Chapter 4

Properties of Mice

4.1 Solidity

In §2.5.3 we introduced the notion of *soundness*. Given a sound M, we were then able to define the n-th projectum $\rho_M^n (n < \omega)$. We then defined the n-th *reduct* $M^{n,a}$ with respect to a parameter a (consisting of a finite set of ordinals). We then defined the n-th set P_M^n of *good parameters* and the set R_M^n of *very good* parameters. (Soundness was, in fact, equivalent to the statement: $P^n = R^n$ for $n < \omega$). We then defined the n-th *standard parameter* $p_M^n \in R_M^n$ for $n < \omega$. This gave us the classical fine structure theory, which was used to analyze the constructible hierarchy and prove such theorems as $\Box$ in L. Mice, however, are not always sound. We therefore took a different approach in §2.6, which enabled us to define $\rho_M^n, M^{n,a}, P_M^n, R_M^n$ for all acceptable M. (In the absence of soundness we could, of course, have: $R_M^n \neq P_M^n$). In fact R_M^n could be empty, although P_M^n never is. P_M^n was defined in §2.6.

P_M^n is a subset of $[\mathrm{On}_M]^{<\omega}$ for acceptable $M = \langle J_\alpha^A, B\rangle$. Moreover, the reduct $M^{n,a}$ is defined for any $n < \omega$ and $a \in [\mathrm{On}_M]^{<\omega}$. The definition of P_M^n, M^n are recapitulated in §3.2.5, together with some of their consequences. R_M^n is defined exactly as before, taking $= R_M^n = \varnothing$ if n is not weakly sound. At the end of §2.6 we then proved a very strong downward extension lemma, which we restate here:

Lemma 4.1.1. *Let $n = m + 1$. Let $a \in [\mathrm{On}_M]^{<\omega}$. Let $N = M^{n,a}$. Let $\overline{\pi} : \overline{N} \longrightarrow_{\Sigma_j} N$ where $\overline{N}$ is a J-model and $j < \omega$. Then:*

(a) *There are unique $\overline{M}, \overline{a}$ such that $\overline{a} \in R_{\overline{M}}^n$ and $\overline{M}^{n,\overline{a}} = \overline{N}$.*

(b) *There is a unique* $\pi \supset \overline{\pi}$ *such that:*

$$\pi : \overline{M} \longrightarrow_{\Sigma_0^{(m)}} M \text{ strictly and } \pi(\bar{(a)}) = a.$$

(c) $\pi : \overline{M} \longrightarrow_{\Sigma_j^{(n)}} M$.

In §2.6. we also proved:

Lemma 4.1.2. *Let* $n = m + 1$. *Let* $a \in R_M^n$. *Then every element of* M *has the form* $F(\xi, a)$ *where* $\xi < \rho_M^n$ *and* F *is a good* $\Sigma_1^{(m)}$ *function.*

Corollary 4.1.3. *Let* $n, a, \overline{\pi}, \pi$ *be as in Lemma 4.1.1, wehere* $j > 0$. *Then*

$\operatorname{rng}(\pi) =$ *The set of* $F(\xi, a)$ *such that* F *is a good* $\Sigma_1^{(m)}$ *function and* $\xi \in \operatorname{rng}(\overline{\pi}) \cap \rho_M^n$

Proof.. Let Z be the set of such $F(\xi, a)$.

Claim 1. $\operatorname{rng}(\pi) \subset Z$.

Proof. Let $y = \pi(\overline{y})$. Then $\overline{y} = \overline{F}(\xi, \overline{a})$ where $\overline{F}$ is a good $\Sigma_1^{(n)}(\overline{M})$ function and $\xi < \rho_{\overline{M}}^n$ by Lemma 4.1.2. Hence $y = F(\pi(\xi), a)$, where F has the same good $\Sigma_1^{(n)}$ definition in M.

QED(Claim 1.)

Claim 2. $Z \subset \operatorname{rng}(\pi)$.

