in I_η, since otherwise, by the fact that $\kappa > \omega$ is regular, there would be $\lambda \in b$ such that $h \cap \lambda = T^\eta``\{\lambda\}$ has infinitely many drop points. Contradiction! Let $i \in b$ such that $b \setminus i$ has no drop points. Using the fact that $\kappa > \omega$ is regular, it follows easily that

$$\langle M_h : h \in b \setminus i\rangle, \langle \pi_{h,j} : h \leq j \text{ in } b \setminus i\rangle$$

has a well founded limit. (If $x_{n+1} \in x_n$ is the limit, these would be a $\xi \in b \setminus i$ such that $x_n = \overline{N}_\xi(\overline{x}_n)$ for $n < \omega$. Hence $\overline{x}_{n+1} \in \overline{x}_n$ in N_ξ. Contradiction!)

QED(Case 1)

Case 2. $\mu = \kappa$.

I has only finitely many drop points, since otherwise these would be $\xi < \kappa$ such that $I|\xi$ has infinitely many drop points. Contradiction! Let the interval (i, κ) be drop free. Since $\kappa > \omega$ is regular, it again follows that:

$$\langle M_h : i \leq h < \kappa\rangle, \langle \pi_{h,j} : i \leq h \leq j < \kappa\rangle$$

has a well founded limit.

QED(Case 2)

This proves Theorem 3.6.2.

3.8 Unique Iterability

3.8.1 One small mice

Although we have thus far developed the theory of mice in considerable generality, most of this book will deal with a subclass of mice called *one small*. These mice were discovered and named by John Steel. It turns out that a great part of many one small mice are uniquely normally iterable. Using the notion of Woodin cardinal defined in the preliminaries we define:

Definition 3.8.1 (1-small)**.** A premouse M is one small iff whenever $E^M_\nu \neq \varnothing$, then

$$\text{no } \mu < \kappa = \text{crit}(E^M_\nu) \text{ is Woodin in } J^{E^M}_\kappa$$

Note. Since J^E_κ is a ZFC model, we can employ the definition of "Woodin cardinal" given in the preliminaries. An examination of the definition shows that the statement "μ is Woodin" is, in fact, first order over H_τ where $\tau = \mu^+$. Thus the statement "μ is Woodin in M" makes sense for any transitive ZFC^- model M. It means that $\mu \in M$ and "μ is Woodin" hold in H^M_τ where $\tau = \mu^{+M}$ (taking $\tau = \text{card}\, M$ if no $\xi > \mu$ is a cardinal in M). We then have:

Lemma 3.8.1. *Let M be a premouse such that $E_\nu^M \neq \varnothing$ and let us set:*

$$\kappa = \mathrm{crit}(E_\nu^M), \lambda = \lambda(E_\nu^M) =: E_\gamma^M(\kappa), \tau = \tau(E_\gamma^M) =: \kappa^{+E^M}.$$

The following are equivalent:

(a) *No $\mu < \kappa$ is Woodin in J_κ^E*

(b) *No $\mu \leq \kappa$ is Woodin in J_τ^E*

(c) *No $\mu < \lambda$ is Woodin in J_λ^E*

(d) *No $\mu \leq \lambda$ is Woodin in J_γ^E.*

Proof: (d)→(c)→(b)→(a) is clear. We now show (a)→(d). Assume (a). Since $J_\kappa^E \prec J_\lambda^E$ we have (c). But then (b) holds. Since $\pi : J_\tau^E \longrightarrow J_\nu^E$ cofinally, we conclude that π is elementary on J_τ^E. Hence (d) holds. QED (Lemma 3.8.1).

Recalling the typology developed in §3.3, we have:

Lemma 3.8.2. *Every active one-small premouse is of type 1.*

Proof: Suppose not. Let $M = \langle J_\nu^E, F\rangle$ be a counterexample. We derive a contradiction by proving:

Claim. κ is Woodin in M, where $\kappa = \mathrm{crit}(F)$.

Proof: Let $A \subset \kappa$, $A \in M$. We show that some $\tau < \kappa$ is A-strong on J_κ^E. It is easily seen that $\langle J_\kappa^E, B\rangle \prec \langle J_\lambda^E, F(B)\rangle$ whenever $B \subset \kappa$, $B \in M$. Hence it suffices to find a $\tau < \lambda$ such that τ is $F(A)$-strong in J_λ^E.

Claim. κ is $F(A)$-strong in J_λ^E.

Proof: Suppose not. Then there is $\xi < \lambda$ such that whenever $G \in J_\lambda^E$ is an extender at κ on J_λ^E, then $F(A) \cap \xi \neq G(A) \cap \xi$ (where $A = F(A) \cap \kappa$). Let ξ be the least such. Since M is not of type 1, there is $\bar\lambda < \lambda$ such that $\bar F = F \restriction \lambda$ is a full extender at κ in M. Hence $\bar F \in J_\lambda^E$. But:

$$\langle J_{\bar\lambda}^E, \bar F(A)\rangle \prec \langle J_\lambda^E, F(A)\rangle$$

Since for $\alpha_1, \ldots, \alpha_n < \bar\lambda$ we have:

$$\begin{aligned}\langle J_{\bar\lambda}^E, \bar F(A)\rangle \models \varphi[\vec\alpha] &\longleftrightarrow \langle J_\lambda^E, F(A)\rangle \models \varphi[\vec\alpha] \\ &\longleftrightarrow \langle \vec\alpha\rangle \in F(e)\end{aligned}$$

where $e = \{\langle\vec\xi\rangle < \kappa : \langle J_\kappa^E, A\rangle \models \varphi[\vec\xi]\}$. Hence $\xi < \bar\lambda$ by minimality. Hence $\bar F \in J_\lambda^E$ and $F(A) \cap \xi = \bar F(A) \cap \xi$. Contradiction! QED (Lemma 3.8.2).

We leave it to the reader to show:

- If M is one small and $\mu \in M$, then $M||\mu$ is one small (for limit μ).
- Let $\langle M_i : i < \lambda \rangle$ be a sequence of one small premice. Let $\pi_{ij} : M_i \longrightarrow_{\Sigma^*} M_j$ for $i \leq j < \lambda$, where the π_{ij} commute. Let $M_\lambda, \langle \pi_{i\lambda} : i < \lambda \rangle$ be the direct limit of $\langle M_i : i < \lambda \rangle, \langle \pi_{ij} : i \leq j < \lambda \rangle$. Then M_λ is one small.

It then follows easily that:

Lemma 3.8.3. *Any full iterate of a small mouse is one small.*

In particular, any normal iterate of a one small mouse is one small.

In §3.8.2 we shall show that there is a large class of one small premice, all of which have the normal uniqueness property. That will be our main result in this section.

3.8.2 Woodiness and non unique branches

In the preliminaries we defined the notion of A-strong. We now adapt these notion to certain admissible structures in place of V.

Definition 3.8.2. $N = J^E_\alpha$ is a *limit structure* iff N is acceptable and there are arbitrarily large $\tau \in N$ such that $N \models \tau$ is a cardinal.

Definition 3.8.3. Let $N = J^E_\alpha$ is a limit structure. $\kappa \in N$ is *strong in N* iff for arbitrarily large $\xi \in N$ there is $F \in N$ such that:

- F is an extender at κ on N of length $\geq \xi$.
- N is extendible by F.
- Let $\pi : N \longrightarrow N' = J^{E'}_{\alpha'}$. Then $J^{E'}_\xi = J^E_\xi$.

Hence, if ξ is a cardinal in N, it follows that $H^N_\xi = H^{N'}_\xi$.

Definition 3.8.4. Let $A \subset N$, where $N = J^E_\alpha$ is as above, $\kappa \in N$ is A-strong in N iff $\langle N, A \rangle$ is amenable and for arbitrarily large $\xi \in N$ there is $F \in N$ such that

- F is an extender at κ of length $\geq \xi$
- N is extendible by F (hence so is $\langle N, A \rangle$)

- Let $\pi : \langle N, A\rangle \longleftrightarrow \langle N', A'\rangle = \langle J_\alpha^{A'}, A'\rangle$. Then $J_\xi^E = J_\xi^{E'}$ and $A \cap J_\xi^E = A' \cap J_\xi^E$.

Definition 3.8.5. N is Woodin for $A \subset N$ iff there are arbitrarily large $\kappa \in N$ which are A-strong in N.

Hence if $N = J_\xi^{E^M}, \xi \in M$, then $M \models$ "ξ is Woodin" if and only if ξ is Woodin for all $A \in M$ such that $A \subset N$.

In this subsection we shall prove:

Theorem 3.8.4. *Let M be a premouse. Let*

$$I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$$

be an iteration of M of limit length η. Set:

$$\tilde{\eta} = \sup_{i<\eta} \kappa_i = \sup_{i<\eta} \lambda_i;\ N = J_{\tilde{\eta}}^E =: \bigcup_{i<\eta} M_i|v_i$$

Assume that b_0, b_1 are distinct cofinal well founded branches in T (hence $\tilde{\eta} = \sup b_h$ for $h = 0, 1$). Then N is Woodin with respect to every $A \subset N$ such that $A \in M_{b_0}, M_{b_1}$.

