\tau = \text{id}, e^\tau(\tilde{\kappa}_\tau) = \tilde{\eta},$$

since:

$$\tilde{\eta} = \sup_{\tau \in b \setminus \tau_0} \tilde{\kappa}_\tau = \sup_{\tau \in b \setminus \tau_0} \hat{s}_\tau = \sup_{\xi+1 \in b \setminus \tau_0} s_\xi + 1.$$

QED(Case 2.1)

Case 2.2. A is cofinal in b.

We shall make use of the following general lemma on normal reiteration:

Lemma 3.7.24. *Let $\xi \leq_T \mu, i \leq \eta_\xi$ such that $\hat{s}_\mu \leq j < e^{\xi,\mu}(i)$. Then $j \in \mathrm{rng}(e^{\sigma,\mu})$.*

Proof. Suppose not. Let μ be the least counterexample. Then $\mu > \xi$.

Case 1. μ is a limit ordinal.

Let ζ such that $\xi \leq \zeta < \mu$ and $j = e^{\zeta,\mu}(j')$. Then $j' \geq \tilde{\kappa}_\zeta$, since otherwise:

$$j = j' < \kappa_\zeta < \tilde{\lambda}_\zeta < \hat{s}_\mu.$$

Contradiction! Thus $\hat{s}_\zeta \leq j' \leq e^{\zeta,\mu}(i)$. By the minimality of μ we conclude:

$$j' \in \mathrm{rng}(e^{\zeta,\mu});$$

hence $j = e^{\zeta,\mu}(j') \in \mathrm{rng}(e^{\zeta,\mu})$. Contradiction!

Case 2. $\mu = \zeta + 1$ is a successor.

Let $\tau = T(\zeta+1)$. Then $j \geq \hat{s}_\mu = s_\zeta + 1 = \tilde{\lambda}_\tau$. Moreover:

$$e^{\tau,\mu}(\tilde{\kappa}_\tau + h) = \tilde{\lambda}_\tau + h \text{ for } h \leq \eta_\tau.$$

Let $j = \tilde{\lambda}_\tau + h, e^{\xi,\mu}(i) = \tilde{\lambda}_\tau + k$. Hence $h < k$. Set $j' = \tilde{\kappa}_\tau + h$. Then $e^{\tau,\mu}(j') = j$, where $\hat{s}_\tau \leq \tilde{\kappa}_\tau \leq j' < e^{\xi,\mu}(i)$. By the minimality of μ we conclude: $j' \in \mathrm{rng}(e^{\xi,\tau})$. Hence $j = e^{\tau,\mu}(j') \in \mathrm{rng}(e^{\xi,\mu})$. Contradiction!

QED(Lemma 3.7.24)

Let $\tau_0 \in b$ such that $\tilde{\eta} \in \mathrm{rng}(\tilde{e}^{\tau_0})$. Then $\tilde{\eta} \in \mathrm{rng}(e^\tau)$ for all $\tau \in b \setminus \tau_0$. Set:

$$\tilde{\eta}_\tau = (e^\tau)^{-1}(\tilde{\eta}) \text{ for } \tau \in b \setminus \tau_0.$$

Then:

(4) $e^\tau(\tilde{\kappa}_\tau) < \tilde{\eta}$ for $\tau \in b \setminus \tau_0$.

Proof. Let $\tau < \gamma \in A$. Then $e^{\tau,\gamma}(\tilde{\kappa}_i) \leq \hat{s}_\gamma < \tilde{\kappa}_\gamma$ by Lemma 3.7.20. Hence:

$$e^\tau(\tilde{\kappa}_\tau) = e^\gamma \cdot e^{\tau,\gamma}(\tilde{\kappa}_\tau) = e^{\tau,\gamma}(\tilde{\kappa}_\tau) < \tilde{\kappa}_\gamma < \tilde{\eta}.$$

QED(4)

Now set:

$$B =: \bigcup_{\tau \in b \setminus \tau_0} [\tilde{s}_\tau, \tilde{\kappa}_\tau).$$

Note. $[\hat{s}_\tau, \tilde{\kappa}_\tau) = \varnothing$ if $\tau \notin A$.

(5) Let $\tau_0 \leq \tau \in b$. Then $B \subset \text{rng}(e^\tau)$.

Proof. Let $\tau \leq \gamma \in A$. Let $j \in [\hat{s}_\gamma, \tilde{\kappa}_\gamma)$. Then

$$\hat{s}_\gamma \leq j \leq \tilde{\eta}_\gamma = e^{\tau,\gamma}(\tilde{\eta}_\gamma).$$

But then by Lemma 3.7.24 we have:

$$[\hat{s}_\gamma, \tilde{\kappa}_\gamma) \subset \text{rng}(e^{\tau,\gamma}).$$

But $e^\gamma \upharpoonright [\hat{s}_\gamma, \tilde{\kappa}_\gamma) = \text{id}$. Hence:

$$[\hat{s}_\gamma, \tilde{\kappa}_\gamma) \subset \text{rng}(e^\tau) = \text{rng}(\tilde{e}^\gamma \tilde{e}^\tau).$$

QED(5)

Since B is cofinal in $\tilde{\eta}$, we conclude:

(6) $e^\tau``\tilde{\eta}_\tau$ is cofinal in $\tilde{\eta}$ for $\tau \in b \setminus \tau_0$. Using this we then get:

(7) Let $\tau \in b \setminus \tau_0$. Then:

$$\tilde{b} \cap (\text{rng}(e^\tau) \cup \text{rng}(\hat{e}^\tau))$$

is cofinal in $\tilde{\eta}$.

Proof. Suppose not. Then there is a $i_0 < \tilde{\eta}$, such that

$$\tilde{b} \cap (\text{rng}(e^\tau) \cup \text{rng}(\hat{e}^\tau)) \subset i_0.$$

Note that if $\gamma \in A$, then $[\hat{s}_\gamma, \tilde{\kappa}_\gamma) \subset \text{rng}(e^\tau)$. Hence $(\hat{s}_\gamma, \tilde{\kappa}_\gamma] \subset \text{rng}(\hat{e}^\tau)$. We shall derive a contradiction by showing that A is not cofinal in b. In particular, we show:

Claim. Let $i_0 < j \in \tilde{b}$. Let $\gamma_0 = \gamma(j)$. Assume that $\gamma \leq \delta \in b$. Then $\hat{s}_\delta = \tilde{\kappa}_\delta \in \tilde{b}$.

Proof. We proceed by induction on δ. There are three cases:

Case 2.2.1. $\delta = \gamma_0$.

It suffices to show: $\gamma_0 \notin A$, since then $\hat{s}_{\gamma_0} \leq j < \tilde{\lambda}_\gamma, j \notin (\tilde{\kappa}_{\gamma_0}, \tilde{\lambda}_{\gamma_0})$, where $\hat{s}_{\gamma_0} = \tilde{\lambda}_{\gamma_0}$. Hence $j = \hat{s}_\gamma = \hat{\kappa}_\gamma \in \tilde{b}$. Suppose not. $j \in [\hat{s}_\gamma, \tilde{\kappa}_\gamma]$ since $(\tilde{\kappa}_\gamma, \tilde{\lambda}_\gamma) \cap \tilde{b} = \varnothing$. But $[\hat{s}_\gamma, \tilde{\kappa}_\gamma] \subset \text{rng}(e^\tau) \cup \text{rng}(\hat{e}^\tau)$. Contradiction!, since $j < i_0$.

QED(Case 2.2.1)

Case 2.2.2. $\delta = \xi + 1 > \gamma_0$ is a successor.

Let $\mu = T(\xi + 1)$. Hence, $\gamma_0 \leq \mu \in b$. Then $s_\mu = \tilde{\kappa}_\mu \in \tilde{b}$. Let $j + 1$ be the immediate successor of s_μ in $\tilde{b}$. Then $\tilde{\kappa}_\mu < j + 1$. Hence

$j+1 \geq \tilde{\lambda}_\mu = s_\xi + 1$, since $(\tilde{\kappa}_\mu, \tilde{\lambda}_\mu) \cap \tilde{b} = \varnothing$. Let $\gamma = \gamma(j+1)$. Then $j+1 \in [\hat{s}_\gamma, \tilde{\kappa}_\gamma]$. Hence, as in Case 2.2.1, $\tilde{\kappa}_\gamma = \hat{s}_\gamma$, since otherwise:

$$[\hat{s}_\gamma, \tilde{\kappa}_\gamma] \subset \mathrm{rng}(e^\tau) \cup \mathrm{rng}(\hat{e}^\tau).$$

Then $j+1 = \hat{s}_\gamma = \tilde{\kappa}_\gamma$ and $\hat{s}_\gamma = s_\xi + 1$, where $\gamma = \zeta + 1$. Let $\xi = T(\zeta+1)$. Then $\tilde{\kappa}_\eta = \tilde{T}(j+1)$, where $j = s_\zeta$. Hence $\tilde{\kappa}_\eta = \hat{s}_\mu = \tilde{T}(j+1)$. Hence $\eta = \mu$, since otherwise $\eta > \mu$ and $\hat{s}_\mu < \hat{s}_\eta = \tilde{\kappa}_\eta$. Hence $\xi = \zeta$, since $\xi + 1 = \zeta + 1 =$ the immediate successor of μ in b. Hence $\hat{s}_\delta = \tilde{\kappa}_\delta \in \tilde{b}$.

QED(Case 2.2.2)

Case 2.2.3. $\delta > \gamma_0$ is a limit ordinal.

Then $\hat{s}_\delta = \sup_{i<\delta} \hat{s}_i \in \tilde{b}$, since $\tilde{b}$ is closed in $\tilde{\eta}$. But then $\hat{s}_\delta = \tilde{\kappa}_\delta$, since otherwise:

$$[\hat{s}_\delta, \tilde{\kappa}_\delta) \subset \mathrm{rng}(e^\tau), \text{ where } \hat{s}_\delta > i_0.$$

QED(Case 2.2.3)

This proves (7).

We now show that (*) holds for all $\tau \in b \setminus \tau_0$.

(8) Let $\tau \in b \setminus \tau_0$. If $i <_{T^\tau} \tilde{\eta}_\tau$, then $\hat{e}^\tau(i) \in \tilde{b}$.

Proof. Set: $\overline{b} = (\hat{e}^\tau)^{-1}{}''\tilde{b}$.

Claim 1. $\overline{b}$ is cofinal in $\tilde{\eta}_\tau$.

Proof. Let $i < \tilde{\eta}_\tau$. Set $i' = e^\tau(i)$. By (7) there is $j' \in \tilde{b}$ such that

$$j' > i' \text{ and } j' \in \mathrm{rng}(e^\tau) \cup \mathrm{rng}(\hat{e}^\tau).$$

If $e^\tau(j) = j'$, then $j > i$ and $\hat{e}^\tau(j) \leq_{\tilde{T}} j' \in \tilde{b}$. Hence $\hat{e}^\tau(j) \in \tilde{b}$ and $j \in \overline{b}$. If $\hat{e}^\tau(j) = j'$, then $\hat{e}^\tau(i) < j' \in \tilde{b}$. Hence $j > i$ and $j \in \overline{b}$.

QED(Claim 1)

Claim 2. $\overline{b}$ is a branch in T^τ.

Proof. Let $i <_{T^\tau} j \in \overline{b}$. Then $\hat{e}^\tau(i) \leq_{\tilde{T}} \hat{e}^\tau(j) \in \tilde{b}$. Hence $\hat{e}^\tau(i) \in \tilde{b}$ and $i \in \overline{b}$.

QED(Claim 2)

Claim 3. $\overline{b}$ is well founded.

This follows by standard methods, given that $\tilde{b}$ is well founded. But then $\overline{b} = T^{\tau}{}''\{\tilde{\eta}_\tau\}$ by uniqueness.

QED(Case 2)

Case 3. $i > \tilde{\eta}$.

Then $e^\tau(\tilde{\eta}_\tau + i) = \tilde{\eta} + i$ by Lemma 3.7.24. Using this, it follows easily by Lemma 3.7.8 and Lemma 3.7.7 that $I|i+1$ exists. We leave the details to the reader.

QED(Lemma 3.7.23)

This proves the existence part of Theorem 3.7.24. We must still prove uniqueness.

Definition 3.7.12. Let b be a cofinal branch in:

$$R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle,$$

where R is a reiteration of limit length η. b is *good for* R iff R extends to R' of length $\eta+1$ with $b = T``\{\eta\}$.

We have proven the existence of a good branch b. Now we must show that it is the only one. Suppose not. Let b^* be a second good branch, inducing R^* of length $\eta+1$ with: $b^* = T^*``\{\eta\}$. Since b, b^* are distinct cofinal branches in T, there is $\tau_0 < \eta$ such that:

$$(b \setminus \tau_0) \cap (b^* \setminus \tau_0) = \varnothing.$$

$I' = (I^\eta)^{R'}$ has length $\hat{\eta}$ and $I^* = (I^\eta)^{R^*}$ has length η^*. However:

$$\tilde{\eta} = \bigcup_{i<\eta} s_i + 1,\ \tilde{I} = \bigcup_{i<\eta} I|s_i + 1$$

remain unchanged. Moreover $I = I'|\tilde{\eta} = I^*|\tilde{\eta}$. Since $\hat{b}$ is the unique cofinal well founded branch in $\tilde{I}$, we must have:

$$\tilde{b} = T'``\{\tilde{\eta}\} = T^*``\{\tilde{\eta}\}.$$

Now let $\gamma > \tau_i$ such that:

$$\gamma = \gamma(i) \in \hat{b} = \{\gamma(i) : i \in \tilde{b}\}$$

Then $\gamma \in b \setminus \tau_0$. Let $\gamma = \gamma(i)$ where $i \in \tilde{b}$. Then $\hat{s}_\gamma \leq i \leq s_\gamma$.

Let δ be least such that $\delta \in b^*$ and $\delta > \gamma_0$. Then $\delta = \xi + 1$ and $\tau =: T^*(\xi+1) < \gamma$. Then $t_\xi \leq s_\tau$. But

$$s_\tau < \hat{s}_\gamma \leq i \leq s_\gamma, \text{ where } s_\gamma + 1 = \hat{s}_{\gamma+1} \leq \hat{s}_\xi = s_\xi + 1.$$

Hence $i \in (t_\xi, s_\xi]$. But then:

$$i < s_\xi + 1 = \tilde{\lambda}^*_\tau \leq \tilde{\kappa}^*_\delta = \mathrm{crit}(e^{*\delta})$$

Hence $e^{*\delta}(i) = i \in b^*$. But $i <_{T^*} \tilde{\eta}$, since $i \in \tilde{b}$. Hence, letting $e^{*\delta}(\tilde{\eta}^*_\delta) = \tilde{\eta}$, we have:

$$i <_T \eta^*_\delta, \text{ where } \tilde{\eta}^*_\delta \geq \hat{s}_0 = s_\xi + 1.$$

But this is impossible, since $(t_\xi, s_\xi]$ is in limbo at δ. Contradiction!

QED(Theorem 3.7.14)

We have shown that, if M is uniquely normally iterable, then it is uniquely normally iterable in the sense that every normal reiteration of limit length has exactly one good branch. As we stated at the outset, the result can be relativized to a regular $\theta > \omega$. In this case we restrict ourselves to θ-reiterations.

Definition 3.7.13. Let $\theta > \omega$ be regular. A normal reiteration $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle$ is called a *θ-reiteration* iff $\mathrm{lh}(R) < \theta$ and $\mathrm{lh}(I^i) < \theta$ for all i. M is *uniquely normally θ-reiterable* iff every θ-reiteration of limit length $< \theta$ has one good branch.

We have shown that, if M is uniquely normally θ-iterable, then it is uniquely normally θ-reiterable. But what if M is, in fact, $\theta + 1$ iterable? Can we strengthen the the conclusion correspondingly? We define:

Definition 3.7.14. Let θ, R be as above. R is a θ+1-reiteration iff $\mathrm{lh}(R) \leq \theta$ and $\mathrm{lh}(I^i) < \theta$ for all i. M is *uniquelly normally $\theta + 1$ reiterable* iff every θ-reiteration of length $\leq \theta$ has a unique good branch.

Now suppose M be normally $\theta + 1$-iterable. Let R be a $\theta + 1$ reiteration of length θ. Define $\tilde{I}, \tilde{b}, \hat{b}, b$ exactly as before. Then b is a cofinal branch in T. (It is also the unique such branch, since if b' were another such, then $b \cap b'$ s club in θ. Hence $b = b'$). b has at most finitely many drop points, since otherwise some proper segment of b would have infinitely many drop points. Suppose that $\gamma \in b$ and $b \setminus \gamma$ has no drop points. Then:

$$\langle\langle I^i : i \in b\setminus\gamma\rangle, \langle e^{i,j} : i < j \in b\setminus\gamma\rangle\rangle$$

has a unique good limit:

$$\langle I, \langle e^i : i \in b\setminus\gamma\rangle\rangle$$

by Lemma 3.7.9. Hence b is a good branch. Thus we have:

Lemma 3.7.25. *If M is uniquely normally iterable, then it is uniquelly normally reiterable. Moreover if $\theta > \omega$ is regular, then:*

(a) *If M is uniquely normally θ-iterable, then it is uniquely normally θ-reiterable.*

(b) *If M is uniquely normally $\theta + 1$-iterable, then it is uniquely normally $\theta + 1$-reiterable.*

Remark. The assumption that M is uniquely normally iterable can be weakened somewhat. We define:

Definition 3.7.15. Let S be a normal iteration strategy for M. S is *insertion stable* iff whenever I is an S-conforming iteration of M and e inserts $\overline{I}$ into I, then $\overline{I}$ is an S-conforming iteration.

Now suppose that M is iterable by an insertion stable strategy S. We can define the notion of a *normal reiteration on* $\langle M, S\rangle$ exactly as before, except that we require each of the component normal iterations I^i to be S-conforming. (We could also call this an *S-conforming normal reiteration on* M). All of the assertions we have proven in this subsection go through for reiterations on $\langle M, S\rangle$, with nominal changes in formulation and proofs. For instance, if we alter the definition of *good branch* mutatis mutandis, our proofs give:

> $\langle M, S\rangle$ is *uniquely reiterable* in the sense that every reiteration of limit length has exactly one good branch.

We close this section with two technical lemmas which will be of use later. Both assume the unique iterability (or θ-iterability) of M.

Lemma 3.7.26. *Let I, I' be normal iterations of M. There is at most one pair $\langle R, \xi\rangle$ such that*

$$R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle,$$

is a reiteration of $M, \mathrm{lh}(R) = \xi + 1, I = I^0, I' = I^\xi$.