Proof. Let $y = F(\pi(\xi), a)$, where F is a good $\Sigma_1^{(m)}(M)$ function. Then the $\Sigma_1^{(n)}$ statement:

$$\bigvee y\, y = F(\pi(\xi), a)$$

holds in M. Hence, there is $\overline{y} \in \overline{M}$ such that $\overline{y} = \overline{F}(\xi, a)$ where $\overline{F}$ has the same good $\Sigma_1^{(m)}$ definition in $\overline{M}$. Hence

$$\pi(\overline{y}) = F(\pi(\xi), a) = y.$$

QED(Corollary 4.1.3)

Note. $\operatorname{rng}(\pi) \subset Z$ holds even if $j = 0$.

Lemma 4.1.1 shows that a great deal of the theory developed in §2.5.3 for sound structures actually generalizes to arbitrary acceptable structures. This is not true, however, for the concept of *standard parameter*.

In our earlier definition of standard parameter, we assumed the soundness of M (meaning that $P^n = R^n$ for $n < \omega$). We defined a well ordering $<_*$ of $[\mathrm{On}]^{<\omega}$ by:

$$a <_* b \longleftrightarrow \bigvee \xi (a \setminus \xi = b \setminus \xi \wedge \xi \in b \setminus a).$$

We then defined the n-th *standard parameter* p^n_M to be the $<_*$-least $a \in M$ with $a \in P^n$. This definition stil makes sense even in the absence of soundness. We know that $p^n \setminus \rho^i \in P^i$ for $i \leq n$. Hence by $<_*$-minimality we get: $p^n \setminus \rho^n = \varnothing$. For $i \leq n$ we clearly have $p^i \leq_* p^n \setminus \rho^i$ by $<_*$-minimality. However, it is hard to see how we could get more than this if our only assumption on M is acceptability.

Under the assumption of soundness we were able to prove:

$$p^n \setminus \rho^i = p^i \text{ for } i \leq n.$$

It turns out that this does still holds under the assumption that M is fully $\omega_1 + 1$ iterable. Moreover if $\pi : M \longrightarrow N$ is an iteration map, then $\pi(p^n_M) = P^n_N$. The property which makes the standard parameter so well behaved is called *solidity.* As a preliminary to defining this notion we first define:

Definition 4.1.1. Let $a \in M$ be a finite set of ordinals such that $\rho^\omega \cap a = \varnothing$ in M. Let $\nu \in a$. The ν-th *witness to* a in M (in symbols M^ν_a) is defined as follows:

Let $\rho^{i+1} \leq \nu < \rho^i$. Let $b = a \setminus (\nu + 1)$. Let $\overline{M} = M^{i,b}$ be the i-th reduct of M by b. Set: $X = h(\nu \cup (b \cap \overline{M}))$, i.e. $X =$ the closure of $\nu \cup (u \cap \overline{M})$ under $\Sigma_1(M)$ functions. Let:

$$\overline{\sigma} : \overline{W} \longleftrightarrow \overline{M}|X$$

be the transitivation of $\overline{M}|X$. By the extension of embedding lemma there are unique $W, n, \sigma \supset \overline{\sigma}$ such that:

$$\overline{W} = W^{i,\overline{b}}, \sigma : W \longrightarrow_{\Sigma_1^{(i)}} M, \sigma(\overline{b}) = b.$$

Set: $M^\nu_a = W$. σ is called the *canonical embedding* for a in M and is sometimes denoted by σ^ν_a.

Note. Using Lemma 4.1.3 it follows that $\mathrm{rng}(\pi)$ is the set of all $F(\vec{\xi}, b)$ such that $\xi_1, \ldots, \xi_n \subset \nu$, $b = a \setminus (\nu + 1)$ and F is good $\Sigma_1^{(i)}(M)$ function. This is a more conceptual definition of M^ν_a, σ.

Definition 4.1.2. M is *n-solid* iff $M^\nu_a \in M$ for $\nu \in a = p^n_M$ it is solid iff it is n-solid for all n.

P^n was defined as the $<_*$- least element of P^n. Offhand, this seems like a rather arbitrary way of choosing an element of P^n. Solidity, however, provides us with a structural reason for the choice. In order to make this clearer, let us define:

Definition 4.1.3. Let $a \in M$ be a finite set of ordinals. a is *solid for M* iff for all $\nu \in a$ we have

$$\rho_M^\omega \leq \nu \text{ and } M_a^\nu \in M$$

.