The proof will require many steps. We first prepare the ground by reformulating the definition of "strong" and "A-strong".

Note that if $A \subset \mathsf{ON}$, then $A \cap J_\xi^E = A \cap \xi$ for $\xi \in N$. Thus, if $F \in N$ verifies A-strongness, then so does $F|\xi$. In the following we shall make frequent use of this fact. Since, in the book, we have generally worked with full extenders, we pause now to remind ourselves what it means to say:

$$F \text{ is an extender at } \kappa \text{ on } M \text{ of length } \xi$$

We take M as being acceptable. The above statement then means that the following hold:

(a) $\xi > \kappa$ is Gödel closed (i.e. closed under Gödel pairs $\prec\, , \succ$).

(b) $\kappa \in M$ and $\mathbb{P}(\kappa) \cap M \in M$

(c) $F : \mathbb{P}(\kappa) \cap M \longrightarrow \mathbb{P}(\xi)$

(d) F has an *extension* $\tilde{\pi}$ characterized by:

- $\tilde{\pi} : H_\kappa^M \longrightarrow_{\Sigma_0} H$ cofinally, where H is transitive

- $F(X) = \tilde{\pi}(X) \cap \xi$ for $X \in \mathbb{P}(\kappa) \cap M$
- Each $x \in H$ has the form $\tilde{\pi}(f)(\bar{\xi})$, where $\bar{\xi} < \xi$ and $f \in H_\kappa^M$ is a function on κ.

Then $\tilde{\pi}$ is uniquely characterized by F. Moreover, $\tilde{\pi}$ is definable from F by an "ultrapower" construction which is absolute in ZFC^- models. Thus $\tilde{\pi} \in M$ if $F \in M$ and $M \models \mathsf{ZFC}^-$. But then $\tilde{\pi} \in M$ if $F \in M$ and M is a limit structure in the above sense, since then M is a union of transitive ZFC^- models.

$\pi : M \longrightarrow_F M'$ here means that $\langle M', T\rangle$ is the Σ_0 lift-up of $M, \tilde{\pi}$. We say that M is *extendable by* F if $\langle M', \pi\rangle$ exists.

Definition 3.8.6. Let $M = \langle J_\alpha^E, B\rangle$ be acceptable. Let F be an extender on M at $\kappa \in M$ of length $\xi \leq \alpha$. Let $\tilde{\pi}$ be the extension of F and let $\tilde{\pi}(J_\kappa^E) = J_\lambda^{E'}$. F is *strong* with respect to M iff $J_\xi^E = J_\xi^{E'}$. If F is strong, we define a function $\tilde{F}$ on $\mathbb{P}(J_\kappa^E) \cap M$ by $\tilde{F}(a) =: \tilde{\pi}(a) \cap J_\xi^E$.

Note that $\tilde{F}(a) = F(a)$ for $a \subset \kappa$.

Note. If M is a premouse, $E_\nu \neq \varnothing$ and τ_ν is a cardinal in M, then E_ν is a strong extender on M at κ of length λ_ν. If $\nu \in M$, then $E_\nu \in M$, but the case $\nu = \alpha$ can give us trouble.

Definition 3.8.7. Let M, F, κ, ξ be as above. Let $A \subset M$. F is A-strong in M iff

- $\langle M, A\rangle$ is amenable
- F is strong in M
- $\tilde{F}(A \cap J_\kappa^E) = A \cap J_\xi^E$.

We note:

Fact. Let F be an extender on M at $\kappa \in M$ of length η. Let $\kappa < \mu < \xi$, where μ is Gödel closed. Define $F' = F|\mu$ by:

$$F'(X) = F(X) \cap \mu \text{ for } X \in \mathbb{P}(\kappa) \cap M.$$

Then:

(a) F' is an extender on M at κ of length μ

(b) If F is strong in M, so is F'

(c) If F is A-strong in M, so is F'

(d) If M is extendible by F, then it is extendible by F'.

We sketch the proof of (b). Let π be the extension of F with:

$$\pi : J_\tau^E \longrightarrow_{\Sigma_0} H \text{ cofinally, where } \tau = \kappa^{+M}.$$

Similarly for π', F'. Let:

$$\pi' : J_\tau^E \longrightarrow_{\Sigma_0} H' \text{ cofinally}$$

Define:

$$k : H' \longrightarrow_{\Sigma_0} H \text{ cofinally}$$

by $k(\pi'(f)(\xi)) = \pi(f)(\xi)$ where $\xi < \mu$ and $f \in J_\kappa$ is a function on κ. Then $k \restriction \mu = \mathrm{id}$, since:

$$k(\xi) = k(\pi'(\mathrm{id} \restriction \tau)(\xi)) = \pi(\mathrm{id} \restriction \tau)(\xi) = \xi$$

But then $\bar{k} = k \restriction J_\mu^{E'}$ maps $J_\mu^{E'}$ cofinally to J_μ^E, since $k(J_\xi^{E'}) = J_\xi^E$ for limit $\xi < \mu$. Now let h', h be the Σ_1 Skolem function of $J_{\mu'}^{E'}, J_\mu^E$ respectively. Then

$$\bar{k}(h'(i, \langle \vec{\xi} \rangle)) = h(i, \langle \vec{\xi} \rangle)$$

for $i < \omega$, $\xi_1, \ldots \xi_n < \mu$. It follows easily that $\bar{k}$ is an isomorphism of $J_\mu^{E'}$ onto J_μ^E. Hence $\bar{k} = \mathrm{id}$, $J_\mu^{E'} = J_\mu^E$. QED (part (b)).

We shall sometimes make use of the following:

Lemma 3.8.5. *Let M be a premouse. Let $F = E_\nu^M \neq \varnothing$, where $\kappa = \kappa_\nu$, $\tau = \tau_\nu$, $\lambda = \lambda_\nu$ and τ is a cardinal in M. Hence F is strong at κ of length λ in M. Let $G \in M$ be am extender at $\bar{\kappa} < \kappa$ on M of length κ. Let $\kappa < \mu \leq \lambda$, where μ is Gödel closed. Set:*

$$F' = F|\mu, \; D = F' \circ G$$

Then:

(a) *$D \in M$ is an extender on M at $\bar{\kappa}$ of length μ.*

(b) *If G is strong in M, so is D. Moreover we then have $\tilde{D} = \tilde{F}' \circ \tilde{G}$.*

(c) *If $A \subset M$ and G, F' are A-strong in M, then so is D.*

Note that we do not assume $F \in M$. **Proof:** We first prove (a). Obviously $G \in J^E_\tau$ is an extender on J^E_τ at $\bar\kappa$ of length κ. But this is expressed by $J^E_\tau \models \varphi[G, \bar\kappa, \kappa]$, where φ is a first order formula. But $\pi_F : J^E_\tau \prec J^E_\nu$. Hence:

$$J^E_\nu \models [\pi_F(G), \bar\kappa, \lambda]$$

Thus $\pi_F(G)$ is an extender on M at $\bar\kappa$ of length λ, and we set:

$$D = \pi_F(G)|\mu$$

Then:

$$\begin{aligned} D(X) = D'(X) \cap \mu = \pi_F(G)(X) \cap \mu = \pi_F(G(X)) \cap \mu \\ = F(G(X)) \cap \mu = F'(G(X)) \end{aligned}$$

This proves (a). We now prove (b).

Clearly $G \in J^E_\tau$ is strong in J^E_τ, where $\tau = \tau_\nu$. But J^E_ν in a ZFC^- model and the fact that G is strong and expressible by a fourth order statement:

$$J^E_\tau \models G \text{ is strong.}$$

But $\pi_F : J^E_\tau \prec J^E_\nu$. Hence

$$J^E_\nu \models D' = \pi_F(G) \text{ is strong.}$$

Hence D' is strong in M. Hence $D = D'|\mu$ is strong in M. Finally we note that $\tilde E(a) = \pi_F(a)$ for $a \subset J_\kappa$, since $\pi_F(\kappa) = \lambda$ (i.e. F is a full extender). But then

$$\begin{aligned} \tilde D'(a) = \pi_F(\pi_G(A)) \cap J^E_\lambda = \pi_F(\pi_G(a) \cap J^E_\kappa) \\ = \pi_F(\tilde G(a)) = \tilde F \tilde G(a) \end{aligned}$$

Hence $\tilde D(a) = \tilde D'(a) \cap J^E_\mu = \tilde F(\tilde G(a)) \cap J^E_\mu = \tilde F' \tilde G(a)$. This proves (b). To prove (c) we note that, if both G, F' are A-strong, then:

$$\tilde F' \tilde G(A \cap J^E_{\bar\kappa}) = \tilde F'(A \cap J^E_\kappa) = A \cap J^E_\mu$$

QED (Lemma 3.8.5)

Lemma 3.8.6. *Let $N = J^E_\alpha$ be a limit structure. Let $F \in N$ be a strong extender at κ on N of length η, where η is regular in N. Then N is extendible by F.*

Proof: Suppose not. Let

$$D = \{\langle f, \alpha\rangle \in N : \alpha < \xi \text{ and } f \text{ is a function on } \kappa = \mathrm{crit}(F)\}$$