Proof. Assume such R, ξ to exist. Ww show that R, ξ are defined by a recursion:

$$R|i + 1 \cong F(R|i)$$

where ξ is least such that $F(R|\xi + 1)$ is undefined. F will be defined solely by reference to I, I'. We have:

$$R|1 = \langle\langle I\rangle, \varnothing, \langle \mathrm{id}\restriction \mathrm{lh}(I)\rangle, \varnothing\rangle.$$

At limit $\lambda, R\restriction\lambda + 1 = F(R|\lambda)$ is given by the unique good branch in $R|\lambda$. Now let $R|i+1$ be given. If $I^i = I'$, then $F(R|i+1)$ is undefined. If not, let $s = s_i$. Then $I^i|s+1 = I'|s+1$, since $\nu_i = \nu_s^{i+1} = \nu'_s$. If $s+1 < \mathrm{lh}(I^i)$, then $\nu_i = \nu'_s < \nu^i_s$. Hence $I^i|s+2 \neq I'|s+2$. We have shown:

$$\begin{aligned} s = {} & \text{the maximal } s \text{ such that } s+1 \leq \mathrm{lh}(I^i) \\ & \text{and } I^i|s+1 = I'|s+1. \end{aligned}$$

But then $R|i+2$ is uniquely defined from $R|i+1$ and $\nu_i = \nu'_s$.

QED(Lemma 3.7.26)

For later reference we state a further lemma about reiterations:

Lemma 3.7.27. *Let $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle$ be a reiteration of length $\mu+1$. Let I^i be of length η_i for $i \leq \mu$. Set:*

$$A_j = A_j^R =: \{i : i <_T j \text{ and } (i,j]_T \text{ has no drop point in } R\}$$

for $j \leq \mu$. Set:

$$\sigma_{i,j} = \sigma^{i,j}_{\eta_i} \text{ for } i \in A_j \text{ or } i = j$$

. Then:

(a) $e^{i,\mu}(\eta_i) = \eta_\mu$ *for $i \in A_\mu$.*

(b) $\sigma_{i,\mu} : M_{\eta_i} \longrightarrow_{\Sigma^*} M_{\eta_\mu}$ *for $i \in A_\mu$.*

(c) *If μ is a limit ordinal, then*

$$M_\eta = \bigcup_{i\in A_\mu} \mathrm{rng}(\sigma_{i,\mu}).$$

Proof. We prove it by induction on μ.

Case 1. $\mu = 0$. Then $A_\mu = \varnothing$ and there is nothing to prove.

Case 2. $\mu = j+1$ is a successor. If μ is a drop point, then $A_\mu = \varnothing$ and there is nothing to prove. Assume that it is not a drop point. Then $h = T(\mu)$ is the maximal element of A_μ. (c) holds vacuously. We now prove (a), (b) for $i = h$. By our construction, $e^{h,\mu}(\eta_h) = \eta_h$ could only fail if μ is a drop point, so (a) holds. We now prove (b) for $i = h$. If $t_j < \eta_h$, then $\hat{e}^{h,\mu} = e^{h,\mu}$ and:

$$\sigma_{h,\mu} = \hat{\sigma}^{h,\mu}_{\eta_h} = \sigma^{h,\mu}_{\eta_h}.$$

Hence (b) holds. Now let $t_j = \eta_h$. Then $\eta_\mu = s_j + 1$ and:

$$\sigma^{h,\mu}_{\eta_h} : M^h_{\eta_h} \longrightarrow^*_F M^\mu_{\eta_\mu},$$

where $F = E^{M_j}_{\nu_j}$. Hence (b) holds.

Now let $i < h$. Then $i \in A_h^{R|h+1}$. This gives us $\sigma_{ih} = \sigma^{i,h}_{\eta_i}$. Then (a)-(c) holds for $R|h+1$ by the induction hypothesis.

By Lemma 3.7.5 we then easily get:

$$\sigma_{h,\mu}\sigma_{i,h} = \sigma_{i,\mu}.$$

It follows easily that (a), (b) hold at i.

QED(Case 2)

Case 3. μ is a limit ordinal. Then $A_\mu = [i_0, \mu)_T$ for a $i_0 <_T \mu$. We know that:

$$\eta_\mu, \langle e^{i,\mu} : i \in A_\mu \rangle$$

is the transitivized direct limit of:

$$\langle \nu_i : i \in A_\mu \rangle, \langle e^{i,j} : i \leq j \text{ in } A_\mu \rangle$$

Hence (a) holds at μ. But:

$$I^\mu, \langle e^{i,\mu} : i \in A_\mu \rangle$$

is the good limit of:

$$\langle I^i : i \in A_\mu \rangle, \langle e^{i,j} : i \leq j \text{ in } A_\mu \rangle$$

(where $e^{j\mu} e^{ij} = e^{i,\mu}$). But then (c) holds by Lemma 3.7.7. Hence (b) holds, since (b) holds for $R|i+1$ whenever $i \in A_\mu$ (hence $A_i = A_\mu \cap i$).

QED(Lemma 3.7.27)

3.7.3 A first conclusion

In this section we prove:

Theorem 3.7.28. *Let M' be a normal iterate of M. Then M' is normally iterable.*

We prove it in the slightly stronger form:

Lemma 3.7.29. *Let $\tilde{I} = \langle \langle \tilde{M}_i \rangle, \langle \tilde{\nu}_i \rangle, \langle \tilde{\pi}_{i,j} \rangle, \tilde{T} \rangle$ be a normal iteration of M of length $\tilde{\eta} + 1$. Let $\tilde{\sigma} : N \longrightarrow_{\Sigma^*} \tilde{M}_{\tilde{\eta}} \min \tilde{\rho}$. Then N is normally iterable.*

First, however, we prove a technical lemma. Recalling the Definition 3.7.6 of the function $W(I, I', \nu)$, we prove:

Lemma 3.7.30. *Let $W(I, I', \nu) = \langle I^*, I'', e \rangle$, where $F, \nu, \kappa, \tau, \lambda, s, t$ are as in 3.7.6. Let I, I^*, I', I'' be of length $\eta + 1, \eta^* + 1, \eta' + 1, \eta'' + 1$ respectively. Let $\sigma = \tilde{\sigma}_{\eta^*}$ be induced by e. Set:*

$$M_* = M_\eta || \mu \text{ whose } \mu \text{ is maximal such that } \tau \text{ is a cardinal } M_\eta || \mu.$$

(Hence $\mathbb{P}(\kappa) \cap M_* = \mathbb{P}(\kappa) \cap J_{\nu'}^{E^{M'}_{\eta'}}$). Then:

(a) $\sigma : M_* \longrightarrow_{\Sigma^*} M''_{\eta''}$

(b) $\sigma(X) = F(X)$ *for* $X \in \mathbb{P}(\kappa) \cap M^*$ *(hence* $\kappa = \mathrm{crit}(\sigma)$*).*

Proof. Case 1. $t = \eta$ and τ is a cardinal in M_η.

Then $\eta^* = \eta, M_* = M, \eta'' = \eta + 1$ and:

$$\sigma_\eta = \pi_\eta = \pi''_{\eta,\eta+1} : M_\eta \longrightarrow^*_F M''_{\eta+1}$$

QED(Case 1)

Case 2. $t < \eta$ and τ is a cardinal in M_η. Then $\eta^* = \eta, M_* = M_\eta$. Moreover, $\hat{\sigma}_\eta = \sigma_\eta$; hence (a) holds. Set:

$M''_* = M_t||\mu$ where μ is maximal such that τ is a cardinal in $M_t||\mu$.

Then $M''_* = M''^*_s$ and:

$$\sigma_t = \pi_t = \pi''_{t,s+1} : M''_* \longrightarrow^*_F M''_{\eta+1}.$$

Note that $\mu \geq \lambda_t$, since λ_t in inaccessible in M_η and $\tau < \lambda_t$ is a cardinal in M_η. Then $\sigma_\eta \upharpoonright \lambda_t = \sigma_t \upharpoonright \lambda_t$ and $J^{E^{M_t}}_{\lambda_t} = J^{E^{M_\eta}}_{\lambda_t}$. Hence $\sigma_\eta \upharpoonright J^{E^{M_\eta}}_{\lambda_t} = \sigma_t \upharpoonright J^{E^{M_t}}_{\lambda_t}$. Hence:

$$\sigma_\eta(X) = \sigma_t(X) = F(X) \text{ for } X \in \mathbb{P}(\kappa) \cap M.$$

QED(Case 2)

Case 3. τ is not a cardinal in M_η. Then $\eta^* = t, \eta'' = s + 1$, and:

$$\sigma_t = \pi_t : M_* \longrightarrow^*_F M''_{s+1}$$

QED(Lemma 3.7.30)

Corollary 3.7.31. *Let:*

$$R = \langle \langle I^i \rangle, \langle \nu_i \rangle, \langle e^{i,j} \rangle, T \rangle,$$

be a reiteration where:

$$I^i = \langle \langle M^i_k \rangle, \langle \nu^i_k \rangle, \langle \pi^i_{k,l} \rangle, T^i \rangle \text{ is of length } \eta_i + 1.$$

Let $\xi = T(i+1)$*. Let* I^i_* *have length* $\eta^* + 1$*. Set:* $M^i_* = M^\xi_{\eta^*}||\mu$*, where* μ *is maximal such that* τ_i *is a cardinal in* $M^\xi_{\eta^*}$*. Then:*

$$\sigma^{\xi,i+1}_{\eta^*} : M^i_* \longrightarrow_{\Sigma^*} M^{i+1}_{\eta_{i+1}} \text{ and:}$$

$$\sigma^{\xi,i+1}_{\eta^*}(X) = E^i_{\nu_i}(X) \text{ for } X \in \mathbb{P}(\kappa_i) \cap M^i_*.$$

Note. $\mathbb{P}(\kappa_i) \cap M^i_* = \mathbb{P}(\kappa_i) \cap J^{E^{M^i}_{\eta_i}}_{\nu_i}$.

Note. This does not say that $M^{i+1}_{\eta_{i+1}}$ is a $*$-ultrapower of M^i_* by $E^{M'_{\eta_i}}_{\nu_i}$.

We now make use of the notion of *mirror* defined in §3.6.

This suggests the following definition:

Definition 3.7.16. Let $I^* = \langle\langle N_i\rangle, \langle\nu_i\rangle, \langle\pi_{i,j}\rangle, T\rangle$ be a normal iteration of length η.

By a *reiteration mirror* (RM) of I^* we mean a pair $\langle R, I'\rangle$ such that

(a) $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle$ is a reiteration of M of length η, where

$$I^i = \langle\langle M^i_h\rangle, \langle\nu^i_h\rangle, \langle\pi^i_{hj}\rangle, T^i\rangle \text{ is of length } \eta_i.$$

(b) $I' = \langle\langle M'_i\rangle, \langle\pi'_{ih}\rangle, \langle\sigma^i\rangle, \langle\rho^i\rangle\rangle$ is a mirror of I^*. (Hence $\sigma_i(\nu^*_i) = \nu_i$).

(c) $M'_i = M^i_{\eta_i}$.

(d) If $h = T(i+1)$, then

$M'^*_i = M^h_{\eta_h}||\mu$, where μ is maximal such that τ_i is a cardinal in $M^h_{\eta_h}$ and $\pi'_{h,i+1} = \sigma^{h,i+1}_{\eta^*_h}$, where $\eta^*_h + 1 = \mathrm{lh}(I^i_*)$.

Definition 3.7.17. $\langle I^*, R, I'\rangle$ is called an *RM-triple* if $\langle R, I'\rangle$ is an RM of I^*.

We obviously have:

Lemma 3.7.32. *$i+1$ is a drop point in I^* iff it is a drop point in R.*

Moreover:

Lemma 3.7.33. *If $(i,j]_T$ has no drop point, then $\pi'_{ij} = \sigma^{ij}_{\eta_i}$.*

Proof. By induction on j, using Lemma 3.7.27. We leave this to the reader.

Lemma 3.7.34. *Let $\langle I, R, I'\rangle$ be an RM-triple of length $\eta+1$. Let $E^{N_\eta}_\nu \neq \varnothing$, where $\nu > \nu_i$ for $i < \eta$. Then $\langle I, R, I'\rangle$ extends to a triple of length $\eta+2$, with $\nu = \nu_\eta$ (hence $\nu'_\eta = \sigma_\eta(\nu)$).*

Proof. By Lemma 3.7.25, R is uniquely reiterable. Hence R extends to $\dot{R}$ of length $\eta+2$ with $\dot{\nu}_\eta = \sigma_\eta(\nu)$. Set: $M'_{\eta+1} =:$ the final model of $\dot{I}^{\xi+1}, \xi =: \dot{T}(\eta+1), \pi' =: \sigma^{\xi,\eta+1}_{\eta^*}$, where $\eta^* = \mathrm{ln}(I^\eta_*)$. The choice of ν_η determines $\dot{M}^*_\eta = M^\xi_\eta || \mu$. Then:

$$\pi^{-1} : \dot{M}^*_\eta \longrightarrow_{\Sigma^*} M_{\eta+1}, \pi(X) = E^{M'_\eta}_\nu(X) \text{ for } X \in \mathbb{P}(\kappa) \cap \dot{M}^*_\eta.$$

The conclusion then follows by Lemma 3.6.38.

QED(Lemma 3.7.34)

By Lemma 3.7.25 and Lemma 3.6.37 we then have:

Lemma 3.7.35. *Let $\langle I, R, I' \rangle$ be an RM-triple of limit length η. Let b be the unique good branch in R. Then there is a unique extension to an RM-triple of length $\eta+1$. Moreover, $b = T``\{\eta\}$ in the extension.*

Proof. R extends uniquely to $\dot{R}$ of length $\eta+1$. We now extend I' to $\dot{I}'$ by taking $\dot{M}'$ as the final model of $\dot{I}'^\eta$. Pick $i < \eta$ such that $b \setminus i$ has no drop point in R. For $j \in b \setminus i$ set:

$$\dot{\pi}'_{j,\eta} = \dot{\sigma}^{i,\eta}_{\eta_j} \text{ (where } \eta_j + 1 = \mathrm{lh}(I^j) \text{ in } R).$$

By Lemma 3.7.33, we know:

$$\dot{\pi}'_{j,\eta}\pi'_{h,j} = \dot{\pi}'_{h,\eta} \text{ for } h \leq j \text{ in } b \setminus i.$$

By Lemma 3.7.27 it follows that:

$$\dot{M}, \langle \dot{\pi}'_{j,\eta} : j \in b \setminus i \rangle$$

is the direct limit of:

$$\langle M'_h : h \in b \setminus i \rangle,\ \langle \pi'_{h,j} : h \leq j \text{ in } b \setminus i \rangle.$$

(For $h \in b \cap i$, we then set: $\dot{\pi}'_{h,\eta} = \pi'_{i,\eta}\pi'_{h,i}$.)

The conclusion is immediate by Lemma 3.6.37.

(Lemma 3.7.35)

Now let $N, \tilde{I}$ be as in the premise of Lemma 3.7.2. In particular, $\tilde{I}$ is a normal iteration of M of length $\tilde{\eta}+1$ and:

$$\tilde{\sigma} : N \longrightarrow_{\Sigma^*} \tilde{M}_{\tilde{\eta}} \min \tilde{\rho}.$$

Using the last two lemmas, we define a successful strategy for N. We first fix a function G such that whenever $\Gamma = \langle I, R, I' \rangle$ is an RM triple of length $\mu + 1$ and $E_\nu^{M_\mu} \neq \varnothing$ with $\mu > \nu_j$ for $j < \mu$, then $G(\Gamma, \nu)$ is an extension of Γ to an RM triple of length $\mu + 1$ with $\nu_\mu = \nu$. In all other cases $G(\Gamma, \nu)$ is undefined. Now let I be any normal iteration of N. There can obviously be only one RM triple $\Gamma = \langle I, T, I' \rangle$ with the properties:

(a) $I^0 = \tilde{I}, \sigma_0 = \tilde{\sigma}, \rho^0 = \tilde{\rho}$.

(b) If $i + 1 < \mathrm{lh}(I)$, then:

$$\Gamma | i + 2 = G(\Gamma | i + 1, \nu_i),$$

since $\Gamma | \lambda + 1$ is uniquely determined at limit stages λ by Lemma 3.7.35.

Denote this Γ by $\Gamma(I)$ if it exists. We define the strategy S as follows:

Let I of limit length. If $\Gamma(I)$ is undefined, then so is $S(I)$. Now let $\Gamma(I) = \langle I, R, I' \rangle$ be defined. Set:

$$S(I) = \text{ the unique cofinal, well founded branch in } R.$$

(This exists by Lemma 3.7.35). We then get:

Lemma 3.7.36. *Let I be a normal iteration of N. If I is S-conforming, then $\Gamma(I)$ is defined.*

Proof. By induction on $\mathrm{lh}(I)$, using Lemma 3.7.34 and Lemma 3.7.35.

QED(Lemma 3.7.36)

In particular, if I is of limit length, it follows by Lemma 3.7.35 that $S(I)$ is defined and is a cofinal, well founded branch in I. This proves Theorem 3.7.28.

Theorem 3.7.28 is stated under the assumption that M is uniquely normally iterable in V. As usual, we can relativize this to a regular cardinal $\theta > \omega$. We call M' a θ-iterate of M is it is obtained by a normal iteration of length $< \theta$. Modifying our proof slightly we get:

Lemma 3.7.37. *Let $\theta > \omega$ be regular.*

(a) *If M is uniquely normally θ-iterable and M' is a θ-iterate of M then M' is normally θ-iterable.*

(b) *If M is uniquely normally $\theta+1$-iterable and M' is a θ-iterate of M, then M' is normally $\theta+1$-iterable.*

Note. In proving (b) we must restate Lemma 3.7.29 as:

Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{i,j}\rangle, T\rangle$ be a normal iteration of length $\eta+1 < \theta$. Let $\sigma : N \longrightarrow_{\Sigma^*} M_\eta \min\rho$. Then M is normally $\theta+1$-iterable.

Note. In proving Lemma 3.7.37, we restrict ourselves to θ-reiterations $R = \langle\langle I^i\rangle, \dots\rangle$ meaning that $\mathrm{lh}(I^i) < \theta$ for $i < \theta$. Thus we restrict to θ-*reiteration mirror* $\langle R, I'\rangle$, meaning that R is a θ-reiteration. Lemma 3.7.34 is then stated for RM-triples of length $\eta+1<\theta$. Lemma 3.7.35 is stated for RM-triples of length $\eta \leq \theta$. All steps fo through as before.