Lemma 4.1.4. *Let $a \in P^n$ such that $a \cap \rho^n = \varnothing$. If a is solid for M, then $a = p^n$.*

Proof. Suppose not. Then there is $q \in P^n$ such that $q <_* a$. Hence there is ν such that $q \setminus (\nu+1) = a \setminus (\nu+1)$ and $\nu \in a \setminus q$. But then $q \subset \nu \cup (a \setminus (\nu+1)) \subset \mathrm{rng}(\sigma)$ where $\sigma_a = \sigma_a^\nu$ is the canonical embedding. Let A be $\Sigma^{(n)}(M)$ in q such that $A \cap \rho^{n+1} \notin M$. Let $\overline{A}$ be $\Sigma_1^{(n)}(M_a^\nu)$ in $\overline{q} = \sigma^{-1}(q)$ by the same definition. Since $\sigma \upharpoonright \nu = \mathrm{id}$ and $\rho^n \leq \nu$, we have:

$$A \cap \rho^n = \overline{A} \cap \rho^n \in M,$$

since $A \in \underline{\Sigma}_1^n(M_a^\nu) \subset M$. Contradiction!

QED(Lemma 4.1.4)

The same proof also shows:

Lemma 4.1.5. *Let a be solid for M such that $a \cap \rho^n = \varnothing$ and $a \cup b \in P^n$ for a $b \subset \nu$ for all $\nu \in a$. Then a is an upper segment of P^n (i.e. $a \setminus \nu = P^n \setminus \nu$ for all $\nu \in a$.)*

Hence:

Corollary 4.1.6. *If M is n-solid and $i < n$, then M is i-solid and $P^i = P^n \setminus \rho^i$.*

Proof. Set $a = p^n \rho^i$. Then $a \in P^i$ is M-solid. Hence $a = p^i$.

QED(Corollary 4.1.6)

We set $p_M^* =: \bigcup_{n<\omega} p_M^n$. Then $p^* = p^n$ where $\rho^n = \rho^\omega$.

p^* is called the *standard parameter* of M. It is clear that M is solid iff P^* is solid for M.

Definition 4.1.4. Let $a \in [\mathrm{On}_M]^{<\omega}, \nu \in a$ with $\rho^{i+1} \leq \nu < \rho^i$ in M. Let $b = a \setminus (\nu+1)$. By a *generalized witness* to $\nu \in a$ we mean a pair $\langle N, c\rangle$ such that N is acceptable, $\nu \in N$ and for all $\xi_a, \ldots, \xi_r < \nu$ and all $\Sigma_1^{(i)}$ formulae φ we have:

$$M \models \varphi(\vec{\xi}, b) \longrightarrow N \models (\vec{\xi}, c).$$

Lemma 4.1.7. *Let $N \in M$ be a generalized witness to $\nu \in a$. Assume that $\nu \notin \mathrm{rng}(\sigma)$, where $\sigma = \sigma_a^\nu$ is the canonical embedding. Then $M_a^\nu \in M$.*

Proof. Let $W = M_a^\nu, \overline{W}, \overline{\sigma}$ be as in the definition of M_a^ν. Then $\overline{W} = W^{i,\bar{b}}$, where $\rho^{i+1} \leq \nu < \rho^i$ in M, $b = a \setminus (\nu+1)$ and $\sigma(\overline{b}) = b$. Since $\sigma \upharpoonright \nu = \mathrm{id}$, we have:

$$\overline{W} \models \varphi(\vec{\xi}, \overline{b}) \longrightarrow N \models \varphi(\vec{\xi}, c),$$

for $\xi_1, \ldots, \xi_r < \nu$ and $\Sigma_1^{(i)}$ formulae φ. We can then define a map $\tilde{\sigma} : W \longrightarrow_{\Sigma_1^{(i)}} N$ by:

Let $x = F(\vec{\xi}, \overline{b})$ where $\xi_1, \ldots, \xi_r < \nu$ and F is a good $\Sigma_1^{(i)}(W)$ function. Then, letting $\dot{F}$ be a good definition of F we have:

$$W \models \bigvee x(x = \dot{F}(\vec{\xi}, \overline{b})); \text{ hence } N \models \bigvee x(x = \dot{F}(\vec{\xi}, c)).$$

We set $\tilde{\sigma}(x) = y$, where $N \models y = \dot{F}(\vec{\xi}, c)$.