Let $e \subset D^2$ be defined by:

$$\langle f, \alpha\rangle \, e \, \langle g, \beta\rangle \longleftrightarrow \langle\alpha, \beta\rangle \in F(\{\langle\xi, \zeta\rangle : f(\xi) \in g(\zeta)\})$$

Our assumption says that e is ill-founded. Hence there is a sequence $\langle f_i, \alpha_i\rangle_{i<\omega}$ such that

$$\langle f_{i+1}, \alpha_{i+1}\rangle \, e \, \langle f_i, \alpha_i\rangle, \text{ for } i < \omega$$

Let $\langle f_0, \alpha_0\rangle \in J^E_\gamma$ where $\gamma > \xi$ is regular in N. We can assume without lose of generality that $\langle f_i, \alpha_i\rangle \in J^E_\gamma$. If not, replace f_i by f'_i where

$$f'_i(\xi) = \begin{cases} f_i(\xi) & \text{if } f_i(\xi) \in J^E_\gamma \\ 0 & \text{otherwise} \end{cases}$$

But then $e' = e \cap J^E_\gamma$ is ill-founded, where $e' \in N$. Since N is a union of transitive ZFC^- models, it follows by absoluteness that:

$$N \models e' \text{ is ill-founded.}$$

But then there is $\langle\langle f_i, \alpha_i\rangle : i < \omega\rangle \in N$ such that

$$\langle f_{i+1}, \alpha_{i+1}\rangle \, e' \, \langle f_i, \alpha_i\rangle \text{ for } i < \omega$$

Let $\tilde{\pi} \in N$ be the extension of F. Then:

$$\tilde{\pi} : J^E_\tau \longrightarrow_{\Sigma_0} H \text{ cofinally.}$$

Set: $X_i = \{\langle\xi, \zeta\rangle : f_{i+1}(\xi) \in f_i(\xi) \in f_i(\zeta)\}$. Let $\tau = \kappa^{+N}$, we have $\langle X_i : i < \omega\rangle \in J^E_\tau$. Set

$$\langle\tilde{X}_i : i < \omega\rangle = \tilde{\pi}(\langle X_i : i < \omega\rangle)$$

Then $\tilde{X}_i \cap \eta = F(X_i)$ for $i < \omega$. Since η is regular in N and F is strong, we have:

$$\langle\alpha_i : i < \omega\rangle \in J^E_\xi \subset H$$

But $\langle\alpha_{i+1}, \alpha_i\rangle \in F(X_i) \subset \tilde{X}_i$ for $i < \omega$. Hence H satisfies the statement:

There is $g : \omega \longrightarrow \tilde{\pi}(\kappa)$ such that $\langle g(i+1), g(i)\rangle \in \tilde{X}_i$ for $i < \omega$

But then J^E_τ satisfies:

There is $g : \omega \longrightarrow \kappa$ such that $\langle g(i+1), g(i)\rangle \in X_i$ for $i < \omega$

Hence $f_{i+1}(g(i+1)) \in f_i(g(i))$ for $i < \omega$. Contradiction! QED (Lemma 3.8.6)

But then by Fact 1, it follows easily that:

Lemma 3.8.7. *Let N be a limit structure, $\kappa \in N$. Then κ is strong in N iff for arbitrarily large $\eta \in N$ there is $F \in N$ which is strong for N at κ of length η.*

Lemma 3.8.8. *Let N, κ be as above. Let $A \subset N$. Then κ is A-strong in N iff for arbitrarily large $\xi \in N$ there is $F \in N$ which is A-strong for N at κ of length ξ.*

The proofs are left to the reader.

We are now ready to embark upon the proof of Theorem 3.8.4.

The proof will have many steps. We shall in fact, first prove it under a simplifying assumption, in order to display the method more clearly.

Since b_0, b_1 are distinct and T is a tree, there is an $\alpha < \eta$ such that $(b_0 \setminus \alpha) \cap (b_1 \setminus \alpha) = \varnothing$. Define a sequence $\langle \delta_i : i < \omega \rangle$ by:

$$\begin{aligned} \delta_0 = &\text{ the least } \xi \in b_i \setminus (\alpha + 1) \\ \delta_{2i+1} = &\text{ the least } \xi \in b_1 \text{ such that } \xi > \delta_{2i} \\ \delta_{2i+2} = &\text{ the least } \xi \in b_0 \text{ such that } \xi > \delta_{2i+1} \end{aligned}$$

By minimality, each δ_i is a successor ordinal. Note that

$$T(\delta_{2i+1}) < \delta_{2i} < \delta_{2i+1}$$

since otherwise, setting $\xi = T(\delta_{2i+1})$, we would have $\xi \geq \delta_{2i}, \xi \in b_1$; hence $\xi > \delta_{2i}$. But then $\delta_{2i+1} \leq \xi < \delta_{2i+1}$. Contradiction! A similar argument shows:

$$T(\delta_{2i+2}) < \delta_{2i+1} < \delta_{2i+2}$$

Hence:

(1) $T(\delta_{i+1}) < \delta_i < \delta_{i+1}$ for $i < \omega$.

Set

(2) $\gamma_i =: \delta_i - 1$, $\gamma_i^* = T(\delta_i)$.

By (1) we then have

(3) $\kappa_{\gamma_{i+1}} < \lambda_{\gamma_{i+1}^*} \leq \lambda_{\gamma_i} \leq \kappa_{\gamma_{i+2}}$.

We have $\lambda_{\gamma_i} \leq \kappa_{\gamma_{i+2}}$ since $(\gamma_i + 1) T (\gamma_{i+2} + 1)$. Now note that for $n < \omega$ we have:

(4) If n is even, then $\langle \delta_{n+i} : i < \omega$ has the same definition as $\langle \delta_i : i < \omega \rangle$ with δ_n in place of α. Similarly for n odd, with b_0, b_1 reversed.

Hence we may without lose of generality assume α chosen large enough that:

(5) No $\xi \in (b_h \setminus \alpha)$ is a drop point $(h = 0, 1)$. Thus $M_{\gamma_i^*} = M_{\gamma_i}^*$ and we have:

(6) $\pi_{\gamma_i^*,\delta_i} : M_{\gamma_i^*} \longrightarrow^*_{E_{\nu_{\gamma_i}}} M_{\delta_i}$.

Clearly

(7) $\sup_{i<\omega} \gamma_i = \sup_{i<\omega} \delta_i = \nu$, since otherwise $\sup_{i<\omega} \gamma_i \in (b_0 \setminus \alpha) \cap (b_1 \setminus \alpha)$.

By (6) we conclude:

(8) τ_{γ_i} is a cardinal in M_ξ for $\xi \geq \gamma_i^*$.

Set:

(9) $N = J^E_{\tilde{\xi}} =: \bigcup_i J^{E^{M_{\gamma_i}}}_{\kappa_{\gamma_i}} = \bigcup_i J^{E^{M_{\gamma_i}}}_{\nu_{\gamma_i}}$.

Until further notice we make the following simplifying assumption:

$$\text{(SA)} \quad E^{M_{\gamma_i}}_{\nu_{\gamma_i}} | \kappa_{\gamma_{i+1}} \in M_{\gamma_i} \ (i < \omega)$$

This would be true e.g. if M were passive and no truncation occurred in the iteration, since then $E^{M_{\gamma_i}}_{\nu_{\gamma_i}} \in M_{\gamma_i}$.

Using this assumption we get:

(10) $N \models$ there are arbitrarily large strong cardinals.

Proof: Since we can choose α (and hence κ_{γ_0}) arbitrarily large, it suffices by (4) to show:

Claim. κ_{γ_0} is strong in N.

Proof: Set $F_n = E^{M_{\gamma_n}}_{\nu_{\gamma_n}}$, $F'_n = F_n | \kappa_{\gamma_{n+1}}$. Set $G_0 = F'_0, G_{n+1} = F'_{n+1} \circ G_n$. Using Lemma 3.8.5 we get:

$$G_n \in N \text{ is strong in } N \text{ at } \kappa_{\gamma_0} \text{ of length } \kappa_{\gamma_{n+1}}$$

QED (10)

(11) Let $A \in M_{b_0} \cap M_{b_1}$. Then N is Woodin for A_n.

Proof. Assume α is so chosen that $A \in \text{rng}(\pi_{\gamma_0^*,b_0}) \cap \text{rng}(\pi_{\gamma_1^*,b_1})$. It suffices to prove:

Claim. κ_{γ_0} is A-strong in N.

Then F_n is A-strong, since

$$\pi_{\gamma_n,\gamma_{n+1}}(A \cap J^E_{\kappa_{\gamma_n}}) = A \cap J^E_{\lambda_{\gamma_n}}$$

Hence F'_n is A-strong. Hence G_n is A-strong for $n < \omega$. QED (11)

Note. Even if $F_0 \notin N$, it follows that $\tilde{G}_n(A \cap J^E_{\gamma_0}) = A \cap J^E_{\gamma_{n+1}}$, where $\tilde{G}_0 = \tilde{F}'_0$, $\tilde{G}_{n+1} = \tilde{F}'_{n+1} \circ \tilde{G}_n$.