Note. An easy modification of the proof shows that, if M is normally iterable by a insertion stable strategy, then every S-conforming iterate of M is normally iterable.

This is a relatively weak result, and could, in fact, have been obtained without use of the pseudo projecta. (However, we would not know how to do it without the use of reiteration). What we really want to prove is that M is smoothly iterable. The above proof indicates a possible strategy for doing so, however: If M is "smoothly reiterable", and:

$$\sigma : N \longrightarrow_{\Sigma^*} M \min\rho$$

we could use the same procedure to define a successful smooth iteration strategy for N. In §3.7.4 we shall define "smooth reiterability" and show that if holds for M.

3.7.4 Reiteration and Inflation

By a *smooth reiteration* of M we mean the result of doing (finitely or infinitely many) successive normal reiterations. We define:

Definition 3.7.18. A *smooth reiteration* of M is a sequence $S = \langle\langle I_i : i < \mu\rangle, \langle e_{i,j} : i \leq j < \mu\rangle\rangle$ such that $\mu \geq 1$ and the following hold:

(a) I_i is a normal iteration of M of successor length $\eta_i + 1$.

(b) $e_{i,j}$ inserts an $I_i|\alpha$ into I_j, where $\alpha \leq \eta_i + 1$.

(c) $e_{h,j} = e_{i,j} \circ e_{h,i}$.

(d) If $i+1<\mu$, there is a normal reiteration:

$$R_i = \langle\langle I_i^l\rangle, \langle \nu_i^l\rangle, \langle e_i^{k,l}\rangle, T_i\rangle$$

of length $\eta_i + 1$ such that $I_i = I_i^0, I_{i+1} = I_i^{\eta_i}$ and $e_{i,i+1} = e_i^{0,\eta_i}$.

Note. R_i is unique by Lemma 3.7.21. Hence so is $\langle e_{i,j} : i \le j < \mu\rangle$, which we call the *induced sequence*.

Call i a *drop point* in S iff R_i has a truncation on the main branch.

(e) If $\lambda < \mu$ is a limit ordinal, then there are at most finitely many drop points $i < \lambda$. Moreover, if $h < \lambda$ and (h, λ) is free of drop points, then:

$$I_\lambda, \langle e_{i,\lambda} : h \le i < \lambda\rangle$$

is the good limit of:

$$\langle I_i : h \le i < \lambda\rangle, \langle e_{i,j} : h \le i \le j < \lambda\rangle$$

This completes the definition. We call μ the *length* of S.

Note. Since $e_{l,\lambda} = e_{h,\lambda}e_{l,h}$ for $l < h < \lambda$, we follow our usual convention, calling:

$$I^\lambda, \langle e_{i,\lambda} : i < \lambda\rangle$$

the *good limit* of:

$$\langle I^i : i < \lambda\rangle, \langle e_{i,j} : i \le j < \lambda\rangle$$

We call M *smoothly reiterable* if every smooth reiteration of M can be properly extended in any legitimate way. We note:

Fact 1. If I is a normal iteration of M, then $\langle\langle I\rangle, \varnothing, \langle\, \mathrm{id}\restriction I\rangle, \varnothing\rangle$ is a smooth reiteration of M of length 1.

Fact 2. If $S = \langle\langle I_i\rangle, \langle e_{i,j}\rangle\rangle$ is a smooth reiteration of M of length $i+1$, and $R = \langle I^i\rangle, \langle \nu^i\rangle\rangle$ is a normal reiteration of length $\eta + 1$ with $I^0 = I_i$, then S extends to S' of length $i+2$ with $I'_{i+1} = I^\eta$ and $e'_{i,i+1} = e^{0,\eta}$ (hence $R = R_i^{S'}$).

Fact 3. Let $S = \langle\langle I_i\rangle, \langle e_{i,j}\rangle\rangle$ be a smooth reiteration of M of limit length λ. Assume:

(a) S has finitely many drop points.

(b) S has a good limit: $I, \langle e_i : i < \lambda\rangle$.

Then S extends uniquely to S' of length $\lambda + 1$ with $I'_\lambda = I, e'_{i,\lambda} = e_i$.

Clearly, then, saying that M is smoothly reiterable is the same as saying that, whenever S is as in Fact 3, then (a), (b) are true. In the next subsection (§3.7.5) we shall prove the smooth iterability of M. The proof is, in all essentials, due to Farmer Schlutzenberg, and is based on his remarkable theory of *inflations*. This subsection is devoted an exposition of that theory.

Before proceeding to the precise definition of *inflation*, however, we give an introduction to Schlutzenberg's methods. Let $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, \tilde{T}\rangle$ be a reiteration of M. Schultzenberg calls I' an "inflation" of I^0, since it was obtained by introducing new extenders into the original sequence. He makes the key observation that the pair $\langle I^0, I^i\rangle$ determines a unique record of the changes made in passing from I^0 to I^i. We shall call that record the *history* of I^i and denote it by $\mathrm{hist}(I^0, I^i)$.

Definition 3.7.19. Let $\eta_i + 1 = \mathrm{lh}(I^i)$ for $i < \mathrm{lh}(R)$. For $\alpha \leq \eta_i$, set:

$$l(\alpha) = l^i(\alpha) =: \text{ the least } i \text{ such that } I^i|\alpha + 1 = I^l|\alpha + 1.$$

Let $s_i, t_i, \hat{s}_i = \mathrm{lub}_{h<i}\, s_h$ be defined as in §3.7.2. Then:

Lemma 3.7.38. (a) *$l(\alpha) =$ that $l \leq i$ such that $\hat{s}_l \leq \alpha$ and either $l = i$ or $l < i$ and $\alpha \leq s_l$.*

(b) *$I^j|\alpha + 1 = I^l|\alpha + 1$ for $l \leq j \leq i$.*

Proof.

(a) $\hat{s}_l \leq \alpha$, since otherwise $s_j + 1 > \alpha$ for a $j < l$. Hence $I^j|s_j + 1 = I^i|s_j + 1$ where $\alpha + 1 \leq s_j + 1$. Hence $j \geq l$. Contradiction!

Suppose $l \neq i$. Then $\alpha \leq s_l$, since otherwise $s_l + 1 \leq \alpha$ and $I^i|\alpha + 1 \neq I^l|\alpha + 1$, since $\nu^i_{s_l} < \nu^l_{s_l}$.

QED(a)

(b) Suppose not. Then $i \neq l, \alpha \leq s_l$ and $I^l|s_l + 1 = I^j|s_l + 1$ for $l \leq j < \mathrm{lh}(R)$. Contradiction! QED(Lemma 3.7.38)

Hence $\hat{s}_i \leq \alpha \longrightarrow l^i(\alpha) = i$.

Lemma 3.7.39. *If $h \leq i$ and $I^h|\alpha + 1 = I^i|\alpha + 1$ then $\nu^i_\alpha \leq \nu^h_\alpha$ if $\alpha < \eta_h$.*

Proof. By induction on i.

Case 1. $i = 0$ (trivial).

Case 2. $i = h + 1$.

Then $I^i|s_h + 1 = I^h|s_h + 1$ and $\nu^i_{s_h} \leq \nu^h_{s_h}$. Thus it holds for $\alpha \leq s_h$ by the induction hypotheses. But $l(\alpha) = i$ for $\alpha > s_h$.

Case 3. i is a limit.

Then $I^i|s_j + 1 = I^j|s_j + 1$ for $j < i$. Hence it holds for $\alpha < \hat{s}_i = \mathrm{lub}_{j<i}\, s_j$ by the induction hypothesis. But $l(\alpha) = i$ for $\alpha \geq \hat{s}_i$.

QED(Lemma 3.7.39)

The next lemma is crucial to developing the theory of inflations:

Lemma 3.7.40. *Let $\alpha \leq \eta_i, l = l(\alpha)$. Set:*

$$a = \{\gamma \leq \eta_0 : e^{0,l}(\gamma) < \alpha\}.$$

There is a unique e inserting $I^0|a + 1$ into $I^i|\alpha + 1$ such that $e \restriction a = e^{0l} \restriction a$ and $e(a) = \alpha$.

Proof. By induction on i.

Case 1. $i = 0$. Set $a = \alpha, e = \mathrm{id} \restriction \alpha + 1$.

Case 2. $i = h + 1$.

If $\alpha \leq s_h$, then $I^i|\alpha + 1 = I^h|\alpha + 1$. Hence $l = l^h(\alpha)$ and the result holds by the induction hypothesis.

If $\alpha > s_h$, then $l(\alpha) = i$, since $I^i|s_h + 1 \neq I^h|s_h + 1$. Then $\alpha = s_h + 1 + j$. Let $\mu = \tilde{T}(h + 1)$. Then $e^{\mu,i}(\overline{\alpha}) = \alpha$, where $\overline{\alpha} = t_h + j$. But $\hat{s}_\mu \leq t_h \leq s_\mu$ by Lemma 3.7.17. Hence $l^\mu(t_h) = l^\mu(\overline{\alpha}) = \mu$. Clearly:

$$a = \{\gamma \leq \eta_0 : e^{0,\mu}(\gamma) < \overline{\alpha}\}$$

Since $\mu \leq h$, the induction hypothesis gives a unique f inserting $I^0|a + 1$ into $I^\mu|\overline{\alpha} + 1$ such that $f \restriction a = e^{0,\mu} \restriction a$ and $f(a) = \overline{\alpha}$. Thus $e = e^{\mu,l} f$ has the desired properties.

QED(Case 2)

Case 3. i is a limit ordinal.

Then $I^i|s_j + 1 = I^j|s_j + 1$ for $j < i$. Hence the assertion holds for $\alpha < \hat{s}_i = \mathrm{lub}_{j<i}\, s_j$ by the induction hypothesis. But $l(\alpha) = i$ for $\hat{s}_i \leq \alpha$. Then

there is $j <_T i$ such that $\alpha = e^{j,i}(\overline{\alpha})$. Let $j = T(\xi + 1)$ where $\xi + 1 <_T i$. Then $\overline{\alpha} \geq \mathrm{crit}(e^{j,i}) = t_\xi$. But $\hat{s}_j \leq t_\xi \leq s_j$. Hence $l^j(\overline{\alpha}) = l^j(t_\xi) = j$. Since $e^{0,i} = e^{j,i} \circ e^{0,j}$, we conclude as in Case 2 that:

$$a = \{\gamma < \eta : e^{0,j}(\gamma) < \overline{\alpha}\}$$

By the induction hypothesis there is f inserting $I^0|a + 1$ into $I^j|\overline{\alpha} + 1$ such that $\hat{f} \restriction a = e^{0,j} \restriction a$ and $f(a) = \overline{\alpha}$. Hence $e = e^{f,i} \circ f$ has the desired properties.

QED(Lemma 3.7.40)

Definition 3.7.20. For $i < \mathrm{lh}(R), \alpha \leq \eta_0$ set:

$$\begin{aligned} a^i_\alpha &=: \mathrm{lub}\{\xi < \eta_0 : e^{0l}(\xi) < \alpha\} \text{ where } l = l^i(\alpha) \\ e^i_\alpha &=: \text{ the unique } e \text{ inserting } I^0|a^j_\alpha + 1 \text{ into } I^i|\alpha + 1 \text{ such} \\ &\qquad \text{that } e \restriction a^i_j = e^{0,l} \restriction a^i_j \text{ and } e(a^i_\alpha) = \alpha \end{aligned}$$

It follows easily that:

Lemma 3.7.41. (a) *If $l = l^i(\alpha)$, then $\alpha \leq \eta_l$ and $l = l^l(\alpha), a^i_\alpha = a^l_\alpha$ and $e^i_\alpha = e^j_\alpha$.*

(Hence $e^i_\alpha = e^h_\alpha$ and $a^i_\alpha = a^h_\alpha$ whenever $I^i|\alpha + 1 = I^h|\alpha + 1$).

(b) *If $e^{\mu,i}(\overline{\alpha}) = \alpha, \hat{s}_\mu \leq \overline{\alpha}, \hat{s}_i \leq \alpha$, then:*

$$l^\mu(\overline{\alpha}) = \mu, l^i(\alpha) = i, a^\mu_{\overline{\alpha}} = a^i_\alpha, \text{ and } e^{\mu,i} e^\mu_{\overline{\alpha}} = e^i_\alpha.$$

(c) $e^i_{\eta_i} \restriction a^i_{\eta_i} = e^{i,\eta_i} \restriction a^i_{\eta_i}$; $e^i_{\eta_i}(a^i_{\eta_i}) = \eta_i$ *($l^{\eta_i} = \eta_i$, since $\eta_i \geq \hat{s}_i$).*

(d) *If there is no truncation on the main branch of $R|i+1$, then $e^{0,i} = e^i{}_{\eta_i}$ and $a_{\eta_i} = \eta_0$ (since $e^{0,i}(\eta_0) = \eta_i$).*

The proof is left to the reader.

We now fix an $i < \mathrm{lh}(R)$ and set:

$$\begin{aligned} I &= \langle\langle M_\alpha\rangle, \langle\nu_\alpha\rangle, \langle\pi_{\alpha,\beta}\rangle, T\rangle =: I^0 \\ I' &= \langle\langle M'_\alpha\rangle, \langle\nu'_\alpha\rangle, \langle\pi'_{\alpha,\beta}\rangle, T'\rangle =: I^i \\ a &= \langle a^i_\alpha : \alpha \leq \eta_i\rangle, e_\alpha = e^i_\alpha \text{ for } \alpha \leq \eta_i. \end{aligned}$$

$\langle a, \langle e_\alpha : \alpha \leq \eta'\rangle\rangle$ is then called the *history* of I' from I. We shall show that it is completely determined by the pair $\langle I, I'\rangle$. a_α is called the *ancestor* of α in this history.

We prove:

Theorem 3.7.42. *Let $I, I', a, \langle e_\alpha : \alpha \leq \eta_i \rangle$ be as above. Then:*

(1) *$a : \mathrm{lh}(I') \longrightarrow \mathrm{lh}(I)$ and e_α inserts $I|a_\alpha + 1$ into $I'|\alpha + 1$ for $\alpha < \mathrm{lh}(I')$. Moreover, $e_\alpha(a_\alpha) = \alpha$.*

(2) *Let $a_\alpha < \eta$. If $\tilde{\nu}_\alpha = \sigma^{e_\alpha}_{a_\alpha}(\nu_{a_\alpha})$ exists and $\alpha + 1 < \mathrm{lh}(I')$, then $\nu'_\alpha \leq \tilde{\nu}_\alpha$.*

(3) *Let $a_\alpha < \eta, \alpha + 1 < \mathrm{lh}(I'), \nu'_\alpha = \tilde{\nu}_\alpha$. Then:*

$$a_{\alpha+1} = a_\alpha + 1,\ e_{\alpha+1} \restriction a_\alpha + 1 = e_\alpha.$$

For $\alpha + 1 < \mathrm{lh}(I^i)$, define the index *of α ($\mathrm{in}(\alpha) = \mathrm{in}^i(\alpha)$) as:*

$$\mathrm{in}(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is as in (3)} \\ 1 & \text{if not} \end{cases}$$

(4) *If $\mathrm{in}(\alpha) = 1, \gamma = T'(\alpha + 1)$, then $a_{\alpha+1} = a_\gamma$.*

(5) *If $\beta \leq_{T'} \alpha$, then $e^{-1}_\alpha \restriction \beta = e^{-1}_\beta \restriction \beta$.*

Note. *Ignoring our formal definition of $\langle a, e \rangle$ and using only (1), (5), we get:*

- *$e_\alpha \restriction a_\beta = e_\beta \restriction a_\beta$.*
- *$a_\beta \leq_T a_\alpha$ since:*

$$\hat{e}_\alpha(a_\beta) = \hat{e}_\beta(a_\beta) \leq_{T'} e_\beta(\beta) = \beta \leq_{T'} \alpha = e_\alpha(a_\alpha).$$

- *If α is a limit ordinal, then:*

$$a_\alpha = \bigcup_{\beta <_{T'} \alpha} a_\beta \text{ and } e_\alpha \restriction a_\alpha = \bigcup_{\beta <_{T'} \alpha} e_\beta \restriction a_\beta,$$

since $e^{-1}_\alpha \restriction \alpha = \bigcup_{\beta <_{T'} \alpha} e^{-1}_\beta \restriction \beta$.

Note. *By (1), (4) and (5) we get:*

- *If $in(\alpha) = 1$, $\gamma = T'(\alpha + 1)$, then $e_{\alpha+1} \restriction a_{\alpha+1} = e_\gamma \restriction a_\gamma$.*

Note. *Since e_α, e_β are monotone and $a_be = e^{-1}_\beta$ “β, the statement:*

$$e^{-1}_\alpha \restriction \beta = e^{-1}_\beta \restriction \beta$$

is equivalent to:

$$e_\beta \restriction a_\beta = e_\alpha \restriction a_\beta \text{ and } e_\alpha(a_\beta) \geq \beta.$$

(6) *If $R|i+1$ has a truncation on the main branch, then there is $\alpha \in (\hat{e}_{\eta_i}(a_{\eta_i}), \eta_i]_{T'}$ which is a drop point in I'.*

Note. By Lemma 3.7.41 (a) we have:

$$\hat{e}_{\eta_i}(a_{\eta_i}) = \operatorname{lub} e_{\eta_i}``a_{\eta_i} = \operatorname{lub} e^{0,i_0} a_{\eta_0} = \hat{e}^{0,i}(a_{\eta_i}).$$

We prove Theorem 3.7.42 by induction on i:

Case 1. $i = 0$.

Trivial, since $a_\alpha = \alpha, e_\alpha = \mathrm{id} \upharpoonright \alpha + 1$.

Case 2. $i = h + 1$.

(1) is given.