If we set: $\overline{N} = N^{i,c}$, we have:

$$\tilde{\sigma} \upharpoonright \overline{W} : \overline{W} \longrightarrow_{\Sigma_0} \overline{N}.$$

Let $\gamma = \sup \tilde{\sigma}\text{''}\, \mathrm{On}_{\overline{N}}, \tilde{N} = \overline{N} | \gamma$. Then:

$$\tilde{\sigma} \upharpoonright \overline{W} : \overline{W} \longrightarrow_{\Sigma_1} \tilde{N} \text{ cofinally.}$$

Note that, since $\sigma(\nu) > \nu$ and $\sigma \upharpoonright \nu = \mathrm{id}$, we have: ν is regular in M_a^ν. Hence $\sigma(\nu)$ is regular in M and $H_{\sigma(\nu)}^M$ is a ZFC^- model. We now code $\overline{W}$ as follows. Each $x \in \overline{W}$ has the form: $h(j, \prec \xi, \overline{b} \succ)$ where $h = h_{\overline{W}}$ is the Skolem function of $\overline{W}$ and $\sigma < \nu$.

Set:

$$\dot{\in} = \{\prec\prec j, \xi \succ, \prec k, \zeta \succ\succ : h(j, \prec \xi, \overline{b} \succ) \in h(k, \langle \zeta, \overline{b} \rangle)\}$$
$$\dot{A} = \{\prec j, \xi \succ : h(j, \langle \xi, \overline{b} \rangle) \in A\}$$
$$\dot{B} = \{\prec j, \xi \succ : h(j, \langle \xi, \overline{b} \rangle) \in B\}$$

where $\overline{W} = \langle J_\gamma^A, B \rangle$. Let $D \subset \nu$ code $\langle \dot{\in}, \dot{A}, \dot{B} \rangle$. Then:

$$D \in \Sigma_\omega(\tilde{(}N)) \subset M,$$

since e.g.

$$\dot{\in} = \{\langle \prec j, \xi \succ, \prec k, \zeta \succ \rangle : h_{\tilde{N}}(j, \langle \xi, c \rangle) \in h_{\tilde{N}}(k, \langle \zeta, c \rangle)\}$$

But then $D \in H^M_{\sigma(\nu)}$ by acceptability. But $H^M_{\sigma(\nu)}$ is a ZFC^- model. Hence $\overline{W} \in H^M_{\sigma(\nu)}$ is recoverable from D in $H^M_{\sigma(\nu)}$. Hence $W \in H^M_{\sigma(\nu)} \subset N$ is recoverable from W in $H^M_{\sigma(\nu)}$.

QED(Lemma 4.1.7)

We note that:

Lemma 4.1.8. *Let $a \in P^n, \nu \in a, M^\nu_a \in M$. Then $\nu \notin \mathrm{rng}(\sigma^\nu_a)$.*

Proof. Suppose not. Then $a \in \mathrm{rng}(\sigma)$. Let A be $\Sigma_1(M)$ such that $A \cap \rho^n \notin M$. Let $\overline{A}$ be $\Sigma_1(M^\nu_a)$ in $\overline{a} = \sigma^{-1}(a)$ by the same definition. Then:

$$A \cap \rho^n = \overline{A} \cap \rho^n \in \underline{\Sigma}^*(M^\nu_a) \subset M.$$

Contradiction!

QED (Lemma 4.1.8)

But then:

Lemma 4.1.9. *Let $q \in P^n_M$. Let a be an upper segment of q which is solid for M. Let $\pi : M \longrightarrow_{\Sigma^*} N$ such that $\pi(q) \in P^n_N$. Then $\pi(a)$ is solid for N.*

Proof. Let $\nu \in a, W = M^\nu_a, \sigma = \sigma^\nu_a$. Set:

$$a' = \pi(a), \nu' = \pi(\nu), W' = N^{\nu'}_{a'}, \sigma' = \sigma^{\nu'}_{a'}.$$

We must show that $W' \in N$. We first show:

(1) $\nu' \notin \text{rng}(\sigma')$.

Proof. Suppose not. Let $\rho^{i+1} \leq \nu < \rho^i$ in M. Then $\rho^{i+1} \leq \nu' < \rho^i$ in N. Then in N we have: $\nu' = F'(\xi, b')$ where $\xi < \nu', b' = a' \setminus (\nu' + 1)$, and F' is a good $\Sigma_1^{(i)}(N)$ function.