We now face the task of proving (10), (11) without the special assumption (SA). In order to prove (10) it would suffice to find a $\beta + 1 < \eta$ such that

$$\mu_{\gamma_0} \leq \mu_\beta < \mu_{\gamma_1} < \lambda_\beta \text{ and } E^{M_\beta}_{\nu_\beta} | \kappa_{\gamma_1} \in M_\beta$$

since then, setting $\bar{G} = E^{M_\beta}_{\nu_\beta} | \kappa_{\gamma_1}$, we have $G \in N$ is strong in N at κ_β of length κ_{γ_1}.

If we set:

$$G_0 = G, G_{n+1} = F_{n+1} \circ G_n$$

it follows that

$$G_n \in N \text{ is strong in } N \text{ at } \kappa_\beta \text{ of length } \kappa_{\gamma_{n+1}}$$

We now look for such a β. As a first step, however, we choose α large enough to prevent the occurrence of an unfortunate configuration. For active premice M let E^M_{top} denote the topmost extender. Call $n < \omega$ undesirable iff

$$\text{crit}(E^{M_{\delta_n+1}}_{\text{top}}) \in [\gamma_n, \gamma_{n+1})$$

(12) If α is chosen sufficiently large, then no $n < \omega$ is undesirable.

Proof: Suppose not. Then there are infinitely many undesirable n. But then these are undesirable n, m such that $n < m$ and n, m are both add or both even. Then $\delta_{n+1} <_T \delta_{m+1}$. Let $\bar{\kappa} = \text{crit}(E^{M_{\delta_{n+1}}}_{\text{top}})$. Then $\bar{\kappa} < \kappa_{\gamma_{n+1}} = \text{crit}(\pi_{\delta_{n+1}, \delta_{m+1}})$ by undesirability. Hence $\bar{\kappa} = \text{crit}(E^{M_{\delta_{m+1}}}_{\text{top}})$. But $\bar{\kappa} < \kappa_{\gamma_{n+1}} \leq \kappa_m$ by (3). Hence m is not undesirable. Contradiction! QED(12)

From now on let α be chosen as in (12). In the following assume that:

$$(*) \quad \gamma < \eta \text{ and } \kappa_\gamma = \text{crit}(E^{M_\gamma}_{\nu_\gamma}) < \kappa < \lambda_\gamma$$

where κ is inaccessible in M_γ. Later we shall apply our argument to the case $\gamma = \gamma_0$, $\kappa = \kappa_{\gamma_1}$.

We call γ *good for* κ iff $E^{M_\gamma}_{\nu_\gamma} | \kappa \in M_\gamma$.

(13) If γ is not good for κ, then

(a) $E^{M_\gamma}_{\nu_\gamma}$ is the top extender of M_γ

(b) $\rho^1_{M_\gamma} \leq \kappa$.

Proof:

(a) It is immediate, since otherwise $E^{M_\gamma}_{\nu_\gamma} \in M_\gamma$.

(b) Set $F = E^{M_\gamma}_{\nu_\gamma}|\kappa$, $\tilde{F} = \{\langle x, \alpha\rangle : \alpha \in F(x)\}$. Then $\tilde{F}$ is $\Sigma_1(M_\gamma)$, $\tilde{F} \subset J^{E^{M_\gamma}}_\kappa$, $\tilde{F} \notin M_\gamma$. QED (13)

(14) Let γ, κ satisfy $(*)$. Let $\beta + 1 \leq_T \gamma$ such that $\kappa < \lambda_\beta$. Then:

(a) $\text{crit}(\pi_{\beta+1,\gamma}) > \kappa$ if $\beta + 1 \neq \gamma$

(b) If γ is not good for κ, then $\pi_{\beta+1,\gamma}$ is total on $M_{\beta+1}$

Proof:

(a) Let $\beta + 1 = T(\mu + 1)$ where $\mu + 1 \leq_T \gamma$. Then $\beta + 1$ is the least ξ such that $\lambda_\xi > \kappa_\mu$, where $\kappa_\mu = \text{crit}(\pi_{\beta+1,\gamma})$. Hence $\kappa < \lambda_\beta \leq \kappa_\mu$.

(b) Suppose not. Then there is a least truncation point $\xi + 1$ such that $\beta + 1 \leq_T \xi + 1 \leq_T \gamma$. Then $M^*_\xi \in M_{\xi^*}$, where $\xi^* = T(\xi + 1)$. Moreover we have:

$$\pi_{\xi^*,\gamma} : M^*_\xi \longrightarrow_{\Sigma^*} M_\gamma,\ \text{crit}(\pi_{\xi^*,\gamma}) > \kappa,$$

since

$$\beta + 1 \leq \xi^*,\ \text{crit}(\pi_{\beta+1,\gamma}) > \kappa$$

Hence $\rho^1_{M_{\xi^*}} \leq \kappa$. Since M^*_ξ is a segment of M it follows that λ_β is not a cardinal in M_{ξ^*}. But λ_β is a cardinal in M_{ξ^*}, since $\beta + 1 \leq \xi^*$. Contradiction! QED (14)

We now set:

Definition 3.8.8. Let γ, κ satisfy $(*)$. $\gamma^+ \cong \gamma^+(\kappa)$ is defined as follows:

- if γ is not good for κ and there is $\beta + 1 \leq_T \gamma$ such that $\kappa_\beta < \kappa < \lambda_\beta$, set $\gamma^+ = \beta$.
- Otherwise γ^+ is undefined.

Note. If γ^+ is defined, then the pair γ^+, κ satisfies $(*)$.

(15) If $\beta = \gamma^+$, then $\kappa_\gamma < \kappa_\beta$.

Proof: Let $\xi = T(\beta + 1)$. Then $\pi_{\xi,\gamma} : M^*_\beta \longrightarrow_{\Sigma^*} M_\gamma$, since $\pi_{\beta+1,\gamma}$ is total on $M_{\beta+1}$ by (14). Then $\pi_{\beta,\gamma}(\bar{\kappa}) = \kappa_\gamma$, where $\bar{\kappa} = E^{M^*_\beta}_{\text{top}}$, since $\kappa_\gamma = \text{crit}(E^{M_\gamma}_{\text{top}})$. Hence $\kappa_\beta > \bar{\kappa}$, since otherwise $\kappa_\beta \leq \bar{\kappa}$ and

$$\kappa < \lambda_\beta \leq \pi_{\xi,\beta}(\bar{\kappa}) = \kappa_\gamma < \kappa$$

Contradiction! Hence $\kappa_\gamma = \pi_{\xi,\gamma}(\bar{\kappa}) = \bar{\kappa} < \kappa_\beta$. QED(15)

We now iterate the operation γ^+.

Definition 3.8.9. Let γ, κ satisfy $(*)$. We set:

$$\gamma^0 = \gamma,\ \gamma^{n+1} \cong (\gamma^n)^+$$

Note that $\gamma^{n+1} < \gamma^n$ if defined. Hence there is a maximal $n < \omega$ such that γ^n is defined. Hence there is a maximal $n < \omega$ such that γ^n is defined. We set

$$\bar{\gamma} = \bar{\gamma}(\kappa) =: \gamma^n$$

(16) The pair $\bar{\gamma}, \kappa$ satisfies $(*)$. Moreover, $\kappa_\gamma \leq \kappa_{\bar{\gamma}} < \kappa < \lambda_{\bar{\gamma}}$ and $\kappa_\gamma < \kappa_{\bar{\gamma}}$ if $\bar{\gamma} \neq \gamma$.

Definition 3.8.10. $\mu = \mu(\kappa) =$ the least μ such that $\kappa < \lambda_\mu$.

Note. $\mu(\kappa_{\gamma_1}) = \gamma_1^*$.

(17) Either $\bar{\gamma}$ is good for $\bar{\kappa}$ or $\mu = \mu(\kappa) \leq_T \bar{\gamma}$.

Proof: Suppose not. Then $\mu \leq \bar{\gamma}$ but $\mu \nleq_T \bar{\gamma}$.

Claim. There is $\beta + 1 \leq_T \bar{\gamma}$ such that $\kappa < \lambda_\beta$.

Proof: If $\bar{\gamma} = \beta + 1$ is a successor, then $\kappa < \lambda_\mu \leq \lambda_\beta$. Now let $\bar{\gamma}$ be a limit ordinal. Pick $\beta + 1 <_T \bar{\gamma}$ such that $\beta \geq \mu$. Then $\kappa < \lambda_\mu \leq \lambda_\beta$. QED (Claim.)