(2) If $\alpha \leq s_h$, then $I^i|\alpha + 1 = I^h|\alpha + 1$, hence $l^i(\alpha) = l^h(\alpha), e^i_\alpha = e^h_\alpha, \tilde{\nu}^i_\alpha = \tilde{\nu}^h_\alpha$. By the induction hypothesis $\nu^h_\alpha = \tilde{\nu}^h_\alpha$. But $\nu^i_\alpha < \nu^h_\alpha$. Now let $\alpha > s_h$. Then $l(\alpha) = i$ and $\alpha = s_h + 1 + j$ for some j. Let $\mu = \tilde{T}(h+1)$. Then $e^{\mu,i}(\overline{\alpha}) = \alpha$ where $\overline{\alpha} = t_h + 1$. Just as in the proof of Lemma 3.7.40 (Case 2), we have: $\mu = l^\mu(t_h) = l^\mu(\overline{\alpha})$ and $e^{\mu,i} \circ e^\mu_{\overline{\alpha}} = e_\alpha$. Hence:

$$\tilde{\nu}^i_\alpha = \sigma^{e^i_\alpha}_a(\nu^0_a) = \sigma^{\mu,\alpha}_{\overline{\alpha}} \sigma^{e^\mu_{\overline{\alpha}}}(\nu^0_a) = \sigma^{\mu,\alpha}_{\overline{\alpha}}(\tilde{\nu}^\mu_{\overline{\alpha}})$$

(Since if $e = e_1 \circ e_0$, then $\sigma^e_\beta = e^{e_1}_{e_0(\beta)} \circ e^{e_0}_\beta$). By the induction hypothesis: $\nu^\mu_{\overline{\alpha}} \leq \tilde{\nu}^\mu_{\overline{\alpha}}$. Hence:

$$\nu^i_\alpha = \sigma^{\mu,\alpha}_{\overline{\alpha}}(\nu^\mu_{\overline{\alpha}}) \leq \sigma^{\mu,\alpha}_{\overline{\alpha}}(\tilde{\nu}^\mu_{\overline{\alpha}}) = \tilde{\nu}'_\alpha.$$

QED(2)

(3) If $\alpha < s_h$, then $\nu^i_\alpha = \nu^h_\alpha, \tilde{\nu}^h_\alpha = \tilde{\nu}^i_\alpha$, since $I^i|s_h + 1 = I^h|s_h + 1$. Hence $\nu^h_\alpha = \tilde{\nu}^h_\alpha$.

Hence $a^h_{\alpha+1} = a^h_\alpha + 1, e^h_{\alpha+1} \upharpoonright a^h_{\alpha+1} = e^h_\alpha$ by the induction hypothesis. But $l^i(\alpha + 1) = l^h(\alpha + 1)$. Hence: $a^n_{\alpha+1} = a^i_{\alpha+1}, a^h_\alpha = a^i_\alpha, e^h_{\alpha+1} = e^i_{\alpha+1}, e^h_\alpha = e^i_\alpha$. The conclusion is immediate. Now let $\alpha = s_h$. We still have $e^h_\alpha = e^i_\alpha$; hence $\tilde{\nu}^h_\alpha = \tilde{\nu}^i_\alpha$. But $\nu^i_\alpha < \nu^h_\alpha \leq \tilde{\nu}^h_\alpha$. Contradiction! Now let $\alpha > s_h$. We again have: $\alpha = s_h + 1 + j, \alpha = e^{\mu,i}(\overline{\alpha})$, where $\mu = T(h+1)$ and $\overline{\alpha} = t_h + j$. As before, we have $l^i(\alpha) = i, l^\mu(\overline{\alpha}) = \mu$. Moreover $\tilde{\nu}^i_\alpha = \sigma^{\mu,i}_{\overline{\alpha}}(\tilde{\nu}^\mu_{\overline{\alpha}})$ and $\nu^i_\alpha = \sigma^{\mu,i}_{\overline{\alpha}}(\nu^\mu_{\overline{\alpha}})$. Hence $\nu^\mu_{\overline{\alpha}} = \tilde{\nu}^\mu_{\overline{\alpha}}$. Hence:

$$a^\mu_{\overline{\alpha}+1} = a^\mu_{\overline{\alpha}} + 1, e^\mu_{\overline{\alpha}+1} \upharpoonright \overline{\alpha} + 1 = e^\mu_{\overline{\alpha}}.$$

But $i = l'(\alpha) = l^i(\alpha+1), \mu = l^\mu(\overline{\alpha}) = l^\mu(\overline{\alpha}+1)$, and $e^{\mu,i}(\overline{\alpha} + 1) = \alpha+1$. Hence:

$$a = a^i_\alpha = a^\mu_{\overline{\alpha}} \text{ and } a_{\alpha+1} = a^i_{\alpha+1} = a^\mu_{\overline{\alpha}+1} = a + 1.$$

Moreover, we have:

$$e^i_{\alpha+1} \upharpoonright a+1 = e^{\mu,i} e^{\mu}_{\overline{\alpha}+1} \upharpoonright a+1 = e^{\mu,i} e^{\mu}_{\overline{\alpha}} = e_\alpha.$$

QED(3)

(4) If $\alpha < s_n$ the result follows by the induction hypothesis, since $I^i|\alpha+2 = I^h|h+2$. Now let $\alpha = s_h$. Then $\mathrm{in}(\alpha) = 1$ as shown above. Let $\mu = \tilde{T}(h+1), \gamma = t_h$. Then $e^{\mu,i}(\gamma) = \alpha + 1$. Hence $a^{\mu}_{\gamma} = a^i_{\alpha+1}$. But $I^i|\gamma+1 = I^{\mu}|\gamma+1$. Hence $l^{\mu}(\gamma) = l^i(\gamma)$ and $a^i_{\gamma} = a^{\mu}_{\gamma} = a^i_{\alpha+1}$. Now let $\alpha > s_h$. Then $i = h+1$ is not a drop point in R, since otherwise $\eta_i = s_h + 1 = \alpha$. Hence $\alpha + 1 \not< \mathrm{lh}(I^i) = \eta_i + 1$. Contradiction! Then $\alpha = s_h+1+j$ and $\alpha = e^{\mu,i}(\overline{\alpha})$ where $\overline{\alpha} = t_h+j$ and $\mu = \tilde{T}(h+1)$. Note that $e^{\mu,i}(\xi) = \hat{e}^{\mu,i}(\xi) = \mathrm{lub}\, e^{\mu,i}``\xi$ for $\xi > t_h$. Clearly $\alpha+1 = e^{\mu,i}(\overline{\alpha}+1)$. As in the foregoing proofs we have:

$$\sigma^{\mu,i}(\nu^{\mu}_{\overline{\alpha}}) = \nu^i_{\alpha};\ \sigma^{\mu,i}(\tilde{\nu}^{\mu}_{\overline{\alpha}}) = \tilde{\nu}^i_{\alpha}.$$

Hence $\nu^{\mu}_{\overline{\alpha}} < \tilde{\nu}^{\mu}_{\overline{\alpha}}$ and $\mathrm{in}(\overline{\alpha}) = 1$. By the induction hypothesis we conclude: $a^{\mu}_{\overline{\gamma}+1} = a^{\mu}_{\overline{\gamma}}$, where $\overline{\gamma} = T^{\mu}(\overline{\alpha}+1)$. But, as before, $a^{\mu}_{\overline{\alpha}+1} = a^i_{\alpha+1}$, since $e^{\mu,i}(\overline{\alpha}+1) = \alpha+1, l^{\mu}(\overline{\alpha}+1) = \mu$, $l^i(\alpha+1) = i$. Thus it suffices to show:

Claim. $a^{\mu}_{\overline{\gamma}} = a^i_{\gamma}$, where $\gamma = T^i(\alpha+1)$.

We consider two cases:

Case A. $\kappa^{\mu}_{\overline{\alpha}} > \kappa_i$. Then $e^{\mu,i}(\overline{\gamma}) = \gamma$ by Lemma 3.7.10 (1). As before $l^{\mu}(\overline{\gamma}) = \mu, l^i(\gamma) = i$ and $a^{\mu}_{\overline{\gamma}} = a^i_{\gamma}$.

Case B. $\kappa^{\mu}_{\overline{\alpha}} < \kappa_i$. Then $\gamma = \overline{\gamma}$ by Lemma 3.7.10(1). Then $\overline{\gamma} \leq t_h$, where $I^i|t_h+1 = I^{\mu}|t_h+1$. Hence $a^i_{\overline{\gamma}} = a^{\mu}_{\overline{\gamma}}$.

QED (4)

(5) If $\alpha \leq s_h$, then $I^h|\alpha+1 = I^i|\alpha+1$ and $a^h_{\gamma} = a^i_{\gamma}, e^h_{\gamma} = e^i_{\gamma}$ for $\gamma \leq \alpha$. Hence the conclusion follows by the induction hypothesis. Now let $\alpha > s_h$. Then $\alpha = s_h+1+j$ for some j. Let $\mu = \tilde{T}(h+1)$. Then $e^{\mu,i}(\overline{\alpha}) = \alpha$ where $\overline{\alpha} = t_h+1$. But $\overline{\alpha} \geq \mathrm{crit}(e^{\mu,i}) = t_h \geq \hat{s}_{\mu}$. Hence:

$$l^{\mu}(\overline{\alpha}) = \mu, a^{\mu}_{\overline{\alpha}} = a^i_{\alpha}, e^i_{\alpha} = e^{\mu,i} \cdot e_{\overline{\alpha}}.$$

Let $\beta <_{T^i} \alpha$. We consider two cases:

Case A. $\beta > s_h$.

Then $\beta = s_h+1+r$ for an $r < j$. Hence, letting $\overline{\beta} = t_h + r$, we have $e^{\mu,i}(\overline{\beta}) = \beta$ and:

$$l^{\mu}(\overline{\beta}) = \mu, a^{\mu}_{\overline{\beta}} = a^i_{\beta}, e^i_{\beta} = e^{\mu,i} \cdot e_{\overline{\beta}}.$$

It follows easily that $\overline{\beta} <_{T^\mu} \overline{\alpha}$. Hence by the induction hypothesis:

$$(e^\mu_{\overline{\beta}})^{-1} \upharpoonright \overline{\beta} = (e^\mu_{\overline{\alpha}})^{-1} \upharpoonright \overline{\alpha}.$$

Hence:

$$\begin{aligned}(e^i_\beta)^{-1} \upharpoonright \beta &= (e^\mu_{\overline{\beta}})^{-1} \cdot (e^{\mu,i})^{-1} \upharpoonright \beta \\ &= (e^\mu_{\overline{\alpha}})^{-1} \cdot (e^{\mu,i})^{-1} \upharpoonright \beta \\ &= (e^i_\alpha)^{-1} \upharpoonright \beta.\end{aligned}$$

QED(Case A)

Case B. $\beta \leq s_h$.

Then $\beta \leq t_h$, since $(t_h, s_h]$ is in limbo at $\hat{s}_i = s_h+1$. Hence $e^{\mu,i} \upharpoonright \beta = \text{id}$, since $t_h = \text{crit}(e^{\mu,i})$. But then:

$$\beta = \hat{e}^{\mu,i}(\beta) \leq_{T^\mu} \alpha = e^{\mu,i}(\overline{\alpha}).$$

Hence $\beta \leq_{T^\mu} \overline{\alpha}$. Moreover $I^i|\beta + 1 = I^\mu|\beta + 1$, since $\hat{e}^{\mu,i} \upharpoonright \beta + 1 = \text{id}$. Hence $a^\mu_\beta = a^i_\beta$ and $e^\mu_\beta = e^i_\beta$. But:

$$(e^\mu_{\overline{\alpha}})^{-1} \upharpoonright \beta = (e^\mu_\beta)^{-1} \upharpoonright \beta$$

since $\beta \leq_{T^\mu} \overline{\alpha}$. Hence:

$$(e^i_\alpha)^{-1} \upharpoonright \beta = (e^\mu_{\overline{\alpha}})^{-1}(e^{\mu i})^{-1} \upharpoonright \beta \quad = (e^\mu_\beta)^{-1}(e^{\mu i})^{-1} \upharpoonright \beta = (e^i_\beta)^{-1} \upharpoonright \beta$$

QED(Case B)

This proves (5).

(6) If $i = h + 1$ is a drop point on $R|i + 1$, then $M'^{*}_{s_h} \neq M_{t_i}$, where $\eta^i = s_h + 1, t_i = T^i(s_h + 1)$. Hence η_i is a drop point in I^i. Now suppose that $h + 1$ does not drop in $R|i + 1$. Let $\mu = \tilde{T}(h + 1)$. Then there must be a drop point on the main branch of $R|\mu + 1$. Hence I^μ has a drop point in $(\varepsilon, \eta_\mu]_{T^\mu}$ where $\varepsilon = \hat{e}^\mu_{\eta_\mu}(a^\mu_{\eta_\mu})$. Since $e^{\mu,i}(\eta_\mu) = \eta_i$, it follows easily from Lemma 3.7.10(7) that there is a drop point on I^i in $(\hat{e}^{\mu,i}(\varepsilon), t_i]_{T^i}$. Since $\hat{s}_\mu \leq \eta_\mu, \hat{s}_i \leq \eta_i$, we have:

$$\mu = l^\mu =: l^\mu(\eta_\mu),\ i = l^i = l^i(\eta_i).$$

Hence $a^\mu_{\eta_\mu} = a^i_{\eta_i}$. Clearly:

$$\hat{e}^{\mu,i}(\varepsilon) = \text{lub}\, e^{\mu,i}``\varepsilon.$$

Since $e^\mu_{\eta_\mu} \upharpoonright a^\mu_{\eta_\mu} = e^{0,\mu} \upharpoonright a^\mu_{\eta_\mu}$, we have: $\varepsilon = \text{lub}\, e^{0,\mu}``a^\mu_{\eta_\mu}$. Hence:

$$\hat{e}^{\mu,i}(\varepsilon) = \text{lub}\, e^{0,i}``a^i_{\eta_i} = \hat{e}^i_{\eta_i}(a^i_{\eta_i}).$$

Hence I^i has a drop in $(\hat{e}^i_{\eta_i}(a^i_{\eta_i}), \eta_i]_{T^i}$.

QED(6)

This completes Case 2.

Case 3. $i = \lambda$ is a limit ordinal.

(1) is given.

(2) Set $\hat{s} = \hat{s}_\lambda = \mathrm{lub}_{i<\lambda}\, s_i$. Then $I^\lambda | s_i + 1 = I^i | s_i + 1$ for $i < \lambda$. Thus (2) holds by the induction hypothesis for $\alpha < \hat{s}$. Now let $\alpha \geq \hat{s}$ then $l^\lambda(\alpha) = \lambda$. Pick $\mu < \lambda$ such that $\alpha \in \mathrm{rng}(e^{\mu,\lambda})$ and there is no drop in $(\mu, \lambda)_{T^\lambda}$. Let $i = h+1$, where $\mu = T(h+1), h+1 <_{T^\lambda} \lambda$. If $e^{\mu,\lambda}(\hat{\alpha}) = \alpha$, then $\hat{\alpha} \geq t_h$, since $e^{\mu,\lambda} \upharpoonright t_h = \mathrm{id}$. Hence $\overline{\alpha} \geq s_h + 1 = \hat{s}_i$, where $e^{i,\lambda}(\overline{\alpha}) = \alpha$. Hence $l^i =: l^i(\overline{\alpha}) = i$. Hence $a^i_{\overline{\alpha}} = a^\lambda_\alpha$ and $e^\lambda_\alpha = e^{i,\lambda} e^i_{\overline{\alpha}}$. We are assuming that:

$$\tilde{\nu}^\lambda_\alpha = \sigma^{e^\lambda_\alpha}_{a^\lambda_\alpha}(\nu^0_{a^\lambda_\alpha}) \text{ exists.}$$

But then:

$$\tilde{\nu}^i_{\overline{\alpha}} = \sigma^{e^i_{\overline{\alpha}}}_{a^i_{\overline{\alpha}}}(\nu^0_{a^i_{\overline{\alpha}}}) \text{ exists and } \sigma^{i,\lambda}_{\overline{\alpha}}(\tilde{\nu}^i_{\overline{\alpha}}) = \tilde{\nu}^\lambda_\alpha.$$

Clearly: $\nu^\lambda_\alpha = \sigma^{i,\lambda}_\alpha(\nu^i_{\overline{\alpha}})$. But $\nu^i_{\overline{\alpha}} \leq \tilde{\nu}^i_{\overline{\alpha}}$ by the induction hypothesis. Hence $\nu^\lambda_\alpha \leq \tilde{\nu}^\lambda_\alpha$.

QED(2)

(3) For $\alpha < \hat{s}_\lambda$ it holds by the induction hypothesis, so let $\alpha \geq \hat{s}_\lambda$. Let $\mu, h, i, \overline{\alpha}$ be as in (2). Then $l^\lambda(\alpha) = \lambda, l^i(\alpha) = i$. We assume $\mathrm{in}^\lambda(\alpha) = 0$, i.e.:

$$\alpha < \eta_\lambda \text{ and } \nu^\lambda_\alpha = \tilde{\nu}^\lambda_\alpha.$$

But then:

$$\overline{\alpha} < \eta_i \text{ and } \nu^i_{\overline{\alpha}} = \tilde{\nu}^i_{\overline{\alpha}} \text{ hence } \mathrm{in}^i(\overline{\alpha}) = 0$$

Hence $a^i_{\overline{\alpha}+1} = a^i_{\overline{\alpha}} + 1$ and $e^i_{\overline{\alpha}+1} \upharpoonright a^i_{\overline{\alpha}} + 1 = e^i_{\overline{\alpha}}$. But $l^i(\overline{\alpha} + 1) = i, l^\lambda(\overline{\alpha} + 1) = \lambda$. Hence

$$a^\lambda_{\alpha+1} = a^i_{\overline{\alpha}+1} = a^i_{\overline{\alpha}} + 1$$

and

$$\begin{aligned} e^\lambda_{\alpha+1} \upharpoonright a^\lambda_{\alpha+1} &= e^{i\lambda} e^i_{\overline{\alpha}+1} \upharpoonright a^i_{\overline{\alpha}} + 1 \\ &= e^{i\lambda} e^i_{\overline{\alpha}} = e^\lambda_\alpha \end{aligned}$$

QED(3)

(4) For $\alpha < \hat{s}_\lambda$ it holds by the induction hypothesis, so let $\alpha \geq \hat{s}_\lambda$. Let $\mu, h, i, \overline{\alpha}$ be as in (2) with the additional stipulation that $\gamma \in \mathrm{rng}(e^{\mu,\lambda})$ where $\gamma = T^\lambda(\alpha + 1)$. Let $e^{i,\lambda}(\overline{\gamma}) = \gamma$. Then either $\gamma \geq \hat{s}_\lambda$ and $\overline{\gamma} \geq \hat{s}_i = s_h + 1$, or $\gamma < \hat{s}_\lambda$ and $\bar{\gamma} = \gamma$. It follows easily that $\overline{\gamma} = T^i(\overline{\alpha} + 1)$. Moreover $\mathrm{in}^i(\overline{\alpha}) = 1$, since $\mathrm{in}^\lambda(\alpha) = 1$. But then $a^i_{\overline{\alpha}} = a^i_{\overline{\gamma}}$.