Let $\dot{F}$ be a good definition for F'. Then in N the $\Sigma_1^{(i)}$ statement holds:

$$\bigvee \xi' < \nu'(\nu' = \dot{F}(\xi', b')).$$

But then in M we have:

$$\bigvee \xi' < \nu(\nu = \dot{F}(\xi', b))$$

where $b = a \setminus (\nu + 1)$. Hence $\nu \in \text{rng}(\sigma)$. Contradiction!

QED(1)

Now set: $W'' = \pi(W)$. In M we have:

$$\bigwedge \xi < \nu(M \models \varphi(\xi, b) \longrightarrow W \models \varphi(\xi, b))$$

for $\Sigma_1^{(i)}$ formulas φ. But this is a $\Pi_1^{(i)}$ statement in M about ν, b, W. Hence the corresponding statement holds in N:

$$\bigwedge \xi < \nu'(N \models \varphi(\xi, b') \longrightarrow W' \models \varphi(\xi, b'))$$

Hence W'' is a generalized witness for $\nu' \in a'$. Hence $W = N_a^{\nu'} \in N$.

QED(Lemma 4.1.9)

As a corollary we then have:

Lemma 4.1.10. *Let M be n-solid. Let $\pi : M \longrightarrow_{\Sigma^*} N$ such that $\pi(p_M^n) \in P_N^n$. Then N is n-solid and $\pi(P_M^n) = P_N^n$.*

Proof. Let $a = p_M^n$. Then $a' = \pi(a) \in P_N^n$ is solid for N by the previous lemma. Moreover, $a' \cap \rho_N^n = \varnothing$. Hence $a' = p_N^n$.

QED(Lemma 4.1.10)

This holds in particular if $\rho^n = \rho^\omega$ in M. But if $\pi : M \longrightarrow N$ is strongly Σ^*-preserving in the sense of §3.2.5, then $\rho^n = \rho^\omega$ in N and $\pi``(P_M^n) \subset P_M^n$. Hence:

Lemma 4.1.11. *Let M be solid. Let $\pi : M \longrightarrow N$ be strongly Σ^*-preserving. Then N is solid and $\pi(p^i_M) = p^i_N$ for $i < \omega$.*

QED(Lemma 4.1.11)

Corollary 4.1.12. *Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal iteration. Let $h = T(i+1)$ where $i+1 \leq_T j$. Assume that $(i+1, j]_T$ has no drop. If M^*_j is solid, then M_j is solid and $\pi_{h,j}(p^n_{M^*_i}) = p^n_{M_j}$ for $n < \omega_1$.*

Proof. $\pi_{h,j}$ is strongly Σ^*-preserving.

We now define:

Definition 4.1.5. Let M be acceptable. M is a *core* iff it is sound and solid. M is the *core of* N with *core map* iff M is a core and $\pi : M \longrightarrow_{\Sigma^*} N$ with $\pi(p^*_M) = p^*_N$ and $\pi \restriction \rho^\omega_M = \mathrm{id}$.

Clearly M can have at most one core and one core map.

Definition 4.1.6. Let $M = \langle J^E_\alpha, E_\alpha\rangle$ be a premouse. M is *presolid* iff $M||\xi$ is solid for all limit $\eta < \alpha$.

Lemma 4.1.13. *Let M be acceptable. The property "M is presolid" is uniformly $\Pi_1(M)$. Hence, if $\pi : M \longrightarrow_{\Sigma_1} N$, then N is presolid.*

Proof. The function:

$$\langle \Vdash_{M||\xi} : \xi \text{ is a limit ordinal}\rangle$$

is uniformly $\Sigma_1(M)$. But for each $i < \omega$ there is a first order statement φ_i which says that M is "solid above ρ^i", i.e.

$$M^\nu_{P^i_M} \in M \text{ for all } \nu \in p^i_M.$$

The map $i \mapsto \varphi_i$ is recursive. But M is presolid if and only if:

$$\bigwedge \xi \in M \bigwedge i(\xi \text{ is a limit } \longrightarrow \Vdash_{M||\xi} \varphi_i)$$

QED(Lemma 4.1.13)

We shall prove that every fully iterable premouse is solid. But if M is fully iterable, then so is every $M||\eta$. Hence M is presolid.