Let β be the least such. Since $\bar{\gamma}$ is not good but $\bar{\gamma}^+$ is undefinded, we conclude $\kappa \leq_T \kappa_\beta$. Let $\xi = T(\beta + 1)$. Then $\kappa < \lambda_\xi$, since $\kappa_\beta < \lambda_\xi$. Hence $\mu \leq \xi$. But then $\mu < \xi$ since:

$$\xi \leq_T \beta + 1 \leq_T \bar{\gamma} \text{ and } \mu \nleq_T \bar{\gamma}$$

If $\xi = \zeta + 1$ is a successor, then $\kappa < \lambda_\zeta$, since $\lambda_\mu \leq \lambda_\xi$. Thus:

$$\zeta + 1 \leq_T \bar{\gamma},\ \zeta < \beta,\ \kappa < \lambda_\zeta,$$

contradicting the minimality of β. Thus ξ is a limit ordinal. Pick $\zeta + 1 \in (\mu, \xi)_T$. Then $\kappa < \lambda_\mu \leq \lambda_\zeta$ and we again have:

$$\zeta + 1 \leq_T \bar{\gamma},\ \zeta < \beta,\ \kappa < \lambda_\zeta$$

Contradiction! QED (17)

Applying this to the case $\gamma = \gamma_0$, $\kappa = \kappa_{\gamma_1}$, we get:

(18) Let $\gamma = \gamma_0$, $\kappa = \kappa_{\gamma_1}$. Then $\bar{\gamma}$ is good for κ.

Proof: Suppose not. Then $\mu = T(\gamma_1 + 1) \leq_T \bar{\gamma}$ by (17). Hence $M_\mu = M^*_{\gamma_1}$, since $b_1 \setminus \alpha$ has no truncation.

Case 1: $\mu = \bar{\gamma}$.

Then $\pi_{\mu,\gamma_1+1} : M_{\bar{\gamma}} \longrightarrow^*_{E_{\nu_{\gamma_1}}} M_{\gamma_1+1}$, where $\kappa_{\bar{\gamma}} < \kappa = \kappa_{\gamma_1} < \lambda_{\bar{\gamma}}$. But $E^{M_{\bar{\gamma}}}_{\nu_{\bar{\gamma}}}$ is the top extender of $M_{\bar{\kappa}}$ since $\bar{\gamma}$ is not good for κ. Hence $\kappa_{\bar{\gamma}} = \text{crit}(E^{M_{\bar{\gamma}}}_{\text{top}}) = \text{crit}(E^{M_{\gamma_1+1}}_{\text{top}})$, and $\kappa > \kappa_{\bar{\gamma}}$, $\kappa = \kappa_{\gamma_1}$. But then $\kappa_{\gamma_0} \leq \kappa_{\gamma} < \kappa_{\gamma_0}$. This is the undesirable situation which we had eliminated by our choice of α. Contradiction! QED(Case 1.)

Case 2: $\mu \leq \bar{\gamma}$.

Let $\mu = T(\beta+1), \beta+1 \leq \bar{\gamma}$. Then $\kappa < \lambda_\mu \leq \lambda_\beta$. Hence

$$\pi_{\mu,\bar{\gamma}} : M^*_\beta \longrightarrow_{\Sigma^*} M_{\bar{\gamma}},\ \text{crit}(\pi_{\mu,\bar{\gamma}}) = \kappa_\beta$$

since $\pi_{\beta+1,\bar{\gamma}}$ in total on $M_{\beta+1}$ by (14).

Clearly $\kappa_\beta \geq \kappa > \kappa_{\bar{\gamma}}$, since $\bar{\gamma}^+$ does not exist. But then:

$$\kappa_{\bar{\gamma}} = \text{crit}(E^{M^*}_{\text{top}}),\ \text{ since } \kappa_{\bar{\gamma}} = \text{crit}(E^{M_{\bar{\gamma}}}_{\text{top}})$$

$M_\mu = M^*_{\gamma_1}$, since $\mu = T(\gamma_1+1)$ and no truncation occurs above α in b_1. Since $\kappa_\beta \geq \kappa$ and $\rho^1_{M_{\bar{\gamma}}} \leq \kappa$, we have $\rho^1_{M^*_\beta} \leq \kappa$. But then M^*_β is not a proper segment of M_μ, since $\tau_{\gamma_1} < \lambda_\mu \leq \lambda_\beta$ would not be a cardinal in M_μ. Hence $M_\mu = M^*_\beta$. Hence $\kappa_{\bar{\gamma}} = \text{crit}(E^{M_{\gamma_1}}_{\text{top}})$ and $\kappa_{\gamma_0} \leq \kappa_{\bar{\gamma}} \leq \kappa = \kappa_{\gamma_1}$. But this is, again, the undesirable situation. Contradiction! QED (18)

Using this we prove:

(19) $N \models$ there are arbitrarily large strong cardinals.

Proof: Since α (and hence κ_{λ_0}) can be chosen as large as we want, it suffices to show:

Claim. There is a $\kappa' \geq \kappa_{\gamma_0}$ which is strong in N.

Proof: We know that $E^{M_{\bar{\gamma}}}_{\nu_{\bar{\gamma}}}$ is strong in N at $\kappa_{\bar{\gamma}} \geq \kappa_{\gamma_0}$ of length $\lambda_{\bar{\gamma}}$. By (18), $G \in M_{\bar{\gamma}}||\tau_{\gamma_1} \subset N$, where G is strong in N at $\kappa_{\bar{\gamma}}$ of length κ. We again set: $F'_n = E^{M_{\gamma_n}}_{\nu_{\gamma_n}}|\kappa_{\gamma_{n+1}}$. Set

$$G_0 = G,\ G_{n+1} = F'_{n+1} \circ G_n$$

By Lemma 3.8.5 it then follows by induction on n that $G_n \in N$ is strong for N at $\kappa_{\bar{\gamma}}$ of length $\kappa_{\gamma_{n+1}}$. QED(19)

We must still show that N is Woodin for A whenever $A \in M_{b_0} \cap M_{b_1}$. We first prove this for the special case $A \subset \eta$:

(20) Let $A \in M_{b_0} \cap M_{b_1}$ such that $A \subset \tilde{\xi}$. Then N is Woodin for A.

Before proving this, however, we prove an auxiliary lemma:

(21) Let γ, κ satisfy $(*)$. Let $\beta = \gamma^+$ be defined (hence $\kappa_\gamma < \kappa_\beta$). Set:

$$F = E^{M_\gamma}_{\nu_\gamma}|\kappa,\ G = E^{M_\beta}_{\nu_\beta}|\kappa$$

Let $a \subset \kappa$ such that $a \in M_\gamma$ and $\bar{F}(a \cap \kappa_\gamma) = a$. Then $a \in M_\beta$ and $G(a \cap \kappa_\beta) = a$.

Proof: $E^{M_\gamma}_{\nu_\gamma}$ is the top extender of M_γ, since γ^+ exists and $\pi_{\beta+1,\gamma}$ is a total function on $M_{\beta+1}$ by (14). Hence:

$$\pi_{\xi,\gamma} : M^*_\beta \longrightarrow_{\Sigma^*} M_\gamma,\ \kappa_\beta = \mathrm{crit}(\pi_{\xi,\gamma'})$$

where $\xi = T(\beta+1)$. Since $\kappa_\gamma < \kappa_\beta$ we conclude:

$$\kappa_\gamma = \mathrm{crit}(E^{M_\gamma}_{\mathrm{top}}) = \mathrm{crit}(E^{M^*_\beta}_{\mathrm{top}})$$

Set $A = E^{M_\gamma}_{\mathrm{top}}(a \cap \kappa_\gamma)$, $\bar{A} = E^{M^*_\beta}_{\mathrm{top}}(a \cap \kappa_\gamma)$.
Then $\pi_{\xi,\gamma}(\bar{A}) = A$ and:

$$\bar{A} \cap \kappa_\beta = A \cap \kappa_\beta = a \cap \kappa_\beta,\ A \cap \kappa = a$$

Since $\mathrm{crit}(\pi_{\beta+1,\gamma}) \geq \lambda_\beta > \kappa$, we have:

$$G(a \cap \kappa_\beta) = \pi_{\xi,\beta+1}(a \cap \kappa_\beta) \cap \kappa = \pi_{\xi,\gamma}(a \cap \kappa_\beta) = A \cap \kappa = a$$

QED (21)

It is now easy to prove (20). Since α can be chosen as large as we want, it again suffices to show that if $A \in \mathrm{rng}(\pi_{\gamma^*_0,b_0}) \cap \mathrm{rng}(\pi_{\gamma^*_1,b_1})$ in N. We in fact show that $\kappa_{\bar{\gamma}}$ is A-strong, where $\gamma = \gamma_0$. We again define:

$$G_0 = G = E^{M_{\bar{\gamma}}}_{\nu_{\bar{\gamma}}}|\kappa,\ G_{n+1} = F'_{n+1} \circ G_n$$

where $\kappa = \kappa_{\gamma_1}$. By iterated use of (21) we then have: $A \cap \kappa = G(A \cap \kappa_{\bar{\gamma}})$. It then follows inductively that

$$G_n(A \cap \kappa_{\bar{\gamma}}) = A \cap \kappa_{\gamma_{n+1}}$$

since $F'_n(A \cap \kappa_{\gamma_n}) = A \cap \kappa_{\gamma_{n+1}}$. QED(20)

We now show that this implies the full result. We use the fact that any $A \subset N$ can be coded by a set $\tilde{A} \subset \tilde{\eta}$. Let $N = J^E_{\tilde{\eta}}$ and suppose that $\alpha \leq \tilde{\eta}$ is Gödel-closed. By Corollary 2.4.12 we know $M = h_M``(\omega \times \alpha)$, where $M = J^E_\alpha$. Let k_α be the canonical $\Sigma_1(M)$ uniformization of

$$\{\langle \nu, x\rangle : x = h_M((\nu)_0, (\nu)_1)\}$$

Then k_α injects M into α and is uniformly $\Sigma_1(M)$. Set $k = k_{\tilde{\eta}}$. Then:

(a) $k_\alpha = k \restriction \alpha$ if $\alpha < \tilde{\xi}$ is Gödel-closed.