But $a^i_{\overline{\alpha}} = a^\lambda_\alpha$. Moreover $a^i_{\overline{\gamma}} = a^\lambda_\gamma$. (If $\gamma \geq \hat{s}_\lambda$, this is because $l^i(\overline{\gamma}) = i$. If $\gamma < \hat{s}_\lambda$, it is because $I^i|\gamma + 1 = I^i|\overline{\gamma} + 1$).

QED(4)

(5) If $\alpha < \hat{s}$, it follows by the induction hypothesis, since $I^\lambda|\alpha{+}1 = I^i|\alpha{+}1$ for ß $< \lambda, \alpha \leq s_i$. Now let $\alpha \geq \hat{s}$. Fix $\beta <_{T^\lambda} \alpha$. Let $\mu, i, h, \overline{\alpha}$ be as before with μ chosen big enough that $\beta \in \mathrm{rng}(e^{\mu,\lambda})$ and $\beta < t_h = \mathrm{crit}(e^{\mu,\lambda})$ if $\beta < \hat{s}$. Let $\alpha = e^{i,\lambda}(\overline{\alpha}), \beta = e^{i,\lambda}(\overline{\beta})$. Since:

$$e^{i,\lambda}(\overline{\beta}) = \beta <_{T^\lambda} \alpha = e^{i\lambda}(\overline{\alpha}),$$

we conclude: $\overline{\beta} <_{T^i} \overline{\alpha}$. Hence:

$$(e^i_{\overline{\alpha}})^{-1} \restriction \beta = (e^i_{\overline{\beta}})^{-1} \overline{\beta}$$

by the induction hypothesis. Since $\hat{s}_i \leq \overline{\alpha}$, we again have:

$$a^i_{\overline{\alpha}} = a^\lambda_\alpha, e^\lambda_\alpha = e^{i,\lambda} e^i_{\overline{\alpha}}.$$

If $\beta \geq \hat{s}$, then $\hat{s}_i \leq \overline{\beta}$ and we have :

$$a^i_{\overline{\beta}} = a^\lambda_\beta, e^\lambda_\beta = e^{i,\lambda} e^i_{\overline{\beta}}.$$

Hence:

$$\begin{aligned} (e^\lambda_{\overline{\alpha}})^{-1} \restriction \beta &= (e^i_{\overline{\alpha}})^{-1} (e^{i\lambda})^{-1} \restriction \beta \\ &= (e^i_{\overline{\beta}})^{-1} (e^{i\lambda})^{-1} \restriction \beta \\ &= (e_\beta)^{-1} \restriction \beta. \end{aligned}$$

Now suppose that $\beta < i$. Then $\beta = \overline{\beta} < \mathrm{crit}(e^{i\lambda})$. Hence $I^i|\beta + 1 = I^\lambda|\beta + 1$ and:

$$a^i_\beta = a^\lambda_\beta, e^i_\beta = e^\lambda_\beta \text{ where } e^{i\lambda} \restriction \beta + 1 = \mathrm{id}\,.$$

Hence we again have:

$$a^i_{\overline{\beta}} = a^\lambda_\beta, e^\lambda_\beta = e^{i\lambda} e^i_{\overline{\beta}},$$

and we argue exactly as before.

QED(5)

(6) Suppose $R|\lambda + 1$ has a truncation on the main branch. Clearly $\eta_\lambda \geq \hat{s}_\lambda$, so $l^\lambda(\eta_\lambda) = \lambda$. Let $\mu, i, h, \overline{\alpha}$ be as in (2) with $\alpha = \eta_\lambda$. Then $[i, \lambda]_{T^\lambda}$ is free of drops. Hence $e^{i,\lambda}(\eta_i) = \eta_\lambda$. But $R|i + 1$ then has a drop on the main branch. Hence there is a drop in $(\hat{e}^i_{\eta_i}(a^i_{\eta_i}), \eta_i]_{T^{i+1}}$. By Lemma 3.7.1 (7) it follows that there is a drop in $(\hat{e}^{i,\lambda}(\varepsilon), \eta_\lambda]_{T^\lambda}$,

where $\varepsilon = e_{\eta 0}(a^i_{\eta 0})$. But $l^i(\eta_i) = i$, since $\eta_i \geq \hat{s}_i$. Hence $a^i_{\eta_i} = a^\lambda_{\eta\lambda}$ and $\varepsilon = \hat{e}_{\eta_i}(a^i_{\eta_i}) = \operatorname{lub} e^{0,i}``a^i_{\eta_i}$. Moreover $e^{i,\lambda}(\varepsilon) = \operatorname{lub} e^{i,\lambda}``\varepsilon$. Hence $\hat{e}^{i,\lambda}(\varepsilon) = \operatorname{lub} e^{o,\lambda}``a^\lambda_{\eta_\lambda} = \hat{e}^\lambda_{\eta_\lambda}(a^\lambda_{\eta_\lambda})$.

QED(6)

This completes the proof of Lemma 3.7.42.

Inflations

Following Farmer Schlutzenberg we now define:

Definition 3.7.21. Let I be a normal iteration of M of successor length $\eta+1$. Let I' be a normal iteration of M. I' is an *inflation* of I iff there exist a pair $\langle a, e\rangle$ satisfying (1)-(5) in Theorem 3.7.42 (with $e = \langle e_\alpha : \alpha < \operatorname{lh}(I')\rangle$). We call any such pair a *history* of I' from I.

By the remark accompanying the statement of Theorem 3.7.42 we have:

Lemma 3.7.43. *Let I' be an inflation of I with history $\langle a, e\rangle$. Then:*

(a) *If $\beta \leq_{T'} \alpha$, then $a_\beta \leq_T a_\alpha$ and $e_\alpha \restriction a_\beta = e_\beta \restriction a_\beta$.*

(b) *If $\alpha \leq \operatorname{lh}(I')$ is a limit ordinal, then:*

$$a_\alpha = \bigcup_{\beta <_{T'} \alpha} a_\beta \text{ and } e_\alpha \restriction a_\alpha = \bigcup_{\beta_{T'} \alpha} e_\beta \restriction a_\beta.$$

(c) *If $\alpha + 1 < \operatorname{lh}(I'), \operatorname{in}(\alpha) = 1, \gamma = T'(\alpha+1)$, then:*

$$a_{\alpha+1} = a_\gamma \text{ and } e_{\alpha+1} \restriction a_{\alpha+1} = e_\gamma \restriction a_g a.$$

Lemma 3.7.44. *Let I, I' be as above. Then there is at most one history of I' from I.*

Proof. Let $\langle a, e\rangle$ be a history. By the conditions (1)-(5), this history satisfies a recursion of the form:

$$\langle a_\alpha, e_\alpha\rangle = F(\langle\langle a, e\rangle : \xi < \alpha\rangle),$$

where F is defined by reference to the pair $\langle I, I'\rangle$ alone. To see this we note:

(a) $a_0 = \varnothing$, $e_0(\varnothing) = \varnothing$ by (1).

(b) Let a_α, e_α be given. Then:

- $a_{\alpha+1} = \begin{cases} a_\alpha + 1 & \text{if } \mathrm{in}(\alpha) = 0 \\ a_\beta & \text{where } \beta = T'(\alpha+1) \text{ if } \mathrm{in}(\alpha) = 1 \end{cases}$
- $e_{\alpha+1}(a_\alpha + 1) = \alpha + 1$
- $e_{\alpha+1} \restriction a_{\alpha+1} = \begin{cases} e_\alpha & \text{if } \mathrm{in}(\alpha) = 0 \\ e_\beta \restriction a_{\alpha+1} & \text{if } \beta = T'(\alpha+1) \text{ and } \mathrm{in}(\alpha) = 1 \end{cases}$

In order to determine $\mathrm{in}(\alpha)$, however, we need only to know $a_\alpha, e_\alpha, I, I'$.

(c) If λ is a limit ordinal, then:

$$a_\lambda = \bigcup_{\alpha <_{T'} \lambda} a_\alpha;\ e_\lambda \restriction a_\lambda = \bigcup_{\alpha <_{T'} \lambda} e_\alpha \restriction a_\alpha;\ e_\lambda(a_\lambda) = \lambda.$$

QED(Lemma 3.7.44)

Definition 3.7.22. Let I' be an inflation of I. We denote the unique history of I' from I by: $\mathrm{hist}(I, I')$.

Note. Schlutzenberg's original definition replaced (5) in Definition 3.7.21 by the following statement, which we now prove as a lemma:

Lemma 3.7.45. *Let $\mu \leq a_\alpha$ such that $\hat{e}_\alpha(\mu) \leq_{T'} \beta \leq_{T'} e_\alpha(\mu)$. Then $a_\beta = \mu$. Moreover $e_\beta \restriction \mu = e_\alpha \restriction \mu$. (Hence $e_\mu(\mu) = \beta, \hat{e}_\beta(\mu) = \hat{e}_\alpha(\mu) = \sup e_\alpha``\mu$).*

Proof. Suppose not. Let α be the least counterexample. Let $\mu \leq a_\alpha, \hat{e}_\alpha(\mu) \leq_{T'} \beta \leq_{T'} e_\alpha(\mu)$. We derive a contradiction by showing:

$$a_\beta = \mu, e_\beta \restriction a_\beta = e_\alpha \restriction a_\beta.$$

Case 1. $\mu = a_\alpha$.

Then $a_\beta \leq_T a_\alpha$ and $e_\beta \restriction a_\beta = e_\alpha \restriction a_\alpha$. But $a_\beta = a_\alpha = \mu$, since otherwise $e_\alpha(a_\beta) < \hat{e}_\alpha(a_\alpha) \leq \beta$. Hence $a_\beta \in e_\alpha^{-1}``\beta$ but $a_\beta = e_\beta^{-1}``\beta$. Hence $e_\alpha^{-1} \neq e_\beta^{-1} \restriction \beta$. Contradiction!

Case 2. $\mu < a_\alpha$.

Then there is $\gamma < \alpha$ such that:

$$\mu \leq a_\gamma, e_\alpha \restriction a_\gamma = e_\gamma \restriction a_\gamma.$$

(Clearly $\alpha > 0$. This holds by (3) or (4) if α is a successor and by Lemma 3.7.43 if α is a limit.) Hence:

$$\hat{e}_\gamma(\mu) \leq_{T'} \beta \leq_{T'} e_\gamma(\mu).$$

Hence:

$$a_\beta = \mu, e_\beta \restriction a_\beta = a_\gamma \restriction a_\beta = a_\alpha \restriction a_\beta$$

by the minimality of α.

QED(Lemma 3.7.45)

Remark. (5) can be equivalently replaced by Lemma 3.7.45 in the definition of "inflation". It can also be equivalently replaced by the conjunction of (a) and (b) in Lemma 3.7.43.

Extending inflations

By Definition 3.7.21 it follows easily that:

Lemma 3.7.46. *Let I' be an inflation of I with history $\langle a, e\rangle$. Let $1 \leq \mu \leq$ $\mathrm{lh}(I')$. Then $I'|\mu$ is an inflation of I with history $\langle a \restriction \mu, e \restriction \mu\rangle$.*

Proof. (1)-(5) continue to hold.

Taking $\mu = 1$ it becomes evident that an inflation might say very little about the original iteration I. Hence it is useful to have lemmas which enable us to extend a given inflation I' to an I'' of greater length, thus "capturing" more of I. We prove two such lemmas:

Lemma 3.7.47. *Let I be a normal iteration of M of length $\eta' + 1$. Let I' be an inflation of I of length $\eta' + 1$ with history $\langle a, e\rangle$, where $a_{\eta'} < \eta$. Let $\tilde{\nu} = \sigma^{e_{\eta'}}_{a_{\eta'}}(\nu'_{a_{\eta'}})$ be defined with: $\tilde{\nu} > \nu'_i$ for $i < \eta$. Extend I' to I'' of length $\eta' + 2$ by appointing $\nu'_{\eta'} = \tilde{\nu}$. Then I'' is an inflation of I with history $\langle a', e'\rangle$ where:*

- *$a' \restriction \eta' + 1 = a$, $e'_\eta = e_\eta$ for $\eta \leq \eta'$,*
- *$a'_{\eta'+1} = a_{\eta'} + 1, e'_{\eta'+1} \restriction a_{\eta'} + 1 = e_{\eta'}$,*
- *$e'_{\eta'+1}(a_{\eta'} + 1) = \eta' + 1$.*

Proof. We must show that (1)-(5) are satisfied. The only problematical case is (5). We must show that if $\gamma <_{T''} \eta' + 1$, then

$$e_\gamma^{-1} \restriction \gamma = e'^{-1}_{\eta'+1} \restriction \gamma.$$

It suffices to prove it for $\gamma = T''(\eta' + 1)$. Let $\overline{\gamma} = T(a_{\eta'} + 1)$. Then

$$\hat{e}_{\eta'}(\overline{\gamma}) \leq_{T'} \gamma \leq_{T'} e_{\eta'}(\gamma)$$

by Lemma 3.7.1 (3). Hence

$$a_\gamma = \overline{\gamma} \text{ and } e_\gamma \upharpoonright a_\gamma = e_{\eta'} \upharpoonright a_\gamma$$

by Lemma 3.7.46. But then

$$e_\gamma^{-1} \upharpoonright \gamma = e_{\eta'}^{-1} \upharpoonright \gamma = (e'_{\eta'+1})^{-1} \upharpoonright \gamma$$

since $e_{\eta'}(\overline{\gamma}) = e'_{\eta'+1}(\overline{\gamma}) \geq \gamma$.

QED(Lemma 3.7.47)

Lemma 3.7.48. *Let I' be an inflation of I of limit length η'. Let b be the unique cofinal well founded branch in I'. Extend I' to I'' of length $\eta' + 1$ by appointing: $\{\xi : \xi <_{T''} \eta'\} = b$. Then I'' is an inflation of I with history $\langle a', e' \rangle$, where:*

$$a' \upharpoonright \eta' = a,\ a'_{\eta'} = \sup_{\beta \in b} a'_\beta,\ e' \upharpoonright \eta' = e \upharpoonright \eta',$$
$$e'_\eta \upharpoonright a'_{\eta'} = \bigcup_{\beta \in b} e_\beta \upharpoonright a_\beta,\ e'_{\eta'}(a'_\eta) = \eta'.$$

Proof. (1)-(5) are satisfied.

Composing Inflations

We now show that if I' in an inflation of I and I'' is an inflation of I', then I'' is an inflation of I.

Theorem 3.7.49. *Let I, I', I'' be normal iteration of M with:* $\mathrm{lh}(I) = \eta + 1, \mathrm{lh}(I') = \eta' + 1$. *Let I' be an inflation of I with:*

$$\mathrm{hist}(I, I') = \langle a, e \rangle.$$

Let I'' be an inflation of I' with:

$$\mathrm{hist}(I', I'') = \langle a', e' \rangle.$$

Then I'' is an inflation of I with:

$$\mathrm{hist}(I, I'') = \langle a'', e'' \rangle,$$

where: $a''_\alpha = a_{a'_\alpha}$, $e''_\alpha = e'_\alpha e_{a'_\alpha}$.

Proof. We verify (1)-(5).

(1) $a'' = a \cdot a'$ clearly maps $\mathrm{lh}(I'')$ into $\mathrm{lh}(I)$. Since e'_α inserts $I'|a'_\alpha + 1$ into $I''|\alpha + 1$ and $e_{a'_\alpha}$ inserts $I|a''_\alpha + 1$ into $I'|a'_\alpha + 1$, then $e'_\alpha \cdot e_{a'_\alpha}$ inserts $I|a''_\alpha + 1$ into $I''|\alpha + 1$.

QED(1)

Now let:

$$\begin{aligned} I &= \langle \langle M_\alpha \rangle, \langle \nu_\alpha \rangle, \langle \pi_{\alpha,\beta} \rangle, T \rangle \\ I' &= \langle \langle M'_\alpha \rangle, \langle \nu'_\alpha \rangle, \langle \pi'_{\alpha,\beta} \rangle, T' \rangle \\ I'' &= \langle \langle M''_\alpha \rangle, \langle \nu''_\alpha \rangle, \langle \pi''_{\alpha,\beta} \rangle, T'' \rangle \end{aligned}$$

We recall by Lemma 3.7.5 that if e inserts I into I' and e' inserts I' into I'' then $e'e$ inserts I into I''. Moreover:

$$\sigma_\xi^{e' \cdot e} = \sigma_{e'(\xi)}^{e'} \cdot \sigma_\xi^e.$$

Thus, in particular:

$$\sigma_\xi^{e''_\alpha} = \sigma_\xi^{e'_\alpha \cdot e_{a'_\alpha}} = \sigma_{e'_\alpha(\xi)}^{e'_\alpha} \cdot \sigma_\xi^{e_{a'_\alpha}} \text{ for } \xi < a''_\alpha.$$

(2) If $\tilde{\nu}''_\alpha = \sigma_{a''_\alpha}^{e_{a''_\alpha}}(\nu_{a''_\alpha})$ exists and $\alpha < \mathrm{lh}(I'')$, then:

$$\tilde{\nu}''_\alpha = \sigma_\alpha^{e_{a'_\alpha}} \cdot \sigma_{a'_\alpha}^{e_{a'_\alpha}}(\nu_{a_{a'_\alpha}}) = \sigma_\alpha^{e_{a'_\alpha}}(\tilde{\nu}'_{a'_\alpha}).$$

But then $\nu'_{a'_\alpha} \leq \tilde{\nu}'_{a'_\alpha}$ and:

$$\nu''_\alpha \leq \sigma_\alpha^{e'_\alpha}(\nu'_{a'_\alpha}) \leq \tilde{\nu}''_\alpha.$$

QED(2)

Now let:

$\mathrm{in}(\alpha) =$ the index of α with respect to I, I',

$\mathrm{in}'(\alpha) =$ the index of α with respect to I', I'',

$\mathrm{in}''(\alpha) =$ the index of α with respect to I, I''.