The *comparison Lemma* (Lemma 3.5.1) tells us that, if we coiterate two premice M^0, M^1 of cardinality less than a regular cardinal θ, then the coiteration will terminate below θ. If both mice are $\theta+1$-iterable, and we use successful strategies, then termination will not occur until we reach $i < \theta$ such that $M_i^0 \triangleleft M_i^1$ or $M_i^1 \triangleleft M_i^0$ ($M \triangleleft M'$ is defined as meaning $\bigvee \xi \leq \mathrm{On}_{M'}, M = M'||\xi$.) If $M_i^0 \triangleleft M_i^1$, we take this as making a statement about the original pair M^0, M^1 to the effect that M^1 contains at least as much information as M_0. However, we may have truncated on the man branch to M_i^1, in which case we have "thrown away" some of the information contained in M_1. If we also truncated on the main branch to M_0, it would be hard to see why the final result tell us anything about the original pair. We now show that, if M^0 and M^1 are both presolid, then this eventually cannot occur: If there is a truncation on the main branch of the M^1-side, there is no such truncation on the other side. (Hence no information was lost in passing from M^0 to M_i^0.) Moreover, we then have $M_i^0 \triangleleft M_1^1$.

Lemma 4.1.14. *Let $\theta > \omega$ be regular. Let $M^0, M^1 \in H_\theta$ be presolid premice which are normally $\theta+1$-iterable. Let:*

$$I^h = \langle \langle M_i^h \rangle, \langle \nu_i^h \rangle, \langle \pi_{ij}^h \rangle, T^h \rangle \ (h = 0, 1)$$

be the coiteration of length $i+1 < \theta$ by successful $\theta+1$ strategies S^0, S^1 (Hence $M_i^0 \triangleleft M_i^1$ or $M_i^1 \triangleleft M_i^0$.) Suppose that there is a truncation on the main branch of I^1. Then:

(a) $M_i^0 \triangleleft M_i^1$.

(b) *There is no truncation on the main branch of I^0.*

Proof. We first prove (a). Let $l_1 + 1 \leq i$ be the least point of truncation in T^1"$\{i\}$. Let $h_1 = T(l_1+1)$. Let $Q^1 = M_{l_1}^{1*}$. Then Q^1 is sound and solid. Let $\pi^1 = \pi_{h_1,i}^1$. By Lemma 4.1.12, M_i' is solid and $\pi^1(p_{Q^1}) = p_{M_i^1}$. Hence $Q^1 = \mathrm{core}(M_i^1)$ and π^1 is the core map. But $\pi^1 \neq \mathrm{id}$. Hence M_i^1 is not sound. If $M_i^0 \ntriangleleft M_i^1$, we would have: $M_i^1 = M_i^0||\eta$ for an $\eta \in M_i^0$. But $M_i^0||\eta$ is sound. Contradiction! This proves (a).

We now prove (b). Suppose not. Let $l_0 + 1$ be the last truncation point in $T^{00}\{i\}$. Let $h_0 = T^0(l_0+1)$. Let Q^0, π^0 be defined as before. Then $Q^0 = \mathrm{core}(M_i^0)$ and $\pi^0 \neq \mathrm{id}$ is the core map. Hence M_i^0 is not sound. Hence, as before, we have: $M_i^1 \triangleleft M_i^0$. Hence $M_i^0 = M_i^1$ and $Q = Q^0 = Q^1$ is the core of $M_i = M_i^0 = M_i^1$ with core map $\pi = \pi^0 = \pi^1$. Set:

$$F^h =: E_{\nu_{l_h}}^{M_{l_h}^h} \ (h = 0, 1).$$

It follows easily that there is κ defined by:

$$\kappa = \kappa_{l_h}^h = \mathrm{crit}(F^h) = \mathrm{crit}(\pi) \ (h = 0, 1)$$

Thus $\mathbb{P}(\kappa_\alpha) \cap M^h_{l_h} = \mathbb{P}(\kappa) \cap Q$. But:

$$\alpha \in F^h[X] \longleftrightarrow \alpha \in \pi(X)$$

for $X \in \mathbb{P}(\kappa) \cap Q, \alpha < \lambda_h = F^h(\kappa)$. Hence $l_0 \neq l_1$, since otherwise $\lambda_0 = \lambda_1$ and $F^0 = F^1$. Contradiction!, since ν_{l_h} is the first point fo difference. Now let e.g. $l_0 < l_1$. Then ν_{l_0} is regular in M^0_j for $l_0 < j \leq i$. But then it is regular in $M^1_{l_1}||\nu_{l_1}$, since $M^1_{l_1}||\nu_{l_1} = M^0_{l_1}||\nu_{l_1}$ and $\nu_{l_1} > \nu_{l_0}$.