(b) $k_\mu^{-1} = k^{-1} \restriction \mu$ if $\mu < \tilde{\eta}$ is a cardinal in N (since J_μ^E is Σ_1-elementary submodel of N).

(c) $k_\alpha \in N$ for Gödel-closed $\alpha < \tilde{\eta}$.

(d) Let $A \subset N$ and set $\tilde{A} = k``A$. If $\mu < \tilde{\eta}$ is a cardinal in $N$, then $\tilde{A} \cap \mu = k``_\mu(A \cap J_\mu^E)$ (hence $\langle N, \tilde{A}\rangle$ is amenable if $\langle N, A\rangle$ is amenable.

Theorem 3.8.4 then follows from

(22) Let $A \subset N$ such that $\langle N, \tilde{A}\rangle$ is amenable and N is Woodin with respect to $\tilde{A}$. Then N is Woodin with respect to A.

Proof: Let $G \in N$ be $\tilde{A}$-strong in N at κ of length μ, where $\mu > \omega$ is regular in N.

Claim. G is A-strong in N (i.e. $\tilde{G}(A \cap J_\kappa^E) = A \cap J_\mu^E$).

Proof: N is extendable by G. Set:

$$\pi : N \longrightarrow_G N' = J_{\tilde{xi}}^{E'}$$

Let k', k'_α be defined over N like k, k_α over N. Since G is strong in N we have: $J_\mu^E = J_\mu^{E'}$ and $k_\mu = k'_\mu$. Let $\nu = \pi(\kappa)$. Then $k'_\nu = k' \restriction J_\nu^{E'}$. Hence for $y \in J_\mu^E$ we have:

$$\begin{aligned}
y \in \tilde{G}(A \cap J_\kappa^E) &\longleftrightarrow k_\mu(y) \in k'_\nu{}''\tilde{G}(A \cap J_\kappa^E) \\
&\longleftrightarrow k_\mu(y) \in k'_\nu{}''\pi(A \cap J_\kappa^E) \\
&\longleftrightarrow k_\mu(y) \in \pi(k'_\nu{}''(A \cap J_\kappa^E)) \\
&\longleftrightarrow k_\mu(y) \in G(\tilde{A} \cap \kappa) \\
&\longleftrightarrow k_\mu(y) \in \tilde{A} \cap \mu = k_\mu{}''(A \cap J_\mu^E) \\
&\longleftrightarrow y \in A \cap J_\kappa^E
\end{aligned}$$

This proves (22) and with it Theorem 3.8.4.

Note. The notion of *premouse* which we develop in this book is based on the notion developed by Mitchell and Steel in [MS]. However, they employ a different indexing of the extenders than we do. Their indexing makes it much easier to prove Theorem 3.8.4, since our special assumption (SA), when reformulated for their premice, turns out to the outright.

We note a further consequence of our theorem:

Lemma 3.8.9. *Let $N = J_{\tilde{\eta}}^E$ be as in Theorem 3.8.4. There are arbitrarily large $\nu \in N$ such that $E_\nu \neq \varnothing$.*

Proof: Suppose not. Let $\alpha < \eta$ be a strict upper bound of the set of such ν. Then N is a constructible extension of J^E_α (in the sense of Definition of E in §2.5). By Theorem 3.8.4 some $\kappa > \alpha$ is strong in N. In particular, there is $F \in N$ which is an extender at κ on N and N is extendible by F. Let $\pi : N \longrightarrow_F N'$. Then $\langle N', \pi \rangle$ is the extension of $\langle N, \bar{\pi} \rangle$ where $\bar{\pi} : J^E_\tau \longrightarrow J^E_\nu$ is the extension of F (with $\tau = \kappa^{+N}$). Then $\bar{\pi} \in N$. Hence ν is not regular in N since $\tau < \nu$ and $\nu = \sup \bar{\pi}"\tau$. Clearly, however, $N' = J^{E'}_{\eta'}$ is a constructible extension of $J^E_{\alpha'}$, where $\alpha' \geq \alpha$. Hence $N \subset N'$. ν is regular in N', since $\nu = \pi(\tau)$. But then ν is regular in N. Contradiction! QED(Lemma 3.8.9)

3.8.3 One smallness and unique branches

We now apply the method of the previous subsection to one small mice. We let $M, b_0, b_1, \alpha, \gamma_n (n < \omega)$, etc. be as before, but also assume that M is one small. It is easily seen that every normal iterate of M must be one small. Hence M_{b_0}, M_{b_1} are one small. Letting $\eta, \tilde{\eta}, N$ be as before, we set:

Definition 3.8.11. $Q =: J^{E^N}_\beta$, where $\beta = \min(\mathrm{On}_{M_{b_0}}, \mathrm{On}_{M_{b_1}})$.

By Theorem 3.8.4 we obviously have:

Lemma 3.8.10. *$\tilde{\eta}$ is Woodin in Q.*

From now on, assume w.l.o.g. that $\mathrm{On}_{M_{b_0}} \leq \mathrm{On}_{M_{b_1}}$ (i.e. $\mathrm{On}_{M_{b_0}} = \beta$). Then:

Lemma 3.8.11. *$M_{b_0} = Q$.*

Proof: Suppose not. Then there is $\nu \geq \tilde{\eta}$ such that $E^{M_{b_0}}_\nu \neq \varnothing$. But then $\nu > \tilde{\eta}$, since $\tilde{\eta}$ is a limit of cardinals in M_{b_0} and ν is not. Taking ν as minimal, we then have $J^{E^{M_{b_0}}}_\nu = J^{E^N}_\nu \models \tilde{\eta}$ is Woodin. Hence M_{b_0} is not one small. Contradiction! QED (Lemma 3.8.11)

But then we can essentially repeat our earlier argument to show:

Lemma 3.8.12. *Let $A \subset N$ be $\Sigma^*(Q)$ such that $\langle N, A \rangle$ is amenable. Then N is Woodin for A.*

Proof: As before, we can assume w.l.o.g. that $A \subset \mathrm{On}_Q$. Let A be $\Sigma^*(Q)$ in a parameter p by Σ^* definition φ. We assume α to be chosen as before, but now large enough that for $h = 0, 1$:

- $p \in \text{rng}(\pi_{\gamma_h^*}, b_h)$
- If $N \neq Q$, then $N \in \text{rng}(\pi_{\gamma_h^*, b_h})$
- If $\text{On}_{M_{b_h}} > \text{On}_Q$ (hence $h = 1$), then $Q \in \text{rng}(\pi_{\gamma_1^*}, b_1)$.

Since $M_{b_0} = Q$ we have

$$\pi_{\gamma_{2i}^*, b_0} : M_{\gamma_{2i}}^* \longrightarrow_{\Sigma^*} Q \text{ with critical point } \kappa_{2i}.$$

Let A_{2i} be defined over $M_{\gamma_{2i}}^*$ in $P_{2i} = \pi_{\gamma_{2i}^*, b_0}^{-1}()$ by φ. Set:

$$N_{2i} = \begin{cases} \pi_{\gamma_{2i}^*, b_i}^{-1}(N) & \text{if } N \in Q \\ M_{\gamma_{2i}^*} & \text{if not} \end{cases}$$

Then $\langle N_{2i}, A_{2i} \rangle$ is amenable and:

$$(\pi_{\gamma_{2i}^*, b} \upharpoonright N_{2i}) : \langle N_{2i}, A_{2i} \rangle \longrightarrow_{\Sigma_0} \langle N, A \rangle$$

It follows easily that $A_{2i} \cap \kappa_{2i} = A \cap \kappa_{2i}$ and

$$E_{\nu_{2i}}(A \cap \kappa_{2i}) = \pi_{\gamma_{2i}^*, \gamma_{2i}+1}(A \cap \kappa_{2i}) = A \cap \lambda_{2i}$$

If $\text{On} \cap M_{b_1} = \text{On} \cap Q$, it follows by symmetry from the proof of Lemma 3.8.11 that $M_{b_1} = Q$. Hence:

$$\pi_{\gamma_{2i}^*, b_1} : M_{\gamma_{2i+1}}^* \longrightarrow_{\Sigma^*} Q \text{ with critical point } \kappa_{\gamma_{2i+1}}$$

If we then define $A_{2i+1}, N_{2i+1}, P_{2i+1}$ as before, we get:

$$E_{\nu_i}(A \cap \kappa_i) = \pi_{\gamma_i^*, \gamma_i+1}(A \cap \kappa_i) = A \cap \lambda_i$$

for $i < \omega$. If $M_{b_1} \neq Q$, we then set:

$$A_{2i+1} = \pi_{\gamma_{2i+1}^*, b_i}^{-1}(A),\ N_{2i+1} = \pi_{\gamma_{2i+1}^*, b_1}^{-1}(N)$$

and get the same result. Defining F_i' as before, we then have:

$$F_i'(A \cap \kappa_{\gamma_i}) = A \cap \kappa_{i+1}, \text{ for } i < \omega$$

Moreover, we can repeat our earlier proof to get $G_0(A \cap \bar{\gamma}) = A \cap \kappa_{\gamma_1^*}$. It then follows by induction on i that

$$G_i(A \cap \kappa_{\bar{\gamma}}) = A \cap \kappa_{\gamma_{i+1}}, \text{ for } i < \omega$$

Hence $\kappa_{\bar{\gamma}} \geq \kappa_{\gamma_0}$ is A-strong in N. But we can choose α and with it $\kappa_{\bar{\gamma}}$ arbitrarily large.