(3) It is easily seen that if $\mathrm{in}''(\alpha) = 0$, then $\mathrm{in}(a'_\alpha) = \mathrm{in}'(\alpha) = 0$. Hence:

$$a'_{\alpha+1} = a'_\alpha + 1, a''_{\alpha+1} = a_{a'_{\alpha+1}} = a_{(a'_\alpha + 1)} = a''_\alpha + 1.$$

Moreover:

$$\begin{aligned} e''_{\alpha+1} \upharpoonright a''_\alpha + 1 &= e'_{\alpha+1} e_{a'_\alpha+1} \upharpoonright a_{a'_\alpha} + 1 \\ &= e'_{\alpha+1} \cdot e_{a'_\alpha} \\ &= e'_{\alpha+1} \upharpoonright (a'_\alpha + 1) \cdot e_{a'_\alpha} \\ &= e'_\alpha \cdot e_{a'_\alpha} = e''_\alpha. \end{aligned}$$

QED(3)

(4) Assume $\mathrm{in}''(\alpha) = 1$. Then either $\mathrm{in}'(\alpha) = 1$ or $\mathrm{in}(a'_\alpha) = 1$.

Case 1. $\mathrm{in}'(\alpha) = 1$.

Let $\gamma = T''(\alpha + 1)$. Thus $a'_\gamma = a'_{\alpha+1}$. Hence

$$a''_\gamma = a_{a'_\gamma} = a_{a'_{\alpha+1}} = a''_{\alpha+1}.$$

Case 2. $\mathrm{in}(a'_\alpha) = 1$ but $\mathrm{in}'(\alpha) = 0$.

Let $\gamma = T'(a'_\alpha + 1)$. Then:

$$a_\gamma = a_{(a'_\alpha+1)} = a_{a'_{\alpha+1}} = a''_{\alpha+1}.$$

Let $\beta = T''(\alpha + 1)$. Then:

$$\hat{e}_\alpha(\gamma) \leq_{T''} \beta \leq_{T''} e_\alpha(\gamma).$$

Hence by Lemma 3.7.45:

$$\gamma = a'_\beta,\ a''_{\alpha+1} = a_\gamma = a_{a'_\beta} = a''_\beta.$$

QED(4)

(5) Let $\beta <_{T''} \alpha$. Then $a'_\beta \leq_{T''} a'_\alpha$ and hence:

$$a''_\beta = a_{a'_\beta} \leq_T a_{a'_\alpha} = a''_\alpha.$$

But then $(e'_\alpha)^{-1} \upharpoonright \beta = (e'_\alpha)^{-1} \upharpoonright \beta$ and

$$(e_{a'_\beta})^{-1} \upharpoonright a'_\beta = (e_{a'_\alpha})^{-1} \upharpoonright a'_\beta.$$

Hence:

$$\begin{aligned} [(e''_{be})^{-1} \upharpoonright \beta &= (e'_{a'_\beta})^{-1} (e'_\beta)^{-1} \upharpoonright \beta \\ &= (e_{a'_\beta})^{-1} (e'_\alpha)^{-1} \upharpoonright \beta \\ &= (e_{a'_\alpha})^{-1} (e'_\alpha)^{-1} \upharpoonright \beta \\ &= (e''_\alpha)^{-1} \upharpoonright \beta. \end{aligned}$$

QED(5)

This proves Theorem 3.7.49.

3.7.5 Smooth Reiterability

In §3.7.2 we proved that if M is uniquely normally iterable, then it is normally reiterable. In this section we prove the fact announced in §3.7.4. that if M is uniquely normally iterable, then it is smoothly reiterable. Just as before, it will also be of interest to know whether this theorem can be relativized to a regular cardinal $\kappa > \omega$. We called a normal reiteration $R = \langle\langle I^i\rangle, \dots\rangle$ a κ-iteration iff each of its component normal iteration I^i has length less than κ. If we are given a smooth κ-reiteration $S = \langle\langle I_i\rangle, \langle e_{i,j}\rangle\rangle$, we call it a *smooth κ-reiteration* iff each of its induced reiteration R_i $(i + 1 < \mathrm{lh}(S))$ is a κ-reiteration of length less than κ. We proved previously that, if M is uniquely normally κ-iterable, then it is normally κ-reiterable. In the present case the proofs are more subtle, and the best we can get is:

Theorem 3.7.50. *Let $\kappa > \omega$ be regular. Let M be uniquely normally $\kappa+1$-iterable. Then it is smoothly $\kappa + 1$-reiterable. (Hence if M is uniquely normally iterable, it is uniquely smoothly reiterable).*

We don't see any way to weaken the hypothesis of this theorem. Thus, for instance, if we only know that M is uniquely normally ω_1-iterable, we have no proof that it is smoothly ω_1-iterable.

We prove Theorem 3.7.50. From now on we take "reiteration" as meaning "κ-reiteration" and "smooth reiteration" as meaning "smooth κ-reiteration". We assume M to be uniquely normally κ+1-iterable. The desired conclusion then is given by:

Lemma 3.7.51. *Let $S = \langle\langle I_i\rangle, \langle e_{i,j}\rangle\rangle$ be a smooth reiteration of M of limit length $\mu \leq \kappa$. Then:*

(a) *S has at most finitely many drop points.*

(b) *S has a good limit $I, \langle e_i : i < \mu\rangle$.*

Proof. Case 1. $\mu = \kappa$.

(a) is immediate by $\mathrm{cf}(\kappa) > \omega$, since if S had infinitely many drop points, then so would $S|\gamma + 1$ for some $\gamma < \kappa$.

To prove (b), let (i, κ) be free of drop points, where $i < \kappa$. We must show that $\langle\langle I_j : i \leq j < \kappa\rangle, \langle e^{hj} : i \leq h \leq j < \kappa\rangle\rangle$ has a good limit:

$$I, \langle e^j : i \leq j < \kappa\rangle.$$

(We then set: $e^h = e^i \cdot e^{h,i}$ for $h < i$). But this is immediate by Lemma 3.7.9.

QED(Case 1)

The hard case is:

Case 2. $\mu < \kappa$.

By induction on μ we prove (a), (b) and:

(c) If $i < \mu$, then I is an inflation of I_i with history $\langle a^i, \langle e^i_\alpha : \alpha \leq \eta_i \rangle\rangle$, where $\eta_i + 1 = \mathrm{lh}(I_i)$.

(d) If $i < \mu$ and (i, μ) has no drop point in S, then $a^i_\mu = \eta_i$ and $e^i_\mu = e_i$.

Assume that this holds at every limit ordinal $\lambda < \mu$. Then:

Claim 1. Let $i \leq j < \mu$. Then

(i) I_j is an inflation of I_i with history $\langle a^{ij}, \langle e^{i,j}_\alpha : \alpha \leq \eta_j \rangle\rangle$.

(ii) If the interval (i, j) has no drop point in S, then $a^{i,j}_{\eta_j} = \eta_i$ and $e_{i,j} = e^{i,j}_{\eta_j}$.

Proof. Suppose not. Let j be the least counterexample. Then $i < j$ since (i), (ii) hold trivially for $i = j$. But j is not a limit ordinal since otherwise (i), (ii) hold by the induction hypothesis. Hence $j = h + 1$. We first show that it holds for $i = h$.

(i) is immediate by Theorem 3.7.42. We now prove (ii) for $i = h$. Let R, ξ be the unique objects such that:

$$R = \langle \langle I^l \rangle, \langle \nu^l \rangle, \langle e^{k,l} \rangle, T \rangle$$

is a normal reiteration of length $\xi + 1$ and $I_h = I^0, I_j = I^\xi$. Then $e_{h,j} = e^{0,\xi}$. Since R has no truncation on its main branch, $e_{h,j}$ inserts I_h into I_j and $e_{h,j}(\eta_h) = \eta_j$. But $a^{h,j}_\alpha = \{a < \eta_h : e_{h,j}(\alpha) < \eta_j\}$. Hence $a^{h,j}_{\eta_j} = \eta_h$. But:

$$e_{h,j} \upharpoonright \eta_h = e^{h_j}_{\eta_j} \upharpoonright \eta_h \text{ and } e_{h,j}(\eta_h) = e^{h,j}_{\eta_j}(\eta_h) = \eta_j$$

Hence $e_{i,h} = e^{h,j}_{\eta_j}$.

But then $i < h$. We know that (i), (ii) hold at h and that

$$a^{i,j}_\alpha = a^{i,h}_{a^{h,j}_\alpha};\ e^{i,j}_\alpha = e^{h,j}_\alpha \cdot e^{i,h}_{a^{h,j}_\alpha},$$

where $a^{h,j}_{\eta_i} = \eta_h, a^{i,h}_{\eta_h} = \eta_i, e_{i,h} = e^{i,h}_{\eta_h}, e_{hj} = e^{h,j}_{\eta_j}$. Thus:

$$\begin{aligned} a^{i,j}_{\eta_i} &= a^{i,h}_{\eta_h} = \eta_i \text{ and} \\ e_{i,j} &= e_{h,j} \cdot e_{i,h} = e^{h,j}_{\eta_j} \cdot e^{i,h}_{\eta_h} = e^{i,j}_{\eta_j} \end{aligned}$$

Contradiction!

QED(Claim 1)

We now attempt to prove (a)-(d), taking an indirect approach. Call I a *simultaneous inflation* if it is an inflation of I_i for each $i < \mu$. Our job is to find a simultaneous inflation which also satisfies the conditions (a), (b) and (d). There is no shortage of simultaneous inflations. For instance the normal iteration of length 1:

$$\langle\langle M\rangle, \varnothing, \langle \mathrm{id} \restriction M\rangle, \varnothing\rangle$$

is a simultaneous inflation. Starting with this, we attempt to form a tower of simultaneous inflations $I^{(i)}$, where $I^{(\xi)}$ is an iteration of length $\xi + 1$ extending $I^{(i)}$ for $i < \xi$. The attempt will have only limited success. If we have constructed $I^{(\xi)}$ for ξ below a limit ordinal λ, we shall, indeed, be able to construct $I^{(\lambda)}$. In attempting to go for $I^{(\xi)}$ to $I^{(\xi+1)}$, however, we may encounter a "bad case", which blocks us from going further. Using the $\kappa+1$-normal iterability of M we can, however, show that, if the bad case does not occur, we reach $I^{(\kappa)}$. But this turns out to be a contradiction. Hence the bad case must have occurred below κ. A close examination of this "bad case" then reveals it to be a very good case, since it gives $I = I^{(\xi)}$ satisfying (a)-(d).

In the following let:

$$I_i = \langle\langle M^i_\alpha\rangle, \langle\nu^i_\alpha\rangle, \langle\pi^i_{\alpha,\beta}\rangle, T^i\rangle \text{ be of length } \eta_i + 1.$$

We attempt to construct:

$$I = \langle\langle M_\alpha\rangle, \langle\nu_\alpha\rangle, \langle\pi_{\alpha,\beta}\rangle, T\rangle \text{ of length } \eta + 1$$

satisfying (a)-(d).

We successively construct:

$$I^{(\xi)} = \langle\langle M^{(\xi)}_\alpha\rangle, \langle\nu^{(\xi)}_\alpha\rangle, \langle\pi^{(\xi)}_{\alpha,\beta}\rangle, T^{(\xi)}\rangle \text{ of length } \eta + 1.$$

The intention is that $I^{(\xi)} = I|\xi + 1$ will be defined up to an $\eta < \theta$ and that $I = I^{(\eta)}$ will have the desired properties (a)-(d). The proof that there is such an η is highly indirect and non constructive. We shall require:

(A) $I^{(\xi)}$ is an inflation of I_i with history

$$\langle a^{(\xi),i}, e^{(\xi),i}\rangle \text{ for } i < \mu.$$

(B) $i < \xi \longrightarrow I^{(i)} = I^{(\xi)}|i+1$.

Note. By (B) we can write $M_\alpha, \nu_\alpha, \pi_{\alpha,\beta}, T, I$ instead of $M_\alpha^{(\xi)}$, etc. without reference to ξ. Similarly we can write a^i, e^i instead of $a^{(\xi),i}, e^{(\xi),i}$. Thus, for $\alpha \leq \xi$ we have:

$$a^i_\alpha \leq \eta_i \text{ and } e^i_\alpha \text{ inserts } I^i|a^i_\alpha + 1 \text{ into } I|\alpha+1.$$

(C) Let $\alpha \leq \xi$. Then $\alpha = \bigcup_{i<\mu} e^i_\alpha``a^i_\alpha$.

By (C) we have:

(1) $\alpha = \sup\{\hat{e}^i_\alpha(a^i_\alpha) : i < \mu\}$, since $\hat{e}^i_\alpha(a^i_\alpha) = \text{lub}\, e^i_\alpha``a^i_\alpha$.

Set: $e^{i,j}_{(\alpha)} = e^{i,j}_{a^i_\alpha}$. Hence by (C) we have:

(2) $I|\alpha+1, \langle e^i_\alpha : i < \mu\rangle$ is the good limit of

$$\langle I^i|a^i_\alpha + 1 : i < \mu\rangle, \langle e^{i,j}_{(\alpha)} : i \leq j < \mu\rangle$$

Now set: $\sigma^i_{(\alpha)} = \sigma^{e^i_\alpha}_{a^i_\alpha}, \sigma^{i,j}_{(\alpha)} = \sigma^{e^{i,j}_{(\alpha)}}_{a^i_\alpha}$. Then: $\sigma^h_{(\alpha)} e^{h,i}_{(\alpha)} = e^h_{(\alpha)}$. We can define $\hat{\sigma}^i_{(\alpha)}, \hat{\sigma}^{(i)}_{(\alpha)}$, similarly. Note, however, that $\sigma^i_{(\alpha)}$ might be a partial function on $M^i_{a^i_\alpha}$, whereas $\hat{\sigma}^i_{(\alpha)}$ is a total function. Nonetheless we do have:

(3) $\sigma^i_{(\alpha)} : M^i_{a^i_\alpha} \longrightarrow_{\Sigma^*} M_\alpha$ for sufficiently large $i < \kappa$.

Proof. $\sigma^i_{(\alpha)} = \pi_{\hat{e}^i_{(\alpha)}(a^i_\alpha),\alpha} \cdot \hat{\sigma}^i_{(\alpha)}$, where:

$$\hat{\sigma}^i_{(\alpha)} : M^i_{a^i_\alpha} \longrightarrow_{\Sigma^*} M_{e^i_{(\alpha)}(a^i_\alpha)}.$$

By (1) we can pick i big enough that there is no truncation in $(e^i_\alpha(a^i_\alpha), \alpha]_T$. Hence $\pi_{e^i_{(\alpha)}(a^i_\alpha),\alpha}$ is Σ^*-preserving.

QED(3)

We construct $I^{(\xi)} = I|\xi+1$ by recursion on ξ as follows:

Case 1. $\xi = 0$.

$I^{(0)} = \langle\langle M\rangle, \varnothing, \langle \text{id} \restriction M\rangle, \varnothing\rangle$ is the 1-step iteration of M. (A)-(C)hold trivially.

Case 2. $\xi = \theta + 1$ and $a^i_\theta < \eta_i$ for arbitrarily large $i < \mu$. Let D be the set of i such that:

$$a^i_\theta < \eta_i \text{ and } \sigma^i_{(\theta)} : M^i_{a^i_\theta} \longrightarrow_{\Sigma^*} M_\theta$$

Then D is unbounded in μ by (3). Clearly:

$$\sigma^{i,j}_{(\theta)} : M^i_{a^i_\theta} \longrightarrow_{\Sigma^*} M^i_{a^i_\theta} \text{ for } i \in D, j \in D \setminus i.$$

Hence:

$$\sigma^{i,j}_{(\theta)}(\nu^i_{a^i_\theta}) \geq \nu^i_{a^i_\theta} \text{ for } i \in D, j \in D \setminus i.$$

But then for sufficiently large $i \in D$ we have:

$$\sigma^{i,j}_{(\theta)}(\nu^i_{a^i_\theta}) = \nu^i_{a^i_\theta} \text{ for } j \in D \setminus i.$$

(To see this, suppose not. Then there is a monotone sequence $\langle i_n : n < \omega \rangle$ such that $i_n \in D$ and

$$\sigma^{i_n, i_{n+1}}_{(\theta)}(\nu^{i_n}_{a^{i_n}_\theta}) > \nu^{i_{n+1}}_{a^{i_{n+1}}_\theta}.$$

Set $\gamma_n = \sigma^{i_n}_{(\theta)}(\nu^{i_n}_{a^{i_n}_\theta})$. Then: $\gamma_n > \gamma_{n+1}$. Hence M_θ is ill founded. Contradiction!)

Let D' be the set of such $i \in D$. Then there is $\nu \in M_\theta$ such that $\nu = \sigma^i_{(\theta)}(\nu^i_{a^i_\theta})$ for $i \in D$.

Claim. $\nu > \nu_\delta$ for $\delta < \theta$.

Proof. Pick an $i \in D$ large enough that $\delta \in e^i_\theta$“a^i_θ. Let $e^i_\theta(\bar{\delta}) = \delta$. Then $\nu^i < \nu^i_{a^i_\theta}$. Hence

$$\nu_\delta = \nu = \sigma^i_{(\theta)}(\nu^i_{\bar{\delta}}) < \sigma^i_{(\theta)}(\nu^i_{a^i_\theta}) = \nu$$

QED(Claim)

We are now in a position to apply the extension lemma Lemma 3.7.47. Extend $I^{(\theta)}$ to $I^{(\theta+1)}$ by setting $\nu_\theta = \nu$. For each $i \in D', I' = I^{(\theta+1)}$ is an inflation of I_i with history $\langle a^{i'}, e^{i'} \rangle$, where:

$$a^{i'} \restriction \theta + 1 = a^i, a^{i'}_{e+1} = a^i_e + 1, e^{i'} \restriction a^i_\theta = e^i \restriction a^i_\theta \text{ and } e^{i'}_{\theta+1}(a^{i'}_{\theta+1}) = \theta + 1.$$

But D' is cofinal in μ. It follows easily that I' is an inflation of each I_i $(i < \mu)$. Thus (A) holds for $\xi = \theta + 1$. (B) follows trivially. (C) holds trivially for $\alpha \leq \theta$. But then (c) holds for $\alpha = \xi = \theta + 1$, since $\sigma^i_\theta(a^i_\theta) = \theta$ for $i < \mu$ and $\theta = \bigcup_{\delta < \mu} e^i_\theta$“$a^i_\theta$.

QED(Case 2)

Case 3. $\xi = \theta + 1$ and Case 2 fails.

Then $a^i_\theta = \eta^i$ for sufficiently large i. This is the "bad case" in which $I^{(\theta+1)}$ is undefined.

Case 4. $\xi = \lambda$ is a limit ordinal.