But $F^0 = F^1|\lambda_{l_0}$ is a full extender. Hence $F^0 \in M_{l_1}||\lambda_{l_1}$ by the initial segment condition. But then $\tilde{\pi} \in M_{l_1}||\lambda_l$, where $\tilde{\pi}$ is the canonical extension of F^0. But $\tilde{\pi}$ maps $\overline{\sigma} = \kappa^{+Q}$ cofinally to ν_{l_0}. Hence ν_{l_0} is not regular in $M^1_{l_1}||\nu_{l_1}$. Contradiction!

Lemma 4.1.14

We remark in passing that:

Lemma 4.1.15. *Each J_α is solid.*

Proof. Suppose not. Let $M = J_\alpha, \nu \in a = p^i_M$, where $\rho^{i+1} \leq \nu < \rho^i$ in M. Let $M^\nu_a = J_{\overline{\alpha}}$ and let $\pi : J_{\overline{\alpha}} \longrightarrow J_\alpha$ be the canonical embedding. Then $\overline{\alpha} = \alpha$, since $J_{\overline{\alpha}} \notin J_\alpha$. Let $b = a \setminus (\nu + 1), \overline{b} = \overline{\pi}^{-1}(b)$. Set $\overline{a} = (a \cap \nu) \cup \overline{b}$. Then $\overline{a} \in P^i$ in M_i. But $\pi"(\overline{a}) = (a \cap \nu) \cup b <_* a$ where π is monotone. Hence $\overline{a} <_* a$. Hence $\overline{a} \notin P^i$ by the $<_*$-minimality of a. Contradiction!

QED(Lemma 4.1.15)

By virtually the same proof:

Lemma 4.1.16. *Let $M = J^A_\alpha$ be a constructible extension of J^A_β (i.e. $A \subset J^A_\beta$, where $\beta \leq \alpha$). Let $\rho^\omega_M \geq \beta$. Then M is solid.*

The solidity Theorem

We intend to prove:

Theorem 4.1.17. *Let M be a premouse which is fully $\omega_1 + 1$-iterable. Then M is solid.*

A consequence of this is:

Corollary 4.1.18. *Let M be a 1-small premouse which is normally $\omega_1 + 1$-iterable. Then M is solid.*

Proof. If M is restrained, then it has the minimal uniqueness property and is therefore fully $\omega_1 + 1$-iterable by Theorem 3.6.1 amd Theorem 3.6.2. But if M is not restrained it is solid by Lemma 4.1.16.

QED(Corollary 4.1.18)

It will take a long time for us to prove Theorem 4.1.17. A first step is to notice that, if $M \in H_\kappa$, where $\kappa > \omega_1$ is regular and $\pi : H \prec H_\kappa$, with $\pi(\overline{M}) = M$, where H is transitive and countable, then M is solid iff $\overline{M}$ is solid, by absoluteness. Moreover, $\overline{M}$ is fully $\omega_1 + 1$-iterable by Lemma 3.5.7. Hence it suffices to prove our Theorem under the assumption: M is countable. This assumption will turn out to be very useful, since we will employ the Neeman-Steel Lemma. It clearly suffices to prove:

(*) If M is presolid, then it is solid.

To see this, let M be unsolid and let η be least such that $M||\eta$ is not solid. Then $M||\eta$ is also fully $\omega_1 + 1$-iterable and ν is also presolid. Hence $M||\eta$ is solid. Contradiction!

Now let N be presolid but not solid. Then there is a least $\lambda \in p_N^*$ such that $N_a^\lambda \notin N$, where $a = p_N^*$. Set: $M = N_a^\lambda$ and let $\sigma : M \longrightarrow_{\Sigma_1^{(n)}} N$, $\sigma \restriction \lambda = \mathrm{id}$ where $\rho_N^{n+1} \leq \lambda < \rho_N^n$ and $a \setminus (\lambda + 1) \in \mathrm{rng}(\sigma)$. We would like to show: $M \in N$, thus getting a contradiction. How can we do this? A natural approach is to coiterate M with N. Let $\langle I^0, I^1 \rangle$ be the coiteration, I^0 being the iteration of M. If we are lucky, it might turn out that $M_\mu \in N_\mu$, where μ is the terminal point of the coiteration. If we are ever luckier, it may turn out that no point below λ was moved in pairing from M to M_μ -i.e. $\mathrm{crit}(\pi^0_{0,\mu}) \geq \lambda$. In this case it is easy to recover M from M_μ, so we have: $M \in N_\mu$, and there is some hope that $M \in N$. There are many "ifs" in this scenario, the most problematical being the assumption that $\mathrm{crit}(\pi^0_{0,\mu}) \geq \lambda$. In an attempt to remedy this, we could instead do a "phalanx" iteration, iterating the pair $\langle N, M \rangle$ against M. If, at some $i < \mu$, we have $F = E_{\nu_i}^{M_i^0} \neq \varnothing$, we ask whether $\kappa_i^0 < \lambda$. If so we apply F to N. Otherwise we apply it in the usual way to M_h, where h is least such that $\kappa_i^0 < \lambda_h$. For the sake of simplicity we take: $N = M_0^0, M = M_1^0$. ν_i is only defined for $i \geq 1$. The tree of I^0 is then "double rooted", the two roots being 0 and 1. (In the normal iteration of a premouse, 0 is the single root, lying below every $i \geq 0$). Here, $i < \mu$ will be above 0 or 1, but not both.

If we are lucky it will turns out the final point μ lies above 1 in T^0. This will then ensure that $\mathrm{crit}(\pi^0_{0,\mu}) \geq \lambda$. It turns out that this -still improbable seeming- approach works. It is due to John Steel.

In the following section we develop the theory of Phalanxes.

4.2 Phalanx Iteration

In this section we develop the technical tools which we shall use in proving that fully iterable mice are solid. Our main concern in this book is with one small mice, which are known to be of type 1, if active. We shall therefore restrict ourselves here to structures which are of type 1 or 2. When we use the term "mouse" or "premouse", we mean a premouse M such that neither it nor any of its segments $M||\eta$ are of type 3.

We have hitherto used the word "iteration" to refer to the iteration of a single premouse M. Occasionally, however, we shall iterate not a single premouse, but rather an array of premice called a *phalanx*. We define:

By a *phalanx* of length $\eta + 1$ we mean:

$$\mathbb{M} = \langle\langle M_i : i \leq \eta\rangle, \langle\lambda_i : i < \eta\rangle\rangle$$

such that:

(a) M_i is a premouse $(i \leq \eta)$

(b) $\lambda_i \in M_i$ and $J_{\lambda_i}^{E^{M_i}} = J_{\lambda_i}^{E^{M_j}}$, $(i < j \leq \eta)$

(c) $\lambda_i < \lambda_j$ $(i < j < \eta)$

(d) $\lambda_i > \omega$ is a cardinal in M_j $(i < j \leq \eta)$.

A *normal iteration* of the phalanx $\mathbb{M}$ has the form

$$I = \langle\langle M_i : i < \mu\rangle, \langle\nu_i : i + 1 \in (\eta, \mu)\rangle, \langle\pi_{i,j} : i \leq_T j\rangle, T\rangle$$

where $\mu > \eta$ is the *length* of I. $\mathbb{M} = I|\eta + 1$ is the first segment of the iteration. Each $i \leq \eta$ is a minimal point in the tree T. As usual, η_i is chosen such that $\eta_i > \eta_h$ for $h < i$. If h is minimal such that $\kappa_i < \lambda_h$ then $h = T(i+1)$ and $E_{\nu_i}^{M_i}$ is applied to an apropiately defined $M_i^* = M_h||\gamma$. But here a problem arises. The natural definition of M_i^* is:

> $M_i^* = M_h||\gamma$, where $\gamma \leq \mathrm{On}_{M_h}$ is maximal such that $\tau_i < \gamma$ is a cardinal in $M_h||\gamma$.