QED (Lemma 3.8.12)

Note that F'_i is strong at κ_{γ_i} of length $\kappa_{\gamma_{i+1}}$, even though we do not know whether $F'_i \in N$. It it also clear that $F'_i(A \cap \kappa_{\gamma_i}) = A \cap \kappa_{\gamma_{i+1}}$, if $A \subset \mathrm{On} \cap N$, $A \in \underset{\sim}{\Sigma}^*(Q)$, $\langle N, A\rangle$ is amenable, and α is chosen as in the proof of Lemma 3.8.12. If, as before, we set $F_0 = F'_0$, $F'_{i+1} \circ F_i$, we get: $F_i(A \cap \kappa_0) = A \cap \kappa_{i+1}$. If drop the requirement $A \subset \mathrm{On}$, permitting only that $A \subset N$, we still have $\tilde{F}'_i(A \cap J^E_{\kappa_{\gamma_i}}) = A \cap J^E_{\kappa_{\gamma_{i+1}}}$ (where $E = E^N$), and $\tilde{F}'_i$ is the associated operation defined in §3.8. If we then set: $\tilde{F}_0 = \tilde{F}'_0, \tilde{F}_{i+1} = \tilde{F}'_{i+1} \circ \tilde{F}_i$, we get:

$$\tilde{F}'_i(A \cap J^E_{\kappa_{\gamma_0}}) = A \cap J^E_{\kappa_{\gamma_{i+1}}}$$

Note. It is not hard to show that F_i is a strong extender at κ on N and that $\tilde{F}_i$ is the associated function defining f earlier. However, we will not need this.

Recapitulating:

Lemma 3.8.13. *Let $A \subset N$, such that A is $\Sigma^*(N)$ in a parameter p. Suppose that $\langle N, A\rangle$ is amenable. Choose α big enough that:*

- $p \in \mathrm{rng}(\pi_{\gamma_h{}^*, b_h})$
- $N \neq Q \longrightarrow N \in \mathrm{rng}(\pi_{\gamma_h{}^*, b_h})$ *for $h = 0, 1$ such that $M_{b_h} = Q$ and:*
- $A, N \in \mathrm{rng}(\pi_{\gamma_1{}^*, b_1})$ *if $M_{b_1} \neq Q$.*

Let $\tilde{F}'_i, \tilde{F}_i\ (i < \omega)$ be defined as above. Then:

$$\tilde{F}_i(A \cap J^E_{\kappa_{\gamma_0}}) = A \cap J^E_{\kappa_{\gamma_{i+n}}},\ \textit{for } i < \omega$$

Note that, by lemma 3.8.12, we san conclude that if $\rho^\omega_Q \geq \tilde{\eta}$ and $A \in \Sigma^*(Q)$ such that $A \subset N$, then N is Woodin with respect to A. We now prove:

Lemma 3.8.14. *$\rho^\omega_Q \geq \tilde{\eta}$.*

Proof: Suppose not. We consider several cases:

Case 1: $\rho^n_Q \geq \tilde{\eta}$ and $\rho^{n+1}_Q < \tilde{\eta}$ for any $n < \omega$. Then there is a $\underset{\sim}{\Sigma}^{(n)}_1(Q)$ set $B \subset \tilde{\eta}$ such that $\langle N, B\rangle$ is not amenable. But B then has the form:

$$B(\xi) \longleftrightarrow \bigvee z A(z, \xi)$$

where $A \subset N = H^n_Q$ is $\Sigma^{(n)}_0$ in a parameter p. Let $\delta < \bar{\eta}$ such that $B \cap \delta \notin N$. Pick α big enough that $\delta < \kappa_{\gamma_h}(h = 0, 1)$ and the conditions in Lemma 3.8.13 are satisfied with respect to A, p. There is $\xi < \delta$ such that

$$\xi \in B \text{ and } \bigwedge z \in J^E_{\kappa_{\gamma_0}} \neg A(z, \xi),$$

since otherwise $B \cap \delta \in N$. Set $\tilde{A} = \{<: A(z, \xi)\}$. Then $\tilde{A} \subset N \in \Sigma^{(n)}_0(Q)$ in $\langle p, \xi\rangle$ and the conditions in Lemma 3.8.13 are satisfied for $\tilde{A}, \langle p, \xi\rangle$ in place of A, p. Hence for a sufficient $n < \omega$ we will have:

$$\varnothing = \tilde{A} \cap J^E_{\kappa_{\gamma_0}} = \tilde{F}_n(\tilde{A} \cap J^E_{\kappa_{\gamma_0}}) = \tilde{A} \cap J^E_{\kappa_{\gamma_{n+1}}} \neq \varnothing$$

Contradiction! QED(Case 1)

Note. The case $N = M_{b_0}$ is included in Case 1.

Case 2: Case 1 fails. Then $\rho^{n+1} < \tilde{\eta} < \rho^n$ in Q. Set: $Q^* = Q^n \circ P^n_Q$. By Lemma 2.5.22 of §2.6., Q is n-sound and:

$$Q^* = h_{Q^*}(\tilde{\eta} \cup p)$$

where $P = P^{n+1}_Q$. Let $\delta = \rho^{n+1}_Q$. Pick α big enough that $\kappa_{\gamma^*_0}, \kappa_{\gamma^*_1} > \delta$ and:

- $p, p^n_Q, \eta \in \text{rng}(\pi_{\gamma^*_h, b_h})$ for $h = 0, 1$
- $Q \in \text{rng}(\pi_{\gamma^*_1, b_1})$ of $Q \neq M_{b_1}$

Each element of Q^* has the form:

$$h_{Q^*}(i, \langle\xi, \tilde{\eta}, p\rangle), \text{ where } i < \omega, \xi < \tilde{\eta}$$

Case 2.1: There is μ such that $\kappa_{\gamma_0} < \mu < \tilde{\eta}$ and

$$h_{Q^*}(i, \langle\xi, \tilde{\eta}, p\rangle) = \mu \text{ where } i < \omega, \xi < \kappa_{\gamma_0}$$

Let:

$$y = h_{Q^*}(i, \langle\xi, \tilde{\eta}, p\rangle) \longleftrightarrow \bigvee z \in Q^* H(z, i, \xi, y)$$

where $H \subset Q^*$ is $\Sigma^{(n)}_0(Q)$ in $\tilde{\eta}, p, P^n_Q$.

Let β be least such that

$$\bigvee z \in S^E_\beta H(z, i, \xi, y)$$

It follows easily that $S^E_\beta \in \text{rng}(\pi_{\gamma^*_h, b_h})$ for $h = 0, 1$. But then $\{\mu\}$ is $\Sigma*^{(Q)}$ in the parameters

$$r = \langle i, \xi, \tilde{\eta}, P, P^n_Q, S^E_A\rangle$$

$$y = \mu \longleftrightarrow \bigvee z \in S^E_\beta H(z, i, \xi, y)$$

But $\langle N, \{\mu\}\rangle$ is obviously amenable. It is easily seen that $\{\mu\}, r$ satisfy the condition in Lemma 3.8.13 in place of A, p. Hence, for sufficient n:

$$\varnothing = \{\mu\} \cap \kappa_{\gamma_0} = F_n(\{\mu\} \cap \kappa_{\gamma_0}) = \{\mu\} \cap \kappa_{\gamma_{n+1}} \neq \varnothing$$

Contradiction! QED(Case 2.1)

Case 2.2. Case 2.1. fails. Set $X = \{h_{Q^*}(i, \langle \xi, \tilde{\xi}, p\rangle) : i < \omega, \xi < \kappa_{\gamma_0}\}$. Since κ_{γ_0} is Gödel-closed, we know that $X = h_{Q^*}(\kappa_{\gamma_0} \cup \langle \bar{\eta}, p\rangle)$. Hence $Q^*|X \prec_{\Sigma_1} Q^*$. Transitivize X to get:

$$\sigma : \bar{Q}^* \overset{\sim}{\longrightarrow} (Q^*|X)$$

Then $\sigma : \bar{Q}^* \longrightarrow_{\Sigma_1} Q^*$. Let $\sigma(\bar{p}) = P$. But the failure of Case 2.1 we know that $X \cap \tilde{\eta} = \kappa_{\gamma_0}$. Since $\tilde{\eta} \in \text{rng}(\sigma)$ we can conclude: $\sigma(\kappa_{\gamma_0}) = \tilde{\eta}$.

σ extends to $\sigma' : \bar{Q} \longrightarrow_{\Sigma_1^{(n)}} Q$, where $\bar{Q}^Q$, where $\bar{Q}^{n, P^n_{\bar{Q}}}$ and $\sigma'(P^n_{\bar{Q}}) = P^n_Q$, $\bar{Q}$ is a constructible extension of $J^E_{\kappa_{\gamma_0}}$, since Q is a constructible extension of $J^E_{\tilde{\eta}} = N$. We now "compare" $\bar{Q}$ with $\bar{N}$. κ_{γ_0} is Woodin in $\bar{Q}$, since $\tilde{\eta}$ is Woodin in Q. Let $\nu < \tilde{\eta}$ be minimal such that $E_\nu \neq \varnothing$ in N and $\nu > \kappa_{\gamma_0}$. Then $J^{E^N}_\nu$ is a constructible extension of $J^E_{\kappa_{\gamma_0}}$. Letting $\beta = \mathsf{ON} \cap \bar{Q}$ then we have $\beta < \nu$, since otherwise κ_{γ_0} would be Woodin in J^E_ν. Hence N would be not one small, contradiction! But then $\bar{Q} \in J^E_\nu \subset N$. There is $B \subset Q^*$ which is $\Sigma_1(Q^*)$ in p such that $B \cap \delta \notin N$. (Recall that $\delta = \rho^{n+1}_Q < \kappa_{\gamma_0}$). Let $\overline{B}$ be $\Sigma_1(\overline{Q^*})$ in $\overline{p}$ by the same definition. Since $\sigma \restriction \kappa_{\gamma_0} = \text{id}$, we then get $B \cap \kappa_{\gamma_0} = \bar{B} \cap \kappa_{\gamma_0}$. But $\bar{B} \in N$, since $Q^* \in N$. Hence $B \cap \delta = \bar{B} \cap \delta \in N$. Contradiction!

QED(Lemma 3.8.14)

Making use of this we prove:

Lemma 3.8.15. *There is no truncation on the branch b_0.*

Proof: Suppose not. Let $\mu + 1$ be the least truncation point. Let $\mu^* = T(\mu + 1)$ (hence $\mu + 1 \leq_T \gamma_0 + 1$ and $\mu^* \leq_T \gamma_0^*$). Then $\rho^\omega_{M^*_\mu} \leq \kappa_\mu$. Hence $\rho^\omega_{M_{b_0}} \leq \kappa_\mu < \tilde{\eta}$, since $\text{crit}(\pi_{\mu^*, b}) = \kappa_\mu$. Contradiction!

QED (Lemma 3.8.15)

Hence $\pi_{0, b_0} : M \longrightarrow_{\Sigma^*} Q$. We shall use this fact to garner information about M. We know:

(a) $Q = J^E_\beta$ is a constructible extension of $N = J^E_{\tilde{\eta}}$.

(b) $\tilde{\eta} = \text{lub}\{\nu : E_\nu \neq \varnothing\}$

(c) $\rho^\omega_Q \geq \tilde{\eta}$ (hence Q is sound).

(d) If $A \subset N = J^E_{\tilde{\eta}}$, $A \in \underline{\Sigma}(Q)$, then N is Woodin for A.

Note. By soundness we have: $\underline{\Sigma}^*(Q) = \underline{\Sigma}_\omega(Q)$.

We shall prove:

Lemma 3.8.16. *Let* $\eta_0 = \text{lub}\{\nu : E^M_\nu \neq \varnothing\}$. *Then:*

(a) $\eta_0 \leq \mathsf{ON}_M$ *is a limit ordinal. Hence* M *is a constructible extension of* $N_0 = J^{E^M}_{\nu_0}$.

(b) $\rho^\omega_M \geq \eta_0$. *Hence* M *is sound.*

(c) *Let* $A \in \underline{\Sigma}_\omega(M)$ *such that* $A \subset N$. *Then* N_0 *is Woodin for* A.

Proof: Set $\pi = \pi_{0,r_0}$. For $i \in b_0$ set: $\pi_i = \pi_{i,b_0}$. Then $\pi_i : M_i \longrightarrow_{\Sigma^i} Q$. We find prove (a). Suppose not $\eta_0 \neq 0$, since otherwise the iteration would be impossible. Hence there is a maximal ν, such that $E^M_\nu \neq \varnothing$. The statement $E^M_\nu \neq \varnothing$ is $\Sigma_r(M)$ in ν and the statement "ν is maximal" is $\Pi_1(M)$. Hence these statement hold in Q of $\pi(\nu)$. But $\pi(\nu) < \tilde{\eta}$ is not maximal. Contradiction! QED(a)

We now prove (b). If not, then $\rho^\omega_M \leq \nu$ where $E^M_\nu \neq \varnothing$. But $\rho^\omega_{M||\nu} \leq \lambda$, where $\kappa = \text{crit}(E^M_\nu)$ and $\lambda = \lambda(E^M_\nu) =: E^M_\nu(\kappa)$. Hence $\rho^\omega_M \leq \lambda < \nu$. Hence

$$\rho^\omega_Q \leq \pi(\rho^\omega_M) \leq \pi(\lambda) < \pi(\nu) < \tilde{\eta}$$

Contradiction! QED(b)

We now prove (c). Let $A \subset N_0$ be $\Sigma_\omega(\,,\,)$. Since M is sound, A is $\underline{\Sigma}^*(M)$ by Corollary 2.6.30. Let A be $\Sigma^*(M)$ in q and let A' be $\Sigma^*(Q)$ in $q' = \pi(q)$ by the same definition. Pick $n < \omega$ such that $\rho^n_M = \eta_0$ and $\rho^n_Q = \tilde{\eta}$. Clearly, every $\Sigma_\omega(H^n_M, A)$ statement translates uniformly into a statement which is $\Sigma^*(M)$ in q. Similarly for Q, A', q'. Hence:

$$\pi \upharpoonright N_0 | \langle N_0, A \rangle \prec \langle N, A' \rangle$$

But the statement "N is Woodin for A'" is elementary in $\langle N, A' \rangle$. Hence N_0 is Woodin for A. QED(Lemma 3.8.16)

We now define:

Definition 3.8.12. A premouse M is *restrained* iff it is one small and does not satisfy the condition (a)-(c) in Lemma 3.8.16.

We have proven:

Theorem 3.8.17. *Every restrained premouse has the minimal uniqueness property.*

By theorem 3.6.1 and theorem 3.6.2 we conclude:

Corollary 3.8.18. *Let $n > \omega$ be regular. Let M be a restrained premouse which is normally $\kappa + 1$-iterable. Then M is fully $\kappa + 1$-iterable.*

Hence, if $\alpha > \omega$ is a limit cardinal and M is normally α-iterable, then M is fully α-iterable. This holds of course for $\alpha = \infty$ as well.

We also note the following fact:

Lemma 3.8.19. *Let M be restrained. Then every normal iterate of M is restrained.*

Proof: Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_i\rangle, T\rangle$ be the iteration of M to $M' = M_\mu$.

Case 1: There is a truncation on the main brach $b = \{i : i \leq_T \mu\}$. Let $i+1$ be the last truncation point. Then $\kappa_i < \lambda_h$ where $h = T(i+1)$. Hence $\rho^\omega_{M^*_h} \leq \lambda_h < \nu_h$. Hence $\rho^\omega_M \leq \pi_{h,\nu}(\rho^\omega_{M^*_h}) < \pi_{h,\mu}(\nu_h)$, where $E^{M'}_{\pi_{h,\mu}(\nu_h)} \neq \varnothing$. Hence M' is restrained.

Case 2: Case 1 fails. Then $\pi_{0,1} : M \longrightarrow_{\Sigma^*} M'$.

Case 2.1: $\rho^\omega_M < \nu$ for a ν such that $E^M_\nu \neq \varnothing$. This is exactly like Case 1. It remains the case:

Case 2.2: Case 2.1 fails. Then $\eta = \text{lub}\{\nu : E^M_\nu \neq \varnothing\}$ is a limit ordinal and M is a constructible extension of $J^{E^M}_\nu$. But then there is $A \subset J^E_\nu$ such that $A \in \underline{\Sigma}_\omega(M)$ and $J^{E^M}_\nu$ is not Woodin for A. Repeating the proof of Lemma 3.8.16, it follows that $\pi_{0,n}$ is an elementary embedding of M into M'. If A is $\Sigma_\omega(M)$ in p and A' is $\Sigma_\omega(M')$ is $\pi(p)$, it follows that $N' = J^{E^{M'}}_{\nu'}$ is not Woodin for A', where

$$\nu' = \text{lub}\{\nu : E^{M'}_\nu \neq \varnothing\} = \pi_{0,\mu}(\eta)$$

Hence M' is restrained. QED(Lemma 3.8.19)

Note. We could also show that every smooth iterate of a restrained premouse is restrained. This does not hold for full iterates, however, since there can be a restrained M such that $M||\mu$ is not restrained for some $\mu \in M$.