Let $\tilde{I} = I|\lambda$ be the componentwise union: $\tilde{I} = \bigcup_{\gamma<\lambda} I^{(\gamma)}$. $\tilde{I}$ is then an inflation of I_i $(i < \mu)$ with history:

$$a^i \restriction \lambda =: \bigcup_{\gamma<\lambda} a^i \restriction \gamma,\ e' \restriction \lambda = \bigcup_{\gamma<\lambda} e^i|\gamma.$$

Let b be the unique well founded cofinal branch in $\tilde{I}$. Extend $\tilde{I}$ to $I' = I^{(\lambda)}$ of length $\lambda + 1$ by setting: $T``\{\lambda\} = b$. By Lemma 3.7.48, I' is then an inflation of each I_i with history $\langle {a'}^i, {e'}^i \rangle$ such that:

$${a'}^i \restriction \lambda = a^i \restriction \lambda, {e'}^i \restriction \lambda = e^i \restriction \lambda,\ {a'}^i_\lambda = \bigcup_{\beta\in b} a^i_\beta,\ \tilde{e}^i_\lambda(a^i_\lambda) = \lambda.$$

(A), (B) are then trivially satisfied. But then so is (C) since

$$\bigcup_{i\in\mu} e^i_i``a^i_\lambda = \bigcup_{i\in\mu}\bigcup_{\beta\in b} e^i_\beta``a^i_\beta = \bigcup_{\beta\in b}\bigcup_{i<\mu} e^i_\beta``a^i_\beta = \bigcup b = \lambda.$$

QED(Case 4)

We note that the construction in Case 4 goes through for $\lambda = \kappa$, since M is $\kappa + 1$-normally iterable. Hence $I^{(\kappa)}$ would exist if the bad case did not occur. This is impossible, however, since:

(4) If λ is a limit ordinal and $I^{(\lambda)}$ exists, then $\mathrm{cf}(\lambda) \leq \mu$ or $\mathrm{cf}(\lambda) \leq \eta_i$ for some $i < \mu$.

Proof. Suppose first that $\lambda > \hat{e}^i_\lambda(a^i_\lambda)$ for all $i < \mu$. Since $\lambda = \mathrm{lub}_{i<\mu}\, \hat{e}^i_\lambda(a^i_\lambda)$ by (1), we conclude that $\mathrm{cf}(\lambda) \leq \mu$. Otherwise $\lambda = \hat{e}^i_\lambda(a^i_\lambda) = \mathrm{lub}\, e^i_\lambda``a^i_\lambda$. Hence a^i_λ is a limit ordinal. Hence $\mathrm{cf}\,\lambda \leq a^i_\lambda \leq \eta_i$.

QED(4)

Hence the "bad case" occurs at $\xi = \delta + 1$, where $\delta < \kappa$. $I = I^{(\delta)}$ is the final element of our tower. For sufficiently large $i < \mu$ we have: $a^i_\delta = \eta_i$. Thus if $i \leq j < \mu$ we have:

$$a^{i,j}_{\eta_j} = a^{i,j}_{a^j_\delta} = a^i_\delta = \eta_i,\ e^{i,j}_{\eta_i} = e^{i,j}_{(\delta)}.$$

We now show:

(5) There are only finitely many drop points $h + 1 < \mu$ in S.

Proof. Suppose not. Since the assertion is true for all $\mu' < \mu$, we conclude that here are cofinally many truncation points $h + 1 < \mu$ in

S. By (1), we can then pick such an $h+1>i$, where i is chosen such that $(\hat{e}^i_\delta(a^i_\delta),\delta)_T$ has no truncation point in I. But we can also choose i large enough that $a^i=\eta_i$. By Theorem 3.7.42(6) there is a drop point:

$$\alpha \in (\hat{e}^{i,i+1}_{\eta_i}(a^i_{\eta_i}),\eta_{i+1}]_{T^{i+1}}.$$

By Lemma 3.7.1(7) we then conclude that there is a drop point in $(\hat{e}^i_{\eta_i}(a^i_{\eta_i}),\delta)_T$. Contradiction!

QED(5)

Now suppose i_0 is chosen large enough that there is no drop point in (i,δ) in S, and that $a^i_\theta=\eta_j$ for $i_0\le j<\theta$. By Claim (1)(ii), we have

$$a^{i,j}_{\eta_i}=\eta_i \text{ and } e_{i,j}=e^{i,j}_{\eta_i}=e^{ij}_{(\theta)}$$

for $i_0\le i\le j<\theta$. By (2) we have:

$$I,\langle e^i_\theta : i_0\le i<\mu\rangle$$

is the good limit of

$$\langle I^i|\eta_i+1 : i_0\le i<\mu\rangle,\ \langle e_{i,j} : i_0\le j<\mu\rangle$$

We have thus proven (a), (b) in Lemma 3.7.51. (c) and (d) are immediate by the construction.

This proves Lemma 3.7.51 and, with it, Theorem 3.7.50.

Note. By the same method we get:

Let S be an insertion stable strategy for M and assume that $\langle M,S\rangle$ is $\kappa+1$-normally-iterable. Then $\langle M,S\rangle$ is κ-smoothly-iterable.

The proofs require only cosmetic changes.

We note the following consequence of Lemma 3.7.51:

Lemma 3.7.52. *Let $S=\langle\langle I_i\rangle,\langle e_{i,j}\rangle\rangle$ be a smooth reiteration of M of length μ, where each I_i is of length η_i+1. For $j<\mu$ set:*

$$A_j=\{i<j : (i,j] \text{ has no drop points in } S\},\ A^*_j=A_j\cup\{j\}.$$

(Hence $i\in A_j\longrightarrow A_i=i\cap A_j$). For $i\in A^*_j$ set: $\pi_{i,j}=\sigma^{e_{i,j}}_{\eta_i}$. Then:

(a) $\pi_{i,j} \cdot \pi_{h,i} = \pi_{h,j}$ *for* $h \leq i \leq j$ *in* A^*_j.

(b) $\pi_{i,j} : M_{\eta_i} \longrightarrow_{\Sigma^*} M_{\eta_i}$.

(c) *If* $j = \lambda$ *is a limit ordinal, then:*

$$M_{\eta_\lambda},\ \langle \pi_{i,\lambda} : i \in A_\lambda \rangle$$

is the direct limit of:

$$\langle M_{\eta_\lambda} : i \in A_\lambda \rangle,\ \langle \pi_{i,j} : i < j \text{ in } A_\lambda \rangle$$

Proof.

(a) Since $e_{h,i}(\eta_h) = \eta_i$ and $e_{i,j}(\eta_i) = \eta_j$, we have: $\sigma^{e_{h,j}}_{(\eta_h)} = \sigma^{e_{i,j}}_{(\eta_i)} \cdot \sigma^{e_{h,i}}_{(\eta_h)}$.

We prove (b), (c) by induction on j as follows:

Case 1. $j = 0$. Then $A_j = \varnothing$ and there is nothing to prove.

Case 2. $j = i + 1$. We must prove (b). If $i + 1$ is a drop point, then $A_j = \varnothing$ and there is nothing to prove. If not, it suffices to prove it for $h = i$, by (a) and the induction hypothesis. Then the main branch of R_i has no drop point in R_i, where R_i is the unique reiteration from I^i to I^{i+1}. Then $\pi_{i,i+1} = (\sigma^{0,\gamma}_{\eta_i})^{R_i}$, where $\gamma + 1 = \text{lh}(R_i)$. But:

$$\sigma^{0,\gamma}_{\eta_i} : M_{\eta_i} \longrightarrow_{\Sigma^*} M_{\eta_{h+1}} \text{ in } R_i.$$

QED(Case 2)

Case 3. $j = \lambda$ is a limit ordinal.

It suffices to prove (c), since (b) then follows by the induction hypothesis. In S we have:

$$I_\lambda,\ \langle e_{i,\lambda} : l \in A_\lambda \rangle$$

is the good limit of

$$\langle I_i : i \in A_\lambda \rangle,\ \langle \pi_{i,j} : i \leq j \text{ in } A_\lambda \rangle$$

But then $M_\eta = \bigcup_{i \in A_\lambda} \text{rng}(\sigma^{i,\lambda}_{\eta_i})$. This implies (c).

QED(Lemma 3.7.52)

3.7.6 The final conclusion

We now apply the method of §3.7.3 to show that M is smoothly iterable. In §3.5.2 we defined a smooth iteration of N to be a sequence $I = \langle I_i : i < \mu \rangle$ of normal iterations, inducing sequences $\langle N_i : i < \mu \rangle$, $\langle \pi_{i,j} : i \leq j < \mu \rangle$ with the following properties:

- N_i is the initial model of I_i. Moreover $N_0 = N$.
- Let $i + 1 < \mu$. Then I_i is of successor length. N_{i+1} is the final model of I_i and $\pi_{i,i+1}$ is the partial embedding of N_i into N_{i+1} determined by I_i.
- $\pi_{i,j}\pi_{h,i} = \pi_{h,i}$.
- Call $i + 1 < \mu$ a *drop point* in I iff I_i has a truncation on its main branch. If the interval $(i, j]$ has no drop point, then:

$$\pi_{i,j} : N_i \longrightarrow_{\Sigma^*} N_j.$$

- If $\lambda < \mu$ is a limit ordinal, $i_0 < \lambda$ and (i, λ) has no drop point, then:

$$N_\lambda, \langle \pi_{i,\lambda} : i_0 \leq i < \mu \rangle$$

  is the direct limit of

$$\langle N_i : i_0 \leq i < \mu \rangle, \ \langle \pi_{i,j} : i \leq j < \mu \rangle.$$

$\langle \langle N_i \rangle, \ \langle \pi_{i,j} \rangle \rangle$ is called the *induced* sequence.

Call a smooth iteration I *critical* if it has successor length $\eta + 1$ and I_η is of limit length. By a *strategy for* N we mean a partial function S defined on critical smooth iterations such that $S(I)$, if defined, is a well founded cofinal branch in I_η, where $\mathrm{lh}(I) = \eta + 1$.

A smooth iteration $I = \langle I_i : i < \mu \rangle$ is S-conforming iff whenever $i < \mu$ and $\lambda < \mathrm{lh}(I_i)$ is a limit ordinal, and $I^* = I \restriction i \cup \{\langle I_i \restriction \lambda, i \rangle\}$, then:

$$T^{i\prime\prime}\{\lambda\} = S(I^*) \text{ if } S(I^*) \text{ is defined.}$$

S is a *successful strategy* for N iff every S-conforming smooth iteration I of N can be properly extended in any legitimate S-conforming way. In other words:

(A) Let I have length $\eta + 1$ and let I_η have length $i + 1$. Let $Q = N_i^\eta$ be the final model of I_η. Let $E_\nu^Q \neq \varnothing$, where ν is greater than all the indices ν_j^η $(j < i)$ employed in I_η. Then Q is $*$-extendible by E_ν^Q.

(B) If I is critical, then $S(I)$ is defined.

(C) Let I have limit length μ. Then there are only finitely many drop points in I. Moreover, if $l_0 < \mu$ and (i_0, μ) is free of drops, then:

$$\langle N_i : i_0 \leq i < \mu \rangle,\ \langle \pi_{i,j} : i \leq j < \mu \rangle$$

has a well founded direct limit:

$$N_\mu, \langle \pi_{i,\mu} : i_0 \leq i < \mu \rangle$$

We say that N is *smoothly iterable* iff it has a successful smooth iteration strategy.

These concepts can, of course, be relativized to an ordinal α. To this end we define the *total length* of $I = \langle I_i : i < \mu \rangle$ to be:

$$\mathrm{tl}(I) = \sum_{i<\mu} \mathrm{lh}(I_i).$$

The notion of *α-successful smooth iteration* strategy is then defined as before, except that we restrict ourselves to iteration of total length less than α.

Note that if $\kappa > \omega$ is regular, then there are only two ways that a smooth iteration $I = \langle I_i : i < \mu \rangle$ can have total length κ. Either $\mu = \kappa$ and $\mathrm{lh}(I_i) < \kappa$ for $i < \kappa$, or else $\mu = \eta + 1 < \kappa$, $\mathrm{lh}(I_\eta) = \kappa$ and $\mathrm{lh}(I_i) < \kappa$ for $i < \eta$.

In this section we shall prove:

Theorem 3.7.53. *Let M be uniquely normally iterable. Then it is smoothly iterable.*

Note. There is of course, considerable interest in relativizing this theorem to $\alpha < \infty$. We shall later show that, if $\kappa > \omega$ is regular, then the theorem can be relativized to $\kappa + 1$. That will require fairly modest changes in the proof we give now.

Until further notice, assume M to be uniquely normally iterable. We prove our Theorem 3.7.53 in the slightly stronger form:

Lemma 3.7.54. *Let I be a normal iteration of M of length $\eta + 1$. Let:*

$$\sigma : N \longrightarrow_{\Sigma^*} M_\eta \min \rho$$

Then N is smoothly iterable.

In §3.7.3 we used the premiss of Lemma 3.7.54 to derive the normal iterability of N. We first briefly review that proof, since our new proof will build upon it. Our main tool was the *reiteration mirror* (RM). Given a normal iteration of N:

$$I = \langle\langle N_i\rangle, \langle\nu_i\rangle, \langle\pi_{i,j}\rangle, T\rangle \text{ of length } \eta,$$

we define a reiteration mirror of I to be a pair $\langle R, I'\rangle$ such that:

(a) $R = \langle\langle I^i\rangle, \langle\nu'_i\rangle, \langle e^{i,j}\rangle, T\rangle$ is a reiteration of M of length η, where:

$$I^i = \langle\langle M^i_h\rangle, \langle\nu^i_h\rangle, \langle\pi^i_{h,j}\rangle, T^i\rangle \text{ is of length } \eta_i + 1$$

(b) $I' = \langle\langle M'_i\rangle, \langle\pi'_{i,h}\rangle, \langle\sigma_i\rangle, \langle\rho^i\rangle\rangle$ is a mirror of I with $\sigma_i(\nu_i) = \nu'_i$.

(c) $M'_i = M^i_{\eta_i}$.

(d) If $h = T(i+1)$, then:

$$M_i^{\prime *} = M'_h||\mu \text{ where } \mu \text{ is maximal such that } \tau'_i \text{ is a cardinal in } M'_h.$$

Moreover:

$$\pi'_{h,i+1} = \sigma^{h,i+1}_{\eta^*_h}, \text{ where } \eta^*_h = \mathrm{lh}(I^i_*).$$

$\langle I, R, I'\rangle$ is called an RM triple of length η if and only if $\langle R, I'\rangle$ is an RM of I.

We observed that:

Lemma 3.7.34 *Let $\Gamma = \langle I, R, I'\rangle$ be an RM triple of length $\eta + 1$. Let $E^{M_\eta}_\nu \neq \varnothing$, where $\nu > \nu_i$ for all $i < \eta$. Then Γ extends to an RM triple $\dot\Gamma = \langle \dot I, \dot R, \dot I'\rangle$ of length $\eta + 2$ with $\dot\nu = \nu$.*

We fixed a function G such that whenever (Γ, ν) is such a pair, then $G(\Gamma, \nu) = \langle \dot I, \dot R, \dot I'\rangle$ is such an extension.

We also observed that:

Lemma 3.7.35. *Let $\Gamma = \langle I, R, I'\rangle$ be an RM-triple of limit length η. Let b be the unique good branch in R. Then there is a unique extension to an RM-triple $\dot\Gamma$ of length $\eta + 1$. Moreover, $b = \dot T``\{\eta\}$ in this extension.*

We also noted that:

Lemma 3.7.32. *$i + 1$ is a drop point in I iff it is a drop point in R.*

Lemma 3.7.33. *If $(i, j]_T$ has no drop point in I, then $\pi'_{i,j} = \sigma^{i,j}_{\eta_i}$.*

Clearly, if $\Gamma = \langle I, R, I' \rangle$ is an RM-triple of length η and $1 \leq i < \eta$, then $\Gamma | i = \langle I|i, R|i, I'|i \rangle$ is a RM triple of length i. Now let:

$$\sigma : N \longrightarrow_{\Sigma^*} \tilde{M}_{\tilde{\eta}} \min \tilde{\rho},$$

where $\tilde{I} = \langle \langle \tilde{M}_i \rangle, \langle \tilde{\nu}_i \rangle, \langle \tilde{\pi}_{i,j} \rangle, \tilde{T} \rangle$ is a normal iteration of M of length $\tilde{\eta} + 1$. We define:

Definition 3.7.23. Let I be a normal iteration of N of length μ. By a *good triple for* I we mean an RM triple $\Gamma(I) = \langle I, R, I' \rangle$ such that:

(a) $R = \langle \langle I^i \rangle, \langle \nu'_i \rangle, \langle e^{ij} \rangle, T \rangle$, $I' = \langle \langle M'_i \rangle, \langle \pi'_{i,j} \rangle, \langle \sigma_i \rangle, \langle \rho^i \rangle \rangle$ with $I^0 = \tilde{I}, \sigma_i = \tilde{\sigma}, \rho^0 = \tilde{\rho}$.

(b) If $i + 1 < \mu$, then $\Gamma | i + 2 = G(\Gamma | i + 1, \nu'_i)$.

By the fact that M is uniquely normally iterable and Γ is an RM-triple, it follows that, if $\eta < \mu$ is a limit ordinal then $\Gamma | \eta + 1$ is obtained from $\Gamma | \eta$ as in Lemma 3.7.35. It follows easily that I can have at most one good triple, which we denote by $\Gamma(I)$, if it exists, we then define a strategy S for N as follows:

Let I be a normal iteration N of limit length. If $\Gamma(I)$ is undefined, then so is $S(I)$. If not, then we let:

$$b = \text{ the unique good branch in } R,$$

where $\Gamma(I) = \langle I, R, I' \rangle$. We set: $S(I) = b$, We then noted:

Lemma 3.7.36. *If I is an S-conforming iteration, then $\Gamma(I)$ is defined.*

But this means that I can be extended one step further, using Lemma 3.7.34 and 3.7.35. Hence S is a successful normal iteration strategy.

Building upon this, we now try to define a successful smooth iteration strategy for N. Note that, given the function G, the operation $\Gamma(I)$ is uniquely characterized by $\tilde{\sigma}, \tilde{I}, \tilde{\rho}$. Thus we can write: $\Gamma_{\tilde{\sigma}, \tilde{I}', \tilde{\rho}}(I)$. We now try to define $\Gamma(I)$ for *smooth iterations* I of N.

Definition 3.7.24. Let $I = \langle I_i : i < \mu \rangle$ be a smooth iteration of N inducing $\langle N_i : i < \mu \rangle, \langle \pi_{i,j} : i \leq j < \mu \rangle$. Let

$$I_i = \langle \langle N^i_h \rangle, \langle \nu^i_h \rangle, \langle \pi^i_{h,j} \rangle, T^i \rangle \text{ be of length } \eta_i.$$

By a Γ-sequence for I, we mean any sequence $\Gamma = \langle \Gamma_i : i < \mu \rangle$ such that:

(a) $\Gamma_i = \Gamma_{\dot{I}_i,\sigma_i,\rho_i}(I_i) = \langle I_i, R_i, I'_i \rangle$ is an RM triple where:

$$\sigma_i : N_i \longrightarrow_{\Sigma^*} \dot{M}_i \min \rho^i$$

and $\dot{I}_i$ is the first iteration in R_i and $\dot{M}_i$ is the final model in $\dot{I}_i$.

We set:

$$R_i = \langle \langle I_i^h \rangle, \langle \nu_i^h \rangle, \langle e_i^{h,j} \rangle, T^i \rangle$$

$$I'_i = \langle \langle M_h^{\prime(i)} \rangle, \langle \pi_{h,j}^{\prime i} \rangle, \langle \sigma_h^i \rangle, \langle \rho^{i,h} \rangle \rangle$$

(Hence $\dot{I}_i = I_i^0, \dot{M}_i = M_0^{\prime i}$.)

(b) $\dot{I} = \langle \dot{I}_i : i < \mu \rangle$ is a smooth reiteration of M such that $R_i =$ the unique reiteration from $\dot{I}_i$ to $\dot{I}_{i+1}$ for $i + 1 < \mu$.

$\dot{I}$ then induces partial insertions $\dot{e}_{i,j}$ with:

$$\dot{I}_{i+1} = I_i^{\eta_i}, \dot{e}_{i,i+1} = e_i^{0,\eta_i} \text{ for } i + 1 < \mu$$

and

$$\begin{aligned} &\dot{I}_\lambda, \langle \dot{e}_{i,\lambda} : i < \lambda \rangle \text{ is the good limit of} \\ &\langle \dot{I}_i : i < \lambda \rangle, \langle \dot{e}_{i,j} : i \leq j < \lambda \rangle \text{ for limit } \lambda < \mu. \end{aligned}$$

(c) There is a commutative system $\langle \dot{\pi}_{i,j} : i \leq j < \mu \rangle$ such that $\dot{\pi}_{i,j}$ is a partial map from $\dot{M}_i$ to $\dot{M}_j$ and:

$$\dot{\pi}_{i,i+1} = \pi_{0,\eta_i}^{\prime i} \text{ for } i + 1 < \mu.$$

Moreover:

$$\begin{aligned} &\dot{M}_\lambda, \langle \dot{\pi}_{i,\lambda} : i < \lambda \rangle \text{ is the limit of} \\ &\langle \dot{M}_i : i < \lambda \rangle, \langle \dot{\pi}_{i,j} : i \leq j < \lambda \rangle \text{ for limit } \lambda < \mu. \end{aligned}$$

(d) $\dot{\sigma}_{i+1} = \sigma_{\eta_i}^i, \rho^{i+1} = \rho^{i,\eta_i}$ for $i + 1 < \mu$.

(e) $\tilde{I} = \dot{I}_0, \tilde{\sigma} = \dot{\sigma}_0, \tilde{\rho} = \dot{\rho}^0$.

(f) Suppose that I has no drop point in $[i, j]$. Then:

(i) $\dot{\pi}_{i,j} : \dot{M}_i \longrightarrow_{\Sigma^*} \dot{M}_j$

(ii) $\dot{\pi}_{i,j} \cdot \sigma_i = \dot{\sigma}_j \pi_{i,j}$

(iii) $\dot{\pi}_{i,j}``\dot{\rho}_n^i \subset \rho_n^j \leq \dot{\pi}_{i,j}(\dot{\rho}_n^i)$ for $n < \omega$.

This completes the definition.

Recall that $h+1$ is a drop point in R_i iff it is a drop point in I_i. We call $i+1$ a drop point in $\dot{I}$ iff r_i has a drop point on its main branch. Similarly, $i+1$ is a drop point in I iff I_i has a drop point on its main branch. Hence $i+1$ is a drop in $\dot{I}$ iff it is a drop point in I.

Lemma 3.7.55. *There is at most one* Γ*-sequence for* I*.*

Proof. By induction on $i < \mu$ we show that the sets:

$$\Gamma_i, \dot{I}_i, \langle \dot{e}_{h,i} : h < i \rangle, \dot{M}_i, \langle \dot{\pi}_{h,i} : h < i \rangle, \sigma_i, \rho^i$$

are uniquely determined by $\Gamma | i = \langle \Gamma_h : h < i \rangle$.

Case 1. $i = 0$.

$\dot{I}_0, \sigma_0, \rho^0$ are explicitly given by (e). Hence so are:

$$\dot{M}_0 = \text{ the final model of } \dot{I}_0, \Gamma_{\dot{I}_0, \dot{\sigma}_0, \rho^0}(I_0)$$

Case 2. $i = h+1$. Then

- $\dot{I}_i = I_h^{\eta_h}, \dot{e}_{j,i} \cdot \dot{e}_{j,h}$ for $h < i$.
- $\dot{M}_i$ is defined from $\dot{I}_{i,j}$ and $\dot{\pi}_{j,i} = \pi'^{h}_{0,\eta_h} \dot{\pi}_{j,h}$ for $h < i$.
- $\sigma_i = \sigma^h_{\eta_h}, \rho^i = \rho^{h,\eta_h}$.
- $\Gamma_i = \Gamma_{\dot{I}_i, \sigma_i, \rho^i}(I_i)$

Case 3. $i = \lambda$ is a limit ordinal.

- $\dot{I}_\lambda, \langle \dot{e}_{h,\lambda} : h < \lambda \rangle$ are given by (b).
- $\dot{M}_\lambda, \langle \dot{\pi}_{h,\lambda} : h < \lambda \rangle$ are given by (c).
- σ_λ is defined by: $\sigma_\lambda \pi_{h,\lambda} = \dot{\pi}_{h,\lambda} \sigma_h$ for $[h, \lambda)$ drop free in I (by (f)).
- By Lemma 3.6.42, ρ^λ is the unique ρ such that

$$\sigma_\lambda : N_\lambda \longrightarrow_{\Sigma^*} \dot{M}_\lambda \min \rho \text{ and}$$

$$\dot{\pi}_{i,\lambda} \text{``} \rho^i \subset \rho \leq \dot{\pi}_{i,\lambda}(\rho^i) \text{ if } (i, \lambda) \text{ is drop free.}$$

- $\Gamma_\lambda = \Gamma_{\dot{I}_\lambda, \sigma_\lambda, \rho^\lambda}(I_\lambda)$.

QED(Lemma 3.7.55)

We denote the unique Γ-sequence for I by $\Gamma(I)$, if it exits. Writing $\dot\sigma_l^{i,j}$ for $\sigma_l^{\dot e_{i,j}}$ and $\dot\eta_i$ for $\mathrm{lh}(\dot I_i)$ we have:

Lemma 3.7.56. *Let $\Gamma = \Gamma(I)$. If $(i, j]$ has no drop point in I, then $\dot\pi_{i,j} = \dot\sigma^{i,j}_{\dot\eta_i}$.*

Proof. We recall that if $i+1$ is not a drop point, then

$$\dot\pi_{i,i+1} = \pi''_{0,\eta_i} = \sigma_{\dot\eta_i}^{e_i^{0,\eta_i}} = \dot\sigma^{i,i+1}.$$

(Here $\eta_i + 1 = \mathrm{lh}(R_i), \dot\eta_i + 1 = \mathrm{lh}(I_i^0)$). Using this and Lemma 3.7.52, we prove the assertion by induction on j.

QED(Lemma 3.7.56)

Lemma 3.7.57. *Let $I = \langle I_i : i < \mu\rangle$ be of limit length μ. Assume that $\Gamma = \Gamma(I)$ exits. Then there are unique: $N_\mu, \langle\pi_{i,\mu}\rangle, \dot I_\mu, \langle\dot e_{i,\mu}\rangle, \dot M_\mu, \langle\dot\pi_{i,\mu}\rangle, \sigma_\mu, \rho^\mu$ such that:*

(a) *$N_\mu, \langle\pi_{i,\mu} : i < \mu\rangle$ is the direct limit of:*

$$\langle N_i : i < \mu\rangle, \langle\pi_{i,j} : i \le j < \mu\rangle.$$

(b) *$\dot I_\mu, \langle\dot e_{i,\mu} : i < \mu\rangle$ is the good limit of*

$$\langle\dot I_i : i < \mu\rangle, \langle\dot e_{i,j} : i \le j < \mu\rangle$$

(c) *$\dot M_\mu$ is the final model of $\dot I_\mu$.*

(d) *$\dot M_\mu, \langle\dot\pi_{i,\mu} : i < \mu\rangle$ is the direct limit of:*

$$\langle\dot M_i : i < \mu\rangle, \langle\dot\pi_{i,j} : i \le j < \mu\rangle.$$

(e) $\sigma_\mu : N_\mu \longrightarrow_{\Sigma^*} \dot M_\mu \min \rho^\mu$.

(f) *For sufficient $i < \mu$ we have:*

$$\sigma_\mu\pi_{i,\mu} = \dot\pi_{i,\mu}\sigma_i; \dot\pi_{i,\mu}``\rho^i \subset \rho^\mu \le \dot\pi_{i,\mu}(\rho^i)$$

Proof. (b) is immediate by Theorem 3.7.50. We let $\dot M_\mu$ be defined as in (c). Let $i < \mu$ such that (i, μ) has no drop points in I, Then (i, μ) has no drop points in $\dot I = \langle\dot I_i : i < \mu\rangle$. By Lemma 3.7.56 we know that $\dot\pi_{h,j} = \dot\sigma^{h,j}_{\dot\eta_h}$ for $i \le h \le j < \mu$. Set: $\dot\pi_{h,\mu} = \dot\sigma^{h,\mu}_{\dot\eta_h}$ for $h \in [i, \mu)$. Then (d) follows by Lemma

3.7.52. We know that $\sigma_j \pi_{hj} = \dot{\pi}_{hj} \sigma_h$ for $i \leq h \leq j < \mu$. Hence we can define σ_μ as in (f). σ_μ is obviously unique. But then there is a unique ρ^μ satisfying (e), (f) by Lemma 3.6.42. QED(Lemma 3.7.57)

We now define the strategy S. Let I be a critical smooth iteration. Then I has length $\eta + 1$ and I_η is of limit length. If $\Gamma(I)$ is undefined, the so is $S(I)$. If not, then:

$$\sigma_\eta : N_\eta \longrightarrow_{\Sigma^*} \dot{M}_\eta \min \rho^\eta$$

where $\dot{I}, \dot{M}_\eta, \sigma_\eta, \rho^\eta$ are as in the definition of "Γ-sequence". Moreover, $\Gamma_\eta = \Gamma_{\dot{I}_\eta, \sigma_\eta, \rho^\eta}(I_\eta)$. We then set:

$$S(I) =: S_\eta(\dot{I}_\eta) = \text{ the unique cofinal, well founded branch in } \dot{I}_\eta.$$

But then:

Lemma 3.7.58. *Let $I = \langle I_i : i < \mu \rangle$ be any S-conforming smooth iteration. Then $\Gamma(I)$ exists.*

Proof. Let $I = \langle I_i : i < \mu \rangle$. Define a partical function on μ by:

$$\Gamma_i =: \text{ the unique } x \text{ such that } \Gamma(I|i+1) = \langle \Gamma_h : h < i \rangle \cup \{\langle x, i \rangle\}.$$

By induction on i we show:

Claim. Γ_i exists.

Case 1. $i = 0$.

Clearly $\Gamma_i = \Gamma_{\tilde{I}, \tilde{\sigma}, \tilde{\rho}}(I_0)$. But this holds for any I_0 which is a normal iteration of N. Hence by induction on $\mathrm{lh}(I_0)$, we have: I_0 is $S_{\tilde{I}, \tilde{\sigma}, \tilde{\rho}}$-conforming, where $S_{\tilde{I}, \tilde{\sigma}, \tilde{\rho}}$ is the normal iteration strategy for N defined from the function $\Gamma_{\tilde{I}, \tilde{\sigma}, \tilde{\rho}}$.

QED(Case 1)

Case 2. $i = h + 1$.

Set $\dot{I}_i = I_h^{\eta_h}, \sigma_i = \sigma^h_{\eta_h}, \rho^i = \rho^{h, \eta_h}$. Clearly, then:

$$\Gamma_i = \Gamma_{\dot{I}_i, \sigma_i, \rho^i}(I_i)$$

where $\dot{I}_i$ is a normal iterate of M and:

$$\sigma : N_i \longrightarrow_{\Sigma^*} \dot{M}_i \min \rho^i,$$

$\dot{M}_i$ being the final model of $\dot{I}_i$. Since this holds for any normal iterate I_i of N_i, we conclude by induction on $\mathrm{lh}(I_i)$ that I_i is $S_{\dot{I}_i, \sigma_i, \rho^i}$-conforming. Hence $\Gamma_i = \Gamma_{\dot{I}_i, \sigma_i, \rho^i}$ exists.

QED(2)

Case 3. $i = \lambda$ is a limit.

It is easily seen that $\langle \Gamma_h : h < \lambda \rangle = \Gamma(I \upharpoonright \lambda)$. Let $\dot{I}_\lambda, \dot{M}_\lambda, \sigma_\lambda, \rho^\lambda$ be as in Lemma 3.7.57. Clearly we have: $\Gamma_\lambda = \Gamma_{\dot{I}_\lambda, \dot{M}_\lambda, \sigma_\lambda, \rho^\lambda}(I_\lambda)$. Exactly as before, we conclude that I_λ is $S_{\dot{I}_\lambda, \dot{M}_\lambda, \sigma_\lambda, \rho^\lambda}$-conforming, hence that Γ_λ exists.

QED(Claim)

But then it is easily seen that $\langle \Gamma_i : i < \mu \rangle = \Gamma(I)$.

QED(Lemma 3.7.58)

But then S is successful, since, if I is S-conforming, then I can be extended un any S-conforming way -i.e. (A)-(C)hold. (A) follows by Lemma 3.7.34. (B) follows by Lemma 3.7.35. (C) follows by Lemma 3.6.47.

This proves Lemma 3.7.54 and with it Theorem 3.7.53. We now show how to relativize this to a regular cardinal $\kappa > \omega$. We assume that M is uniquely $\kappa + 1$-normally iterable. By a κ-reiteration of M we mean a reiteration of length $\leq \kappa$ in which each component normal iteration is of length $< \kappa$. If we understand "reiteration" as meaning a κ-reiteration of length $< \kappa$, and "smooth iteration" as meaning a smooth iteration of total length $< \kappa$, then a literal repetition of the above proof shows:

Lemma 3.7.59. *Let M be uniquely normally $\kappa + 1$-iterable. Let $\tilde{I}$ be a normal iteration of M of length $\eta + 1 < \kappa$. Let*

$$\sigma : N \longrightarrow_{\Sigma^*} \tilde{M}_\eta \min \rho$$

Then N is smoothly κ-iterable.

The following strength of $\kappa+1$-iterability is needed for this, however, in order to justify the use of Theorem 3.7.50. We now show that, under the premises of Lemma 3.7.59, N is in fact, smoothly $\kappa + 1$-iterable. Let $I = \langle I_i : i < \mu \rangle$ be a smooth iteration of N of total length κ. As mentioned earlier, one of two cases hold, which we consider separately:

Case 1. $\mu = \eta + 1 < \kappa$ and I_η is of length κ.

We assume I to be S-conforming. Then $I|\eta$ is S-conforming. Then $I|\eta$ is S-conforming and I_η is $S_{\dot{I}_\eta, \sigma_\eta, \rho^\eta}$-conforming. Hence:

$$\Gamma_{\dot{I}_\eta, \sigma_\eta, \rho^\eta}(I_\eta) = \langle I_\eta, R, I' \rangle \text{ exists,}$$

where R is a reiteration of M of length κ. But then R has a well founded cofinal branch b. Hence b is cofinal in I_η. b has only finitely many drop points in I_η, since otherwise, by the fact that $\kappa > \omega$ is regular, there would be $\lambda \in b$ such that $h \cap \lambda = T^{\eta}``\{\lambda\}$ has infinitely many drop points. Contradiction! Let $i \in b$ such that $b \setminus i$ has no drop points. Using the fact that $\kappa > \omega$ is regular, it follows easily that

$$\langle M_h : h \in b \setminus i \rangle, \langle \pi_{h,j} : h \leq j \text{ in } b \setminus i \rangle$$

has a well founded limit. (If $x_{n+1} \in x_n$ is the limit, these would be a $\xi \in b \setminus i$ such that $x_n = \overline{N}_\xi(\overline{x}_n)$ for $n < \omega$. Hence $\overline{x}_{n+1} \in \overline{x}_n$ in N_ξ. Contradiction!)

QED(Case 1)

Case 2. $\mu = \kappa$.

I has only finitely many drop points, since otherwise these would be $\xi < \kappa$ such that $I|\xi$ has infinitely many drop points. Contradiction! Let the interval (i, κ) be drop free. Since $\kappa > \omega$ is regular, it again follows that:

$$\langle M_h : i \leq h < \kappa \rangle, \langle \pi_{h,j} : i \leq h \leq j < \kappa \rangle$$

has a well founded limit.

QED(Case 2)

This proves Theorem 3.6.2.

3.8 Unique Iterability

3.8.1 One small mice

Although we have thus far developed the theory of mice in considerable generality, most of this book will deal with a subclass of mice called *one small*. These mice were discovered and named by John Steel. It turns out that a great part of many one small mice are uniquely normally iterable. Using the notion of Woodin cardinal defined in the preliminaries we define:

Definition 3.8.1 (1-small)**.** A premouse M is one small iff whenever $E_\nu^M \neq \varnothing$, then

$$\text{no } \mu < \kappa = \text{crit}(E_\nu^M) \text{ is Woodin in } J_\kappa^{E^M}$$

Note. Since J_κ^E is a ZFC model, we can employ the definition of "Woodin cardinal" given in the preliminaries. An examination of the definition shows that the statement "μ is Woodin" is, in fact, first order over H_τ where $\tau = \mu^+$. Thus the statement "μ is Woodin in M" makes sense for any transitive ZFC^- model M. It means that $\mu \in M$ and "μ is Woodin" hold in H_τ^M where $\tau = \mu^{+M}$ (taking $\tau = \text{card}\, M$ if no $\xi > \mu$ is a cardinal in M). We then have: