In the following section we develop the theory of Phalanxes.

4.2 Phalanx Iteration

In this section we develop the technical tools which we shall use in proving that fully iterable mice are solid. Our main concern in this book is with one small mice, which are known to be of type 1, if active. We shall therefore restrict ourselves here to structures which are of type 1 or 2. When we use the term "mouse" or "premouse", we mean a premouse M such that neither it nor any of its segments $M||\eta$ are of type 3.

We have hitherto used the word "iteration" to refer to the iteration of a single premouse M. Occasionally, however, we shall iterate not a single premouse, but rather an array of premice called a *phalanx*. We define:

By a *phalanx* of length $\eta + 1$ we mean:

$$\mathbb{M} = \langle \langle M_i : i \leq \eta \rangle, \langle \lambda_i : i < \eta \rangle \rangle$$

such that:

(a) M_i is a premouse $(i \leq \eta)$

(b) $\lambda_i \in M_i$ and $J_{\lambda_i}^{E^{M_i}} = J_{\lambda_i}^{E^{M_j}}$, $(i < j \leq \eta)$

(c) $\lambda_i < \lambda_j$ $(i < j < \eta)$

(d) $\lambda_i > \omega$ is a cardinal in M_j $(i < j \leq \eta)$.

A *normal iteration* of the phalanx $\mathbb{M}$ has the form

$$I = \langle \langle M_i : i < \mu \rangle, \langle \nu_i : i + 1 \in (\eta, \mu) \rangle, \langle \pi_{i,j} : i \leq_T j \rangle, T \rangle$$

where $\mu > \eta$ is the *length* of I. $\mathbb{M} = I|\eta + 1$ is the first segment of the iteration. Each $i \leq \eta$ is a minimal point in the tree T. As usual, η_i is chosen such that $\eta_i > \eta_h$ for $h < i$. If h is minimal such that $\kappa_i < \lambda_h$ then $h = T(i+1)$ and $E_{\nu_i}^{M_i}$ is applied to an apropiately defined $M_i^* = M_h||\gamma$. But here a problem arises. The natural definition of M_i^* is:

> $M_i^* = M_h||\gamma$, where $\gamma \leq \mathrm{On}_{M_h}$ is maximal such that $\tau_i < \gamma$ is a cardinal in $M_h||\gamma$.

But is there such a γ? If λ_h is a limit cardinal in M_i, then $\tau_i < \lambda_h$ and hence λ_h is such a γ. For $i < \gamma$ we have left the possibility open, however, that λ_h is a successor cardinal in M_i. We could then have: $\tau_i = \lambda_h$. In this case κ_i is the largest cardinal in $J^{E^{M_h}}_{\lambda_i}$. If $E_{\lambda_h} \neq \varnothing$ in M_h, it follows that $\rho^1_{M_h||\lambda_h} \leq \kappa_i < \tau_i$. Hence there is no γ with the desired property and M_i^* is undefined.

In practice, phalanxes are either defined with restrictions which prevent this eventuality, or -in the worst case- a more imaginative definition of M_i^* is applied. If $h = T(i+1)$ and M_i^* is given, then $M_{i+1}, T_{h,i+1}$ are, as usual, defined by:

$$\pi_{h,i+1} : M_i^* \longrightarrow^{(n)}_{E_{\nu_i}} M_{i+1},$$

where $n \leq \omega$ is maximal such that $\kappa_i < \rho^n_{M_i^*}$. In iterations of a single premouse, we were able to show that E_{ν_i} is always close to M_i^*, but there is no reason to expect this in arbitrary phalanx iterations.

We will not attempt to present a general theory of phalanxes, since in this section we use only phalanxes of length 2. We write $\langle N, M, \lambda\rangle$ as an abbreviation for the phalanx $\mathbb{M}$ of length 2 with $M_0 = N, M_1 = M$, and $\lambda_0 = \lambda$. We define:

Definition 4.2.1. The phalanx $\langle N, M, \lambda\rangle$ is *witnessed* (or verified) by σ iff the following hold:

(a) $\sigma : M \longrightarrow_{\Sigma_0^{(n)}} N$ for all $n < \omega$ such that $\lambda < \rho^n_M$

(b) $\lambda = \mathrm{crit}(\sigma)$

(c) σ is cardinal preserving, i.e. if τ is a cardinal in M then $\sigma(\tau)$ is cardinal in N.

Lemma 4.2.1. *Let $\langle N, M, \lambda\rangle$ be witnessed by σ. Then the following hold:*

(1) *λ is a regular cardinal in M.*

Proof. Suppose not. Let $\gamma = \mathrm{dom}(f) < \lambda, f \in M$ such that $\sup f"\gamma = \lambda$. Then $\sigma(f) = f$ since $\mathrm{dom}(\sigma(f)) = \sigma(\gamma) = \gamma$ and $\sigma(f)(\nu) = \sigma(f(\nu)) = f(\nu)$ for $\nu < \gamma$. But then $f : \gamma \longrightarrow \sigma(\lambda)$ cofinally, where $\sigma(\lambda) > \lambda$. Contradiction!

QED(1)

By acceptability it follows that:

(2) *If $\tau < \lambda$ is a (regular) cardinal in $J^{E^M}_\lambda$, then it is a (regular) cardinal in M (hence in N).*

Obviously:

(3) *If λ is a limit cardinal in M, then it is a limit cardinal in N.*

However, λ could be a successor cardinal in M, in which case there is $\tilde{\lambda}$ with $\lambda = \tilde{\lambda}^+$ in M and $\tilde{\lambda} < \lambda < \sigma(\lambda) = \tilde{\lambda}^+$ in N. Theoretically we could have $E^M_\lambda \neq \varnothing$, but we have eliminated that possibility by restricting to premice of type 1 or 2:

(4) $E^M_\lambda = \varnothing$.

Proof. Suppose not. Let $\tilde{\nu} = \sigma(\lambda)$. Let $\tilde{\kappa} = \text{crit}(E^M_\lambda) = \text{crit}(E^N_{\tilde{\nu}})$. Set $\tilde{\lambda} = E^N_{\tilde{\nu}}(\overline{\kappa})$. Then $\tilde{\lambda}^+ = \nu$ and $\tilde{\lambda}$ is regular in N by acceptability, since $\tilde{\nu}$ is a cardinal in N. Working in N, it follows easily that $E^N_{\tilde{\nu}}|\eta$ is a full extender for arbitrarily large $\eta < \tilde{\lambda}$. Hence $N||\tilde{\nu}$ is of type 3. Contradiction!

QED(4)

Hence:

(5) *Let $\kappa < \lambda$ be a cardinal in M. Set $\tau = \kappa^{+M}$. There is $\gamma \in N$ such that $\gamma > \tau$ and τ is a cardinal in $N||\gamma$.*

Proof. If $\tau < \lambda$, take $\lambda = \gamma$. Otherwise $\tau = \lambda$. Then τ is a cardinal in $M||\lambda+\omega$, since $E^M_\lambda = \varnothing$. But $\lambda+\omega < \sigma(\lambda) \in N$ and $J^{E^M}_{\lambda+\omega} = J^{E^N}_{\lambda+\omega}$, since $E^N_\lambda = E^M_\lambda = \varnothing$.

QED(5)

Note. It will follow from (5) that if $h = T(i+1)$ is a normal iteration of $\langle N, M, \lambda\rangle$, then M^*_i is defined.

Following our earlier sketch, we define:

Definition 4.2.2. Let $\langle N, M, \lambda\rangle$ be a phalanx which is witnessed by σ. By a *normal iteration* of $\langle N, M, \lambda\rangle$ of length $\eta \geq 2$ we mean:

$$I = \langle\langle M_i : i < \mu\rangle, \langle \nu_i : i+1 \in (\eta, \mu)\rangle, \langle \pi_{i,j} : i \leq_T j\rangle, T\rangle$$

such that:

(a) T is a tree on η with $iTj \longrightarrow i < j$. Moreover $T\text{"}\{0\} = T\text{"}\{1\} = \varnothing$.

(b) M_i is a premouse for $i < \eta$. Moreover $M_0 = N, M_1 = N$.

(c) If $1 \leq i, i+1 < \eta$, then $M_i||\nu_i = \langle J^E_{\nu_i}, E_{\nu_i}\rangle$ with $E_{\nu_i} \neq \varnothing$. We define $\kappa_i, \tau_i, \lambda_i$ as usual. We also set: $\lambda_0 = \lambda$. We require: $\nu_i > \nu_h$ if $1 \leq h < i$ and $\lambda_h > \lambda$. (Hence $\lambda_i > \lambda_h$ for $h < i$).

(d) Let $i > 0$. Let h be least such that $h = i$ or $h < i$ and $\kappa_i < \lambda_h$. Then $h = T(i+1)$ and $J^{E^{M_h}}_{\tau_i} = J^{E^{M_i}}_{\tau_i}$.

(e) $\pi_{i,j}$ is a partial map of M_i to M_j for $i \leq_T j$. Moreover $\pi_{i,i} = \mathrm{id}$, $\pi_{i,j}\pi_{h,i} = \pi_{h,j}$.

(f) Let $h = T(i+1)$. Set: $M_i^* = M_h||\gamma$, where $\gamma \leq \mathrm{On}_{M_h}$ is maximal such that $\tau_i < \gamma$ is a cardinal in $M_h||\gamma$. (We call it a *drop point* in I if $M_i^* \neq M_k$). Then:

$$\pi_{h,i+1} : M_i^* \longrightarrow^{(n)}_{E_{\nu_i}} M_{i'+1}, \text{ where } n \leq \omega \text{ is maximal s.t.}$$
$$\lambda_h \leq \rho^n_{M_i^*} (\text{where } \lambda_0 = \lambda)$$

(g) If $i \leq_T j$ and $(i,j]_T$ has no drop point, then π_{ij} is a total function on M_i.

(h) Let $\mu < \eta$ be a limit ordinal. Then $T"\mu$ is a club in μ and contains at most finitely many drop points. Moreover, if $i < \mu$ and $(i,\mu)_T$ is drop free, then:

$$M_\mu,\ \langle \pi_{j,\mu} : i \leq_T j <_T \mu \rangle$$

is the transitivized direct limit of

$$\langle M_j : i \leq_T j \leq_T \mu \rangle, \langle \pi_{j,k} : i \leq_T j \leq_T k <_T \mu \rangle.$$

As usual we call M_μ, $\langle \pi_{j,\mu} : j <_T \mu \rangle$ the limit of $\langle M_i : i <_T \mu \rangle$, $\langle \pi_{j,k} : i \leq_T j \leq_T k <_T \mu \rangle$, since the missing points are given by:

$$\pi_{h,j} = \pi_{i,j}\pi_{h,i} \text{ for } h <_T i \leq_T j <_T \mu \rangle$$

This completes the definition. Note that the existence of M_i^* is guaranteed by Lemma 4.2.1(5). We define:

Definition 4.2.3. $i+1$ is an anomaly in I if $i > 0$ and $\tau_i = \lambda$ (hence $0 = T(i+1)$).

Anomalies will cause us some problems. Just as in the case of ordinary normal iterations, we can extend an iteration of length $\eta+1$ to a *potential iteration* of length $\eta+2$ by appointing ν_η such that:

$$E^{M_\eta}_{\nu_\eta} \neq \varnothing, : \nu_\eta > \nu_i \text{ for } i \leq i < \eta, \lambda_\eta > \lambda.$$

This determines M_η^*. In ordinary iterations we know that E_{ν_η} is close to M_η^*. In the present situation this may fail, however, if $\eta+1$ is an anomaly. We, nonetheless, get the following analogue of Theorem 3.4.4:

Theorem 4.2.2. *Let I be a potential normal iteration of $\langle N, M, \lambda \rangle$ of length $i+1$. If $i+1$ is not an anomaly, then $E^{M_i}_{\nu_i}$ is close to M_i^*. If $i+1$ is an anomaly, then $E^{M_i}_{\nu_i,\alpha} \in N$ for $\alpha < \lambda_0$.*

We essentially repeat our earlier proof (but with one additional step). We show that if $A \subset \tau_i$ is $\underline{\Sigma}_1(M_i||\nu_i)$, then it is $\underline{\Sigma}_1(M_i^*)$ if $i+1$ is not an anomaly, and otherwise $A \in N$. Let I be a counterexample of length $i+1$ where i is chosen minimally. Let $h = T(i+1)$. Let $A \subset \tau_i$ be a counterexample. Then:

(1) $h < i$.

We then prove:

(2) $\nu_i = \mathrm{On}_{M_i}, \rho^1_{M_i} \leq \tau_i$.

The first equation is proven exactly as before. The second follows as before if $i+1$ is not an anomaly, since then $\tau_i < \lambda_h$. Now let $i+1$ be an anomaly. Assume $\rho^1_{M_i} > \tau_i$ and let $A \subset \tau_i$ be $\underline{\Sigma}(M_i)$. Then $A \in M_1$, since either $i = 1$ or $A \in J^{E^{M_i}}_{\lambda_1} = J^{E^{M_1}}_{\lambda_1}$ where λ_1 is a cardinal in M_i. Hence $A = \sigma(A) \cap \lambda \in N$. Contradiction!

QED(2)

In an extra step we then prove:

Claim. $i > 1$.

Proof. Suppose not. Then $i = 1$ and $h = 0$. Let:

$$\pi : J^E_{\tau_1} \longrightarrow J^E_{\nu_1},\ \pi' : J^{E'}_{\tau'_1} \longrightarrow J^{E'}_{\nu'_1}$$

be the extensions of M, N respectively. Then π, π' are cofinal and $\sigma\pi = \pi'\sigma$. If $\tau_1 < \lambda$ then $\sigma \restriction \tau_1 + 1 = \mathrm{id}$ and σ takes M cofinally to N. Hence σ in Σ_1-preserving. If A is $\Sigma_1(M)$ in p, then A is also $\Sigma_1(N)$ in $\sigma(p)$, where $N = M_1^*$. Contradiction!

Now let $\tau_1 = \lambda$. Then $i+1$ is an anomaly. Then σ takes ν_1, non cofinally to ν'_1, since $\pi'(\lambda) > \pi(\xi) = \sigma\pi(\xi)$ for $\xi < \lambda$. Let $\tilde{\nu} =: \sup \sigma"\nu_1$. Then:

$$\sigma : M \longrightarrow_{\Sigma_1} \tilde{M} \text{ cofinally,}$$

where $\tilde{M} = \langle J^{E'}_{\tilde{\nu}}, E'_{\nu'_1} \cap J^{E'}_{\tilde{\nu}} \rangle$. Let A' be $\Sigma_1(\tilde{M})$ in $\sigma(p)$ by the same definition as A in p. Then $A' \in N$ and $A = A' \cap \lambda \in N$. Contradiction!

QED(Claim)

(3) i is not a limit ordinal.

Proof. Suppose not. Then as before, we can pick $l <_T i$ such that $\pi_{l,i}$ is a total function on M_l and $l > h$. Hence $\pi_{l,i}$ is Σ_1-preserving. Let $M_i = \langle J^E_{\nu_i}, F \rangle$. We can also pick l big enough that $p \in \mathrm{rng}(\pi_{l,i})$, where A is $\Sigma_1(M_i)$ in p. Hence $A \in \underline{\Sigma}_1(M_l)$, where $M_l = \langle J^{\tilde{E}}_{\tilde{\nu}}, \tilde{F} \rangle$, where $\tilde{\nu} = \mathrm{On}_{M_l} \geq \nu_l$. Extend $I|l+1$ to a potential iteration I' of length $l+2$ by setting: $\nu'_l = \tilde{\nu}$. Since $l > h$, it follows easily that:

$$\kappa'_l = \kappa_i, \tau'_l = \tau_i, h = T'(l+1), M_i^* = M_l'^*.$$

By the minimality of i it follows that $A \in \Sigma_1(M_l^*)$ if $i+1$ is not an anomaly and otherwise $A \in N$. Contradiction!

QED(3)

We then let: $i = j+1, \xi = \tau(i)$. By the claim we have: $j \leq 1$.

But:

$$\pi_{\xi,i} : M_j^* \longrightarrow^{(n)}_{E^{M_i}_{\nu_j}} M_i = \langle J^E_{\nu_i}, E_{\nu_i} \rangle.$$

If $n = 0$, this map is cofinal. Hence in any case $\pi_{\xi,i}$ is Σ_1-preserving. Hence:

(4) $M_j^* = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{E}_{\overline{\nu}} \rangle$ where $\overline{E}_{\overline{\nu}} \neq \varnothing$.

Hence:

(5) $\tau_i < \kappa_j$.

Proof. $\kappa_i < \lambda_h \leq \lambda_j$ where λ_j is inaccessible in M_i (since $j \geq 1$). Hence $\tau_i < \lambda_j$. Moreover, $\kappa_i, \tau_i \in \mathrm{rng}(\pi_{\xi,i})$ by (4). But:

$$\mathrm{rng}(\pi_{\xi,i}) \cap [\lambda_j, \lambda_j) = \varnothing.$$

QED(5)

Exactly as before we get:

(6) $\pi_{\xi,i} : M_j^* \longrightarrow_{E_{\nu_j}} M_i$ is a Σ_0 ultrapower. But then:

(7) i is not an anomaly.

Proof. Let $A \subset \tau_i$ be $\Sigma_1(M_i)$ in the parameter p. By (6) we have: $p = \pi_{\xi,i}(f)(\alpha)$, where $f \in M_j^*, \alpha < \lambda_j$.

Then:

$$A(\zeta) \longleftrightarrow \bigvee u \in M_j^* \bigvee y \in \pi_{\zeta,i}(u) A'(y, \zeta, p)$$

But then:

$$A(\zeta) \longleftrightarrow \bigvee u \in M_j^* \{\gamma < \kappa_j : \overline{A}'(y, \zeta, f(\gamma))\} \in (E_{\nu_j})_\alpha.$$

But since $j < i$ and $j+1$ is an anomaly, we have by the minimality of i that $(E_{\nu_j})_\alpha \in N$. Hence $A \in N$. Contradiction!

QED(7)

Since $j+1$ is not an anomaly, we have $(E_{\nu_j})_\alpha \in \underline{\Sigma}_1(M_j^*)$. Hence $A \in \underline{\Sigma}_1(M_j^*)$. Hence we have shown:

(8) $\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i) \subset \underline{\Sigma}_1(M_j^*)$.

We know that $M_j^* = M_\xi || \overline{\nu} = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{E}_{\overline{\nu}} \rangle$. Moreover, $\overline{\nu} > \nu_l$ for $l < \xi$, since $\lambda_l \leq \kappa_j < \lambda_{\overline{\xi}} < \overline{\nu}$; hence $\nu_l < \lambda_\xi < \overline{\nu}$. Thus we can extend $I|\xi+1$

to a potential iteration I' of length $\xi+2$ by setting: $\nu'_\xi = \overline{\nu}$. Since $\tau_i < \kappa_j$, we then have: $\kappa_i = \kappa'_\xi, \tau_i = \tau'_\xi$. Hence:

$$h = T(i+1) = T'(\xi+1) \text{ and } M_i^* = (M_\xi^*)'.$$

Suppose that $i+1$ is not an anomaly in I. Then neither is $\xi+1$ in I'. By the minimality of i we conclude:

$$\mathbb{P}(\tau_i) \cap \underset{\sim}{\Sigma}_1(M_\xi||\overline{\nu}) \subset \underset{\sim}{\Sigma}_1(M_i^*)$$

where $M_\xi||\overline{\nu} = M_j^*$. Hence by (8):

$$\mathbb{P}(\tau_i) \cap \underset{\sim}{\Sigma}_1(M_i) \subset \underset{\sim}{\Sigma}_1(M_i^*).$$

Contradiction!

Now let $i+1$ be an anomaly. Then so is $\xi+1$ in I'. But then just as before:

$$\mathbb{P}(\tau_i) \cap \underset{\sim}{\Sigma}_1(M_i) \subset \mathbb{P}(\tau_i) \cap \underset{\sim}{\Sigma}_1(M_\xi||\overline{\nu}) \subset N.$$

Contradiction! QED(Theorem 4.2.2)

We now prove:

Lemma 4.2.3. *Let $h = T(i+1)$ in I, where I is a normal iteration of $\langle N, M, \lambda\rangle$. Then:*

$$\pi_{h,i+1} : M_i^* \longrightarrow_{\Sigma^*} M_{i+1} \text{ strongly.}$$

Proof. If $i+1$ is not an anomaly, then $E_{\nu_i}^{M_i}$ is close to M_i^* and the result is immediate. Now let $i+1$ be an anomaly. Then $h = 0, M_i^* = N||\eta$ for an $\eta < \tau'_i = \sigma(\lambda)$, since $\tau_i = \lambda$. $\rho^\omega_{M_i^*} \leq \kappa_i$, since τ_i is not a cardinal in $N|\eta+\omega = J^{E^N}_{\eta+\omega}$. But then $\rho^\omega_{M_i^*} = \kappa_i$, since κ_i is a cardinal in N. Let $\rho^n_{M_i^*} > \kappa_i \geq \rho^{n+1}_{M_i^*}$, where $n < \omega$. Let $\pi = \pi_{h,i+1}$. Since M_{i+1} is the $\Sigma_0^{(n)}$ ultrapower of M_i^*, we know:

$$\pi\text{''}\rho^n_{M_i^*} \subset \rho^n_{M_{i+1}^*} \text{ and } \pi(\rho^j_{M_i^*}) = \rho^j_{M_{i+1}} \text{ for } j < n.$$

Since E_{ν_i} is weakly amenable, Lemma 3.2.16 gives us:

(1) $\sup \pi\text{''}\rho^n_{M_i^*} = \rho^n_{M_{i+1}}$ and π is $\Sigma_1^{(n)}$-preserving.

We now prove:

(2) Let $H =: |J^{E^{M_i}}_{\nu_i}| = |J^{E^{M_{i+1}}}_{\nu_i}|$. Then $\mathbb{P}(H) \cap \Sigma^{(n)}_1(M_{i+1}) \subset N$.

Proof. Let B be $\Sigma^{(n)}_1(M_{i+1})$ in q such that $B \subset H$. Let $q = \pi(f)(\alpha)$ where $f \in \Gamma^*(\kappa_i, M^*_i), \alpha < \lambda_i$. Let:

$$B(x) \longleftrightarrow \bigvee y \in H^n_{M_{i+1}} B'(y, x, q)$$

where B' in $\Sigma^{(n)}_0(M_{i+1})$. Let $\overline{B}'$ be $\Sigma^{(n)}_0(M^*_i)$ by the same definition. Then:

$$B(x) \longleftrightarrow \bigvee u \in H^n_{M^*_i} \bigvee y \in \pi(u) B'(y, x, \pi(f)(\alpha))$$
$$\longleftrightarrow \bigvee u \in H^n_{M^*_i} \{\gamma < \kappa_i : \bigvee y \in u\, \overline{B}'(y, x, f(\gamma))\} \in (E^{M_i}_{\nu_i})_\alpha$$

But $(E^{M_i}_{\nu_i})_\alpha \in N$. Hence $B \in N$.

QED(2)

Clearly, if $A \subset H$ is $\underline{\Sigma}^*(M_{i+1})$, then it is $\underline{\Sigma}_\omega(\langle H, B\rangle)$ where B is $\underline{\Sigma}^{(n)}_1(M_{i+1})$. Hence $A \in N$ and $\langle H, A\rangle$ is amenable, since $H = J^{E^{M^*_i}}_{\kappa_i} = J^{E^N}_{\kappa_i}$, and κ_i is regular in N. But then $\rho^\omega_{M_{i+1}} = \rho^\omega_{M^*_i} = \kappa_i$. It follows that:

(3) π is Σ^*-preserving.

Proof. By induction on j we show that if $R(\vec{x}^j, \vec{z})$ is $\Sigma^{(i)}_1(M^*_i)$ and $R'(\vec{x}^j, \vec{z})$ are $\Sigma^j_1(M_{i+1})$ by the same definition (where $\vec{z} = z^{h_1}_1, \ldots, z^{h_m}_m$ with $h_1, \ldots h_m < j$), then:

$$R(\vec{x}, \vec{z}) \longleftrightarrow R'(\pi(\vec{x}), \pi(\vec{z})).$$

For $j \leq n$ this holds by (1). Now let it hold for $j = m \geq n$. We show that it holds for $j = m + 1$. Then:

$$R(\vec{x}, \vec{z}) \longleftrightarrow H_{\vec{z}} \models \varphi[\vec{x}]$$

where φ is Σ_1 and:

$$H_{\vec{z}} = \langle H, \overline{Q}^1_{\vec{z}}, \ldots, \overline{Q}^P_{\vec{z}}\rangle$$

where $Q^l(\vec{w}, \vec{z})$ is $\Sigma^{(m)}_1(M^*_i)$ and:

$$\overline{Q}^l = \{\langle\vec{w}\rangle \in H : Q^l(\vec{w}, \vec{z})\} \text{ for } l = 1, \ldots, p.$$

Now let Q' be $\Sigma^{(m)}_1(M_{i+1})$ by the same definition and let $H'_{\vec{x}}$ be defined like $H_{\vec{x}}$ with $Q^{l'}$ in place of Q^l $(l = 1, \ldots, p)$. By the induction hypothesis we then have:

$$R(\vec{x}, \vec{z}) \longleftrightarrow H_{\vec{z}} \models \varphi(\vec{x})$$
$$\longleftrightarrow H_{\pi(\vec{z})} \models \varphi(\vec{x})$$
$$\longleftrightarrow R'(\vec{x}, \pi(\vec{z})) \longleftrightarrow R'(\pi(\vec{x}), \pi(\vec{z}))$$

since $\pi(\vec{x}) = \vec{x}$.

QED(3)

But this embedding π is also strong, since if $\rho^{m+1} = \kappa$ and A confirms $a \in P^m$ in M_i^*, then if A' is $\Sigma_{i+1}^{(m)}$ in $\pi(a)$ by the same definition, we have: $A \cap H = A' \cap H$, where $M_i^* \cap \mathbb{P}(H) = M_{i+1} \cap \mathbb{P}(H)$. Hence $A' \cap H \notin M_{i+1}$.

QED(Lemma 4.2.3)

But then:

Lemma 4.2.4. *Let $h = T(i+1)$, where $i+1 \leq_T j$ and $(i+1, j]$ has no drop point. Then:*

$$\pi_{h,j} : M_i^* \longrightarrow_{\Sigma^*} M_j \text{ strongly.}$$

Proof. By Lemma 3.2.27 and Lemma 3.2.28.

QED(Lemma 4.2.4)

Exactly as in Corollary 4.1.12, we conclude that if M_i^* is solid and $i = j+1$, then so is M_j and $\pi(p_i^m) = p_j^m$ for $m < \omega$.

We intend to do comparison iterations in which $\langle N, M, \lambda\rangle$ is coiterated with a premouse. For this we shall again need padded iteration. Our definition of a normal iteration of $\langle N, M, \lambda\rangle$ encompassed only strict iteration, but we can easily change that:

Definition 4.2.4. Let $\langle N, M, \lambda\rangle$ be a phalanx which is witnessed by σ. By a *padded normal iteration* of $\langle N, M, \lambda\rangle$ *of length* $\mu \geq 1$ we mean:

$$I = \langle\langle M_i : i < \mu\rangle, \langle \nu_i : i \in A\rangle, \langle \pi_{i,j} : i \leq_T j\rangle, T\rangle.$$

Where:

(1) $A = \{i :<\leq i+1 < \mu\}$ is the set of *active points.*

(2) (a)-(b) of the previous definition hold. However (f), (d) require that $i \in A$. Moreover:

 (i) Let $1 \leq h < j < \mu$ such that $[h, j) \cap A = \varnothing$. Then:
 - $h <_T j, M_h = M_j, \pi_{h,j} = \mathrm{id}$.
 - $i \leq h \longrightarrow (i \leq_T h \longleftrightarrow i <_T j)$ for $i < \mu$.
 - $j \leq i \longrightarrow (j \leq_T i \longleftrightarrow h <_T i)$ for $i < \mu$.
 (In particular, if $2 \leq i+1 < \mu, i \notin A$. Then $i = T(i+1), M_i = M_{i+1}, \pi_{i,i+1} = \mathrm{id}$).

Note. 0 plays a special role, behaving like an active point in that λ_0 exists, but ν_0 does not exist.

Our previous results go through *mutatis mutandis*. We shall say more about that later.

Definition 4.2.5. Let M^0 be a premouse and $M^1 = \langle M, N, \lambda \rangle$ a phalanx iteration witnessed by σ. By a *coiteration* of M^0, M^1 of length $\mu \geq 1$ with *coindices* $\langle \nu_i : 1 \leq i < \mu \rangle$ we mean a pair $\langle I^0, I^1 \rangle$ such that:

(a) $I^h = \langle \langle M_i^h \rangle, \langle \nu_i^h : i \in A^h \rangle, \langle \pi_{i,j}^h \rangle, T^h \rangle$ is a padded normal iteration of M^h $(h = 0, 1)$.

(b) $M_0^0 = M_1^0$.

(c) $\nu_i =$ the least ν such that $E_\nu^{M_i^0} \neq E_\nu^{M_i^1}$.

(d) If $E_{\nu_i}^{M_i^n} \neq \varnothing$, then $i \in A^h$ and $\nu_i^h = \nu_j$. Otherwise $i \notin A_i^h$.

Note. We always have $M_0^0 = M_1^0$ whereas: $M_0^1 = N, M_1^1 = M$.

Definition 4.2.6. Let $M^0, M^1 \in H_\kappa$, where $\kappa > \omega$ is regular. Let S^h be a successful iteration strategy for M^h $(h = 0, 1)$. The $\langle S^0, S^1 \rangle$*-coiteration of* length $\mu \leq \kappa + 1$ with coindices $\langle \nu_i : 1 \leq i < \mu \rangle$ is the coiteration $\langle I^0, I^1 \rangle$ such that:

- I^h is S^h-conforming.
- Either $\mu = \kappa + 1$ or $\mu = i + 1 < \kappa$ and ν_i does not exist (i.e. $M_1^0 \lhd M_i^1$ or $M_0^1 \lhd M_i^0$).

  Note that $\lhd$ was defined by:

$$P \lhd Q \longleftrightarrow P = Q || \mathrm{On}_P$$

We leave it to the reader to show that the coiteration exists. This is spelled out in §3.5 for coiteration of premice. We obtain the following analogue of Lemma 3.5.1:

Lemma 4.2.5. *The coiteration of $M : M^1$ terminates below κ_1.*

The proof is virtually unchanged. We leave the details to the reader. Using Lemma 4.2.4, we get the following analogue of Lemma 4.1.14:

Lemma 4.2.6. *Let N, M^0 be presolid. (Hence M^1 is presolid). Let $\langle I^0, I^1\rangle$ be the coiteration of M^0, M^1 terminating at $j < \kappa$. Suppose there is a drop on the main branch of I^h. Then there is no drop on the main branch of I^{i-h}. Moreover, $M_i^{i-h} \lhd M_i^h$.*

The proof is virtually the same.

At the end of §4.1 we sketched an approach to proving that fully iterable mice are solid. The basic idea was to coiterate $\langle N, M, \lambda\rangle$ with N, where N is fully iterable and σ witnesses $\langle N, M, \lambda\rangle$. In order to do this, we must know that $\langle N, M, \lambda\rangle$ is normally iterable. (The notions "iteration strategy", "successful iteration strategy" and "iterability" are defined in the obvious way for phalanxes $\langle N, M, \lambda\rangle$. We leave this to the reader.) We prove:

Lemma 4.2.7. *If $\langle N, M, \lambda\rangle$ is witnessed by σ and N is normally iterable, then $\langle N, M, \lambda\rangle$ is normally iterable.*

For the sake of simplicity we shall first prove this under a *special assumption*, which eliminates the possibility of anomalies:

(SA) λ is a limit cardinal in M.

Later we shall prove it without SA.

In §3.4.5 we showed that if $\sigma : M \longrightarrow_{\Sigma^*} N$ and N is normally iterable, then M is normally iterable. Given a successful iteration strategy for N, we defined a successful strategy for M, based on the principle of *copying* the iteration of M onto N. In this case, we "copy" an iteration of $\langle N, M, \lambda\rangle$ onto an iteration of N. It suffices to prove it for strict iterations. Let

$$I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$$

be a strict normal iteration of $\langle N, M, \sigma\rangle$. Its copy will be an iteration of N:

$$I' = \langle\langle N_i\rangle, \langle\nu'_i\rangle, \langle\pi'_{ij}\rangle, T'\rangle$$

of the same length. We will have $N_0 = N_1 = N$. (Thus I' is a padded iteration, even if I is not). There will be *copying maps* $\sigma_i (i < \mathrm{lh}(I))$ with:

$$\sigma_i : M_i \longrightarrow N_i, \sigma_0 = \mathrm{id} \restriction N, \sigma_1 = \sigma.$$

We shall have $\nu'_i \cong \sigma_i(\nu_i)$ for $1 \leq i$. The tree T was "double rooted" with 0, 1 as its two initial points, T', on the other hand, has the sole initial point 0. We can define T' from T by:

$$iT'j \longleftrightarrow (iTj \ \vee \ i < 2 \leq j)$$

In I each point $i < \mu$ has a unique origin $h \in \{0,1\}$ such that $h \leq_T i$. Denote this by: $\text{or}(i)$. Using the function or we can define T from T' by:

$$iTj \longleftrightarrow (iTj \wedge \text{or}(i) = \text{or}(j))$$

Thus, each infinite branch b' in I' uniquely determines an infinite branch b in I defined by:

$$b = \bigcup_{i \in b' \setminus 2} \{\text{or}(i), i\}$$

However, we cannot expect the copying map to always be Σ^*-preserving, since $\sigma_1 = \sigma$ is assumed to be $\Sigma_0^{(n)}$-preserving only for $\rho_M^n > \lambda$. In this connection it is useful to define:

$$\text{depth}(M, \lambda) =: \text{ the maximal } n \leq \omega \text{ s.t. } \rho_M^n > \lambda.$$

Modifying our definition of "copy" in §3.4.5 appropiately we now define:

Definition 4.2.7. Let $\langle N, M, \lambda \rangle$ be witnessed by σ. Let

$$I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$$

be a normal iteration of $\langle N, M, \lambda \rangle$ of length μ. Let:

$$I' = \langle \langle N_i \rangle, \langle \nu_i' \rangle, \langle \pi_{ij}' \rangle, T' \rangle$$

be a normal iteration of N of the same length. I' is a *copy of I onto N* with *copying maps $\sigma_i (i < \mu)$* iff the following hold:

(a) $\sigma_i : M_i \longrightarrow_{\Sigma_*} N_i, \sigma_0 = \text{id} \restriction N, \sigma_1 = \sigma, N_0 = N_1 = N$.

(b) $iT'j \longleftrightarrow (iTj \vee i < 2 \leq j)$

(c) $\sigma_i \restriction \lambda_h = \sigma_h \restriction \lambda_h$ for $h \leq i < \mu$

(d) $\sigma_i \pi_{hi} = \pi_{hi}' \sigma_h$ for $i \leq_T h$.

(e) $\nu_i' \cong \sigma_i(\nu_i)$

(f) Let $1 \leq_T i$. If $(1, i]_T$ has no drop point in I, then σ_i is $\Sigma_0^{(n)}$-preserving for all n such that $\lambda \leq \rho_M^n$. If $(1, i]_T$ has a drop point in I. Then σ_i is Σ^*-preserving.

(g) If $0 \leq_T i$ then σ_i is Σ^*-preserving.

Note: $N_0 = N_1$, since $0 \notin A$.

Our notion of copy is very close to that defined in §3.4.5. The main difference is that σ_i need not always be Σ^*-preserving. Nonetheless we can imitate the theory developed in §3.4.5. Lemma 3.4.14 holds literally as before. In interpreting the statement, however, we must keep in mind that if $i \in A$ and $T(i+1) = 0$, then $T'(i+1) = 1$. In this case $\tau_i < \lambda$ is a cardinal in N. Hence $M_i^* = N$. Moreover $\tau_i' = \sigma(\tau_i) = \tau_i$. Hence τ_i' is a cardinal in $N^* = N$ and $N_i^* = N$. In all other cases $T'(i+1) = T(i+1)$. Clearly $\pi'_{0j} = \pi'_{ij}$ for all $j \geq 1$. Lemma 3.4.14 then becomes:

Lemma 4.2.8. *Let $I, I', \langle \sigma_i : i < \mu \rangle$ be as in the above definition. Let $h = T(i+1)$. Then:*

(i) *If $i+1$ is a drop point in I, then it is a drop point in I' and $N_i^* = \sigma_h(M_i^*)$.*

(ii) *If $i+1$ is not a drop point in I, then it is not a drop point in I' and $N_i^* = N_h$.*

(iii) *If $F = E_{\nu_i}^{M_i}, F' = E_{\nu_i'}^{N_i}$. Then:*

$$\langle \sigma_h \restriction M_i^*, \sigma_i \restriction \lambda_i \rangle : \langle M_i^*, F \rangle \longrightarrow \langle N_i^*, F' \rangle$$

(iv) *$\sigma_{i+1}(\pi_{h,i+1}(f)(\alpha)) = \pi'_{h,i+1}\sigma_h(f)(\sigma_i(\alpha))$ for $f \in \Gamma^*(\kappa_i, M_i^*), \alpha < \lambda_i$.*

(v) *$\sigma_j(\nu_i) \cong \nu_i'$ for $j > i$.*

(vi) *σ_i is cardinal preserving.*

Note. In the general case, where anomalies can occur, Lemma 3.4.14 will not translate as easily.

Proof. In §3.4.5 we proved this under the assumption that each σ_i is Σ^*-preserving. We must now show that the weaker degree of preservation which we have posited suffices. The proof of (i)-(ii) are virtually unchanged. We now show that Σ_0-preservation is sufficient to prove (iii). Set: $\overline{M} = M_i||\nu_i, \overline{N} = N_i||\nu_i'$. Then $\sigma_i \restriction \overline{M}$ is a Σ_0 preserving map to $\overline{N}$. Let $\alpha < \lambda, X \in \mathbb{P}(\kappa_i) \cap \overline{M}$. The statement $\alpha \in F(X)$ is uniformly $\Sigma_1(\overline{M})$ in α, X. But it is also $\Pi_1(\overline{M})$ since:

$$\alpha \in F(X) \longleftrightarrow \alpha \notin F(\kappa_i \setminus X)$$

Hence:

$$\alpha \in F(X) \longleftrightarrow \sigma(\alpha) \in F'(\sigma(X))$$

by Σ_0-preservation. Finally we note that $\sigma_i \restriction (M_i \restriction \lambda_i)$ embeds $M_i||\lambda_i$ elementarily into $\sigma_i(M_i||\lambda_i) = N_i||\lambda_i'$. Hence:

$$\sigma_i(\prec \vec{\alpha} \succ) = \prec \sigma_i(\vec{\alpha}) \succ \text{ for } \alpha_1, \ldots, \alpha_n < \lambda_i.$$

Thus all goes through as before, which proves (iii).

In our previous proof of (iv) we need that $\sigma_h \upharpoonright M_i^*$ is Σ^*-preserving. This can fail if $1 \leq_T h$ and $[1, h]_T$ has no drop point. But then σ_h is $\Sigma_0^{(n)}$-preserving for $\lambda < \rho^M$ in M, where $\lambda \leq \kappa_i$. Hence the preservation is sufficient. Finally, (v) is proven exactly as before.

(vi) is clear if σ_i is Σ_1-preserving. If not, then $1 \leq i$ and $(1, i]$ has no drop. Hence $\pi_{1,i}$ is cofinal, since only Σ_0-ultraproducts were involved. If α is a cardinal in M_i, then $\alpha \leq \beta$ for a β which is a cardinal in M. By acceptability it suffices to note that $\sigma_i \pi_{1i}(\beta) = \pi'_{1i}\sigma(\beta)$ is a cardinal in N_i.

QED(Lemma 4.2.8)

Exactly as before we get the analogue of Lemma 3.4.15:

Lemma 4.2.9. *There is at most one copy I' of I induced by σ. Moreover, the copy maps are unique.*

As before we define:

Definition 4.2.8. Let $\langle N, M, \lambda\rangle$ be a phalanx witnessed by σ. $\langle I, I', \langle\sigma\rangle\rangle$ is a *duplication induced by σ* iff I is a normal iteration of $\langle N, M, \lambda\rangle$ and I' is the copy of I induced by σ with copy maps $\langle\sigma_i : i < \mu\rangle$.

We also define:

Definition 4.2.9. $\langle I, I', \langle\sigma_i : i \leq \mu\rangle\rangle$ is a *potential duplication of length $\mu + 2$ induced by σ* iff:

- $\langle I|\mu + 1, I'|\mu + 1, \langle\sigma_i : i \leq \mu\rangle\rangle$ is a duplication of length $\mu + 1$ induced by σ.
- I is a potential iteration of length $\mu + 2$.
- I' is a potential iteration of length $\mu + 2$.
- $\sigma_\mu(\nu_\mu) = \nu'_\mu$.

To say that an actual duplication of length $\mu + 2$ is the *realization* of a potential duplication means the obvious thing. If it exists, we call the potential duplication *realizable.*

Our analogue of Theorem 3.4.16 is somewhat more complex. We define:

Definition 4.2.10. i is an exceptional point ($i \in$ EX) iff:

$$1 \leq_T i, (1, i]_T \text{ has no drop point, and } \rho^1 \leq \lambda \text{ in } M.$$

Note. Suppose $\rho^1 \leq \lambda$ in M. For $j \in$ EX we have: $\rho^1_{M_j} \leq \lambda$, as can be seen by induction on j.

Our analogue of Theorem 3.4.16 reads:

Lemma 4.2.10. *Let $\langle I, I', \langle \sigma_i \rangle \rangle$ be a potential duplication of length $i+2$, where $h = T(i+1)$. Suppose that $i+1 \notin EX$. Then:*

$$\langle \sigma_h \restriction M^*_i, \sigma_i \restriction \lambda_i \rangle : \langle M^*_i, F \rangle \longrightarrow^* \langle N^*_i, F' \rangle$$

where $F = E^{M_i}_{\nu_i}, F' = E^{N_i}_{\nu'_i}$.

Before proving this we note some of its consequences. Just as in §3.4.5 it provides exact criteria for determining whether the copying process can be carried one step further. We have the following analogue of Lemma 3.4.17:

Lemma 4.2.11. *Let $\langle I, I', \langle \sigma_i : i \leq \mu \rangle \rangle$ be a potential duplication of length $\mu + 2$ (where $\mu \geq 1$). It is realizable iff N^*_μ is $*$-extendible by $E^{N_\mu}_{\nu'_\mu}$.*

Proof. If N^ν_μ is not $*$-extendable, then no realization can exist, so suppose that it is. Form the realization $\hat{I}'$ of I' by setting:

$$\pi'_{h,i+1} : N^*_\mu \longrightarrow^*_{F'} N_{\mu+1},$$

where $h = T(\mu + 1), F' = E^{N_\mu}_{\nu'_\mu}$. We consider three cases:

Case 1. $\sigma_h \restriction M^*_\mu$ is Σ^*-preserving.

Bu Lemma 4.3.2 we have:

$$\langle \sigma_h \restriction M^*_\mu, \sigma_\mu \restriction \lambda_\mu \rangle \ \langle M^*_\mu, F \rangle \longrightarrow^* \langle N^*_\mu, F' \rangle,$$

where $\sigma_h \restriction M^*_h$ is Σ^*-preserving. By Lemma 3.2.23 this gives us:

$$\pi_{h,\mu+1} : M^*_\mu \longrightarrow^*_F M_{\mu+1},$$

and a unique:

$$\sigma_{\mu+1} : M_{\mu+1} \longrightarrow_{\Sigma^*} N_{\mu+1}$$

such that $\sigma_{mu+1}\pi_{h,\mu+1} = \pi'_{h,\mu+1}\sigma_h, \sigma_{\mu+1} \restriction \lambda_\mu = \sigma_\mu \restriction \lambda_\mu$.

The remaining verification are straightforward.

Case 2. Case 1 fails and $\eta + 1 \notin$ EX.

By Lemma 4.3.2 we again have:

$$\langle \sigma_h, \sigma_\mu \restriction \lambda_\mu \rangle : \langle M_h, F \rangle \longrightarrow^* \langle N_h, F' \rangle.$$

Moreover σ_h is $\Sigma_0^{(m)}$-preserving, where $m \leq \omega$ is maximal such that $\lambda < \rho^m$ in M. Now let $n \leq \omega$ be maximal such that $\kappa_i < \rho^n$ in M_h. Then $n \leq m$, since $\lambda \leq \kappa_i$. By Lemma 3.2.19 M_h is n-extendible by F. But then it is $*$-extendible, since F is close to M_h. Set:

$$\pi_{h,\mu+1} : M_h \longrightarrow_F^* M_{\mu+1}.$$

Since σ is $\Sigma_0^{(m)}$-preserving, it follows by Lemma 3.2.19 that there is a unique:

$$\sigma_{\mu+1} : M_{\mu+1} \longrightarrow_{\Sigma_0^{(n)}} N_{mu+1},$$

such that $\sigma'_{\mu+1}\pi_{h,\mu+1} = \pi'_{h,\mu+1}\sigma_h$ and $\sigma'\lambda_\mu = \sigma_n \restriction \lambda_\kappa$. But σ' is, in fact, $\Sigma_0^{(m)}$-preserving. If $n = m$, this is trivial. If $n < m$, it follows by Lemma 3.2.24. We let $\sigma_{\mu+1} = \sigma'$. The remaining verification are straightforward.

QED(Case 2)

Case 3. The above cases fail.

Then $\mu + 1 \in$ EX and $\rho^1 \leq \lambda$ in M. Thus $\rho^1 \leq \lambda \leq \kappa_i$ in M_h. By Lemma 4.2.8 we have:

$$\langle \sigma_h, \sigma_\mu \restriction \lambda_\mu \rangle : \langle M_h, F \rangle \longrightarrow \langle N_h, F' \rangle.$$

Hence by Lemma 3.2.19, there are π, σ' with:

$$\pi : M_h \longrightarrow_F M_{\mu+1}, \sigma' : M_{\mu+1} \longrightarrow_{\Sigma_0} N_{\mu+1}$$

such that $\sigma'\pi = \pi'_{h,\mu+1}\sigma_h$ and $\sigma' \restriction \lambda_\mu = \sigma_\mu \restriction \lambda_\mu$. But $M_{\mu+1}$ is the $*$-ultrapower of M_h, since $\rho^1_{M_h} \leq \kappa_i$ and F is close to M_h. We set: $\pi_{h,\mu+1} = \pi, \sigma_{\mu+1} = \sigma'$. The remaining verifications are straightforward.

QED(Lemma 4.3.3)

Our analogue of Lemma 3.4.18 reads:

Lemma 4.2.12. *Let $\langle I, I', \langle \sigma_i : i < \mu \rangle \rangle$ be a duplication of limit length μ. Let b' be a well founded cofinal branch in I'. Let $b = \bigcup_{i \in b' \setminus 2} \{\text{or}(i), i\}$ be the induced cofinal branch in I. Our duplication extends to one of length $\mu + 1$ with:*

$$T\text{"}\{\mu\} = b, T\text{"}\{\mu\} = b'$$

and $\sigma_\mu \pi_{i,\mu} = \pi'_{i\mu} \sigma_i$ for $i \in b$.

The proof is left to the reader.

With these two lemmas we can prove Lemma 4.2.7:

Fix a successful normal iteration strategy for N. We construct a strategy S^* for $\langle N, M, \lambda \rangle$ as follows: Let I be a normal iteration of $\langle N, M, \lambda \rangle$ of limit length μ. If I has no S-conforming copy, then $S^*(I)$ is undefined. Otherwise, let I' be an S-conforming copy. Let $S(I') = b'$ be the cofinal well founded branch given by S. Set $S^*(I) = b$, where b is the induced branch in I. Clearly if I is S^*-conforming, then the S-conforming copy I' exists. If I is of length $\mu + 1 (\mu \geq 1)$, then by Lemma 4.3.3, if $\nu \in M_\mu, \nu > \nu_i$ for $i < \mu$, then I extends to an S^*-conforming iteration of length $\mu + 2$ with $\nu_\mu = \nu$. By Lemma 4.3.4, if I is of limit length μ, then $S^*(I)$ exists. Hence S^* is successful.

QED(Lemma 4.2.7)

We still must prove Lemma 4.3.2. This, in fact turns out to be a repetition of Lemma 3.4.16 in §3.4. As before we derive it from:

Lemma 4.2.13. *Let $\langle I, I', \langle \sigma_j \rangle \rangle$ be a potential duplication of length $i + 1$ where $h = T(i + 1)$. Suppose that $i + 1 \notin EX$. Let $A \subset \tau_i$ be $\Sigma_1(M_i || \nu_i)$ in a parameter p. Let $A' \subset \tau'_i$ be $\Sigma_1(N_i || \nu'_i)$ in $\sigma_i(p)$ by the same definition. Then A is $\Sigma_1(M^*_i)$ in a parameter q and A' is $\Sigma_1(N^*_i)$ in $\sigma_h(q)$ by the same definition.*

Proof. The proof is a virtual repetition of the proof of Lemma 3.4.20 in §3.4. As before we take $\langle I, I', \langle \sigma_j \rangle \rangle$ as being a counterexample of length $i + 1$, where i is chosen minimally for such counterexamples. The proof is exactly the same as before. The only difference is that σ_j may not be Σ^*-preserving if $j \in \text{EX}$. But in the case where we need it, we will have that σ_j is $\Sigma_0^{(1)}$-preserving, which suffices.

QED(Lemma 4.3.5).

Hence Lemma 4.2.7 is proven.

However, we have only proven this on the special assumption that λ is a limit cardinal in M. We now consider the case: $\lambda = \kappa^+$ in M. This will require a radical change in the proof. Set:

$$N^* =: N||\gamma \text{ where } \gamma \text{ is maximal such that } \lambda \text{ is a cardinal in } N||\gamma.$$

Then $\lambda = \kappa^{+N^*} < \sigma(\lambda) = \kappa^{+N}$. An anomaly occurs at $i+1$ whenever $\tau_i = \lambda$. Then $0 = T(i+1)$ and $\kappa = \kappa_i$. Clearly $N^* = M_j^*$. Thus M_{i+1} is the ultraproduct of N^* by $F = E_{\nu_i}^{M_i}$ and N_{i+1} is the ultraproduct of N_i^* by $F' = E_{\nu_i}^{N_i}$. In order to define σ_{i+1}, we require:

$$\sigma(M_i^*) = N_i^*.$$

This is false however, since $\sigma_i \upharpoonright \lambda_0 = \sigma \upharpoonright \lambda_i$ where $\tau_i < \lambda_i$. Hence:

$$\tau_i' = \sigma_i(\tau_i) = \sigma(\tau_i) = \tau^{+N}.$$

Hence $N_i^* = N \ni \sigma(N^*)$.

The answer to this conundrum is to construct two sequences I' and $\hat{I}$. The sequence:

$$\hat{I} = \langle\langle\hat{N}_i\rangle, \langle\hat{\nu}_i : i \in A\rangle, \langle\pi_{ij} : \hat{i \leq_T} j\rangle, \hat{T}\rangle$$

will be a padded iteration of N of length μ in which many points may be inactive. The second sequence:

$$I' = \langle\langle N_i\rangle, \langle{\nu'}_i : i \in A\rangle, \langle\pi_{ij}' : i \leq_T j\rangle, T'\rangle$$

will have most of the properties it had before, but, in the presence of anomalies, it will not be an iteration . If no anomalies occurs, we will have: $I' = \hat{I}$. If $i+1$ is an anomaly, then $\pi_{0,i+1}$ will not be an ultrapower and N_i will be a proper segment of $\hat{N}_i = \hat{N}_{i+1}$. (Hence i is passive in $\hat{I}$). To see how this works, let $i+1$ be the first anomaly to occur in I, then $I'|_{i+1} = \hat{I}|_{i+1}$, but at $i+1$ we shall diverge. Under our old definition we would have taken $N_i^* = N$ and $\pi_{i,i+1}' = \pi''$, where:

$$\pi'' : N \longrightarrow_F^* N'', \ F = E_{\nu_i'}^{N_i}.$$

We instead take:

$$N_i^* = N^*, \ N_{i+1} = \pi''(N^*), \ \pi_{i,i+1} = \pi'' \upharpoonright N^*.$$

Note that $\pi''(N^*) = \pi'(N^*)$, where π' is the extension of $\langle J_{\nu_i}^{E^{M_i}}, F\rangle$. But then N_{i+1} is a proper segment of $J_{\nu_i}^{E^{N_i}}$ hence of $N_i = \hat{N}_i$.

We can then define:

$$\sigma_{i+1} : M_{i+1} \longrightarrow N_{i+1}$$

by:

$$\sigma_{i+1}(\pi_{0,i+1}(f)(\alpha)) =: \pi'(f)(\sigma_i(\alpha))$$

for $f \in \Gamma^*(\kappa, N^*), \alpha < \lambda_i$. σ_{i+1} will then be $\Sigma_0^{(n)}$−preserving, where $n \leq \omega$ s maximal such that $\kappa < \rho^n$ in N^*. To see that this is so, let φ be a $\Sigma_0^{(n)}$ formula. Let $f_1, \ldots, f_n \in \Gamma^*(\kappa, N^*)$ and let $\alpha_1, \cdots, \alpha_n < \lambda_i$. Let:

$$x_j = \pi_{0,i+1}(f_j)(\alpha_j), y_j = \pi'(f_j)(\sigma_i(\alpha_j)) \ (j = 1, \ldots, n).$$

Let $X =: \{\prec \xi_1, \ldots, \xi_m \succ : N^* \models \varphi[f_1(\xi_1), \ldots, f_n(\xi_n)]\}$. Then $\sigma_i F(X) = F'(X)$, since $\sigma_i \upharpoonright H_\lambda^M = \sigma_0 \upharpoonright H_\lambda^M = \text{id}$. Hence:

$$\begin{aligned} M_{i+1} \models \varphi[\vec{X}] &\longleftrightarrow \prec \vec{\alpha} \succ \in F(X) \\ &\longleftrightarrow \prec \sigma_i(\vec{\alpha}) \succ \in F'(X) = \pi'(X) \\ &\longleftrightarrow \sigma(N^*) \models \varphi[\vec{y}]. \end{aligned}$$

Since we had no need to form an ultraproduct at $i+1$, we set: $\hat{N}_{i+1} = \hat{N}_i$. i is then an inactive point in $\hat{I}$ and N_{i+1} is a proper segment of $\hat{N}_{i+1}$.

We continue in this fashion: The active points in $\hat{I}$ are just the points $i > 0$ such that $i + 1 < \mu$ is not an anomaly. If i is active, we set $\hat{\nu}_i = \nu_i'$. (This does not, however, mean that $\hat{N}_i = N_i'$.) If i is any non anomalous point, we will have: $N_i = \hat{N}_i$. If $h < i$ is also non anomalous, thus $\pi'_{hi} = \hat{\pi}_{hi}$. If i is an anomaly, we will have: N_i is a proper segment of $\hat{N}_i$. If μ is a limit ordinal it then turns out that any cofinal well founded branch b' in I', which, in turn, gives us such a branch b in I. This enables us to prove iterability.

We now redo our definition of “copy” as follows:

Definition 4.2.11. Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a strict normal iteration of $\langle N, M, \lambda\rangle$, where $\langle N, M, \lambda\rangle$ is a phalanx witnessed by σ.

$$I' = \langle\langle M_i\rangle, \langle\nu_i'\rangle, \langle\pi_{ij}'\rangle, T'\rangle$$

is a copy of I with *copy maps* $\langle\sigma_i : i < \mu\rangle$ induced by σ if and only if the following hold:

(I) (a) T' is a tree such that $iT'j \longrightarrow i < j$.

(b) Let μ be the length of I. Then N_i is a premouse and

$$\sigma_i : M_i \longrightarrow_{\Sigma_0} N_i \text{ for } i < \mu$$

(c) $\pi'_{ij}(i \leq_T j)$ is a commutative system of partial maps from N_i to N_j.

(II) (a)-(f) of our previous definition hold. Moreover:

(g) Let $0 \leq_T j$. If $(0, i]_T$ have no anomaly, then σ_i is Σ^*-preserving.

(h) Let $h = T(i+1)$. Set:

$$N_i^* = \begin{cases} \sigma_h(M_i^*) & \text{if } M_i^* \in M_h \\ N_h & \text{if not} \end{cases}$$

Then $\pi'_{h,i+1} : N_i^* \longrightarrow_{\Sigma^*} N_{i+1}$.

(i) Let h, i be as above. If $i+1$ is not an anomaly, then:

$$\pi'_{h,i+1} : N_i^* \longrightarrow^*_{F'} N_{i+1}$$

where $F' = E^{N_i}_{\nu'_i}$.

(j) Let $i+1$ be an anomaly. (Hence $\tau_i = \lambda = \kappa^{+M}$, where $\kappa = \kappa_i$ is a cardinal in M, hence in N.)

We then have:

$$M_i^* = N^* =: N||\gamma,$$

where γ is maximal such that λ is a cardinal in $N||\gamma$. Let π be the extension of $N_i||\nu_i = \langle J^E_{\nu'}, F' \rangle$. Then:

$$N_{i+1} = \pi(N^*) \text{ and } \pi'_{0,i+1} = \pi \restriction N^*.$$

Moreover, $\sigma_{i+1} : M_{i+1} \longrightarrow N_{i+1}$ is defined by:

$$\sigma_{i+1}(\pi_{0,i+1}(f)(\alpha)) = \pi'(f)(\sigma_i(\alpha))$$

where $f \in \Gamma^*(\kappa, N^*), \alpha < \lambda_i$. (Hence σ_{i+1} is $\Sigma_0^{(n)}$-preserving for $\kappa < \rho^n_{N^*}$.)

(k) Let $h \leq_T i$, where h is an anomaly. If $(h, i]_T$ has no drop point, then σ_i is $\Sigma_0^{(n)}$-preserving for $\kappa < \rho^n$ in N^*. If $(h, i]_T$ has a drop point, then σ_i is Σ^*-preserving.

(III) There is a *background iteration*:

$$\hat{I} = \langle \langle \hat{N}_i \rangle, \langle \hat{\nu}_i \rangle, \langle \hat{\pi}_{ij} \rangle, \hat{T} \rangle$$

with the properties.

(a) $\hat{I}$ is a padded normal iteration of length μ.

(b) $i < \mu$ is active in $\hat{I}$ iff $0 < i+1 < \mu$ and $i + \mu$ is not an anomaly in I. In this case: $\hat{\nu}_i = \nu'_i$.

(c) If i is not an anomaly in I, then $\hat{N}_i = N'_i$. Moreover, if $h < i$ is also not an anomaly, then:

$$h <_{\hat{T}} i \longleftrightarrow h <_{T'} i,\ \hat{\pi}_{h,i} = \pi'_{h,i} \text{ if } h <_{T'} i.$$

This completes the definition. In the special case that λ is a limit cardinal in M, we of course have: $I' = \hat{I}$ and the new definition coincides with the old one. We note some simple consequence of our definition:

Lemma 4.2.14. *The following hold:*

(1) *If $i < j < \mu$, then $\sigma_j(\lambda_i) = \lambda_i$. (Hence $\lambda'_i < \lambda'_j$ for $j + 1 < \mu$.)*

Proof. By induction on j. For $j = 0$ it is vacuously true. Now let it hold for j.

$$\sigma_{j+1}(\lambda_j) = \sigma_{j+1}\sigma_{h,i+1}(\kappa_j) = \pi'_{h,j+1}\sigma_h(\kappa_j) = \pi'_{h,j+1}(\kappa'_j) = \lambda_j.$$

(Here $\sigma_h(\kappa_j) = \sigma_j(\kappa_j) = \lambda'_j$, since $\kappa_j < \lambda_h$ and $\sigma_j||\lambda_h = \sigma_h \restriction \lambda_h$.)

For $i < j$ we then have:

$$\sigma_{j+1}(\lambda_i) = \sigma_j(\lambda'_i)(\text{since } \lambda_i < \lambda_j).$$

QED(1)

(2) *σ_i is a cardinal preserving for $i < \mu$.*

Proof. If σ_i is Σ_1-preserving, this is trivial, so suppose not. Then one of two cases hold:

Case 1. $1 \leq_T i, (1, i]_T$ has no drop, and $\rho^1 \leq \lambda$ in M.

Then $\pi_{hj} : M_h \longrightarrow_{\Sigma^*} M_j$ is cofinal for all $h \leq_T j \leq_T i_\eta$ since each of the ultrapower involved is a Σ_0-ultrapower. Hence, if α is a cardinal in M_i, then $\alpha \leq \pi_{1,i}(\beta)$ where β is a cardinal in M_1. By acceptability it suffices to show that $\sigma_i\pi_{1,i}(\beta)$ is a cardinal in N_i. But $\sigma_i\pi_{1,i}(\beta) = \pi'_{1t}\sigma(\beta)$, where σ and π'_{1i} are cardinal preserving.

Case 2. $h \leq_T i$ where h is an anomaly, $(h, i]_T$ has no drop and $\rho^1 \leq k = k_i$ in N^*.

The proof is a virtual repeat of the proof in Case 1, with $(0, i]_T$ in place of $(1, i]_T$.

QED(2)

(3) I' behaves like an iteration at limits. More precisely:

Let $\eta < \kappa$ be a limit ordinal. Let $i_0 <_T \eta$ such that $b = (i_0, \eta)_T$ is free of drops. Then

$$N_\eta, \langle \pi_{i\eta} : i \in b \rangle$$

is the direct limit of:

$$\langle N_i : i \in b \rangle, \langle \pi_{ij} : i \leq j \text{ in } b \rangle.$$

Proof. No $i \in b \cup \{\eta\}$ is an anomaly since every anomaly is a drop point. Hence:

$$N'_i = \hat{N}_i, \pi'_{i,j} = \hat{\pi}_{i,j} \text{ for } i \leq j \text{ in } b \cup \{\eta\}.$$

Since I is an iteration, the conclusion is immediate.

QED(3)

(4) Let $i < \mu$. If $i + 1$ is an anomaly, then:

(a) N_{i+1} is a proper segment of $N_i || \nu'_i$. (Hence $\nu'_{i+1} < \nu'_i$).

(b) $\rho^\omega = \lambda'_i$ in N_{i+1}.

Proof. (a) is immediate by II (i) in the definition of "copy". But $N_{i+1} = \pi(N^*)$ where π is the extension of $N_i || \nu'_i$. By definition, $N^* = N || \gamma$, where $\gamma < \sigma(\lambda) = \kappa^{+N}$ is the maximal γ such that $\tau_i = \lambda$ is a cardinal in $N || \gamma$. Hence $\rho^\omega = \kappa$ in N^*. But then $\rho^\omega = \lambda'_i$ in N_{i+1}.

QED(4)

(5) Let $i < \mu$. There is a finite n such that $i + n + 1$ is not an anomaly. (This includes the case: $i + n + 1 = \mu$.)

Proof. If not then $\nu_{i+n+1} < \nu_{i+n}$ for $n < \mu$ by(4). Contradiction!

(6) Let $i < \mu$. There is a maximal $j \leq i$ such that j is not an anomaly.

Proof. Suppose not. Then $i \neq 0$ is an anomaly and for each $j < i$ there is $j' \in (j, i)$ which is an anomaly. But then i is a limit ordinal, hence not an anomaly.

By(5) and (6) we can define:

Definition 4.2.12. Let $i < \mu$. We define:

- $l(i) =$ the maximal $j \leq i$ such that j is not an anomaly.
- $r(i)$ the least $j \geq i$ such that $j + 1$ is not an anomaly.

Definition 4.2.13. An interval $[l, r]$ in μ is called *passive* iff i is an anomaly for $l < i \leq r$. A passive interval is called *full* if it is not properly contained in another passive interval.

It is then trivial that:

(7) $[l(i), r(i)]$ = the unique full I such that $i \in I$.

(8) Let $[l, r]$ be a full passive interval. Then, for all $i \in [l, r]$:

(a) $N_l = N_i$.

(b) If $j \leq l$ and $j \leq_{\hat{T}} i$, then $j \leq_{\hat{T}} l$.

(c) If $j \geq r$ and $i \leq_{\hat{T}} j$, then $r \leq_{\hat{T}} j$.

Proof. This follows by induction on j, using the general fact about padded iterations that if j is not active, then:

- $\hat{N}_j = \hat{N}_{j+1}$
- $h \leq_{\hat{T}} j \longleftrightarrow h <_{\hat{T}} j+1$
- $j <_{\hat{T}} h \longleftrightarrow j+1 \leq_{\hat{T}} h$. QED(8)

(9) Let b be a branch of limit length in $\hat{I}$. There are cofinally many $i \in b$ such that i is not an anomaly.

Proof. Let $j \in b$. Pick $i \in b$ such that $i > r(j)$. Then $l(i) > r(j)$, since $r(j)+1 \leq i$ is not an anomaly. Hence $l(i) \in b$ and $l(i) > j$ is not an anomaly.

QED(9)

We define N_i^* for $i < \mu$ exactly as if I' were an iteration: Let $h = T'(i+1)$. Then:

$$N_i^* =: N_i||\gamma \text{ where } \gamma \text{ is maximal such that } \tau_i' \text{ is a cardinal in } N_i||\gamma.$$

We then get the following version of Lemma 4.2.8.

Lemma 4.2.15. *Let I' be a copy of I induced by σ. Let $h = T(i+1)$. If $i+1$ is not an anomaly. Then the conclusion (i)-(vi) of Lemma 4.2.8 hold. If $i+1$ is an anomaly, then (v), (vi) continue to hold.*

Proof. If $i+1$ is not an anomaly, the proof are exactly as before. Now let $i+1$ be an anomaly. (iv) is immediate by II (j) in the definition of "copy". But then (vi) follows as before.

QED(Lemma 4.2.15)

Lemma 3.3.19 is strengthened to:

Lemma 4.2.16. *I has at most one copy I'. Moreover the background iteration $\hat{I}$ is unique.*

Proof. The first part is proven exactly as before (we imagine I'' to be a second copy and show by induction on i that $I'|i = I''|i$). The second part is proven similarly, assuming $\hat{I}'$ to be a second background iteration.

QED(Lemma 4.2.16)

The concept *duplication induced by* σ is defined exactly as before. Now let:

$$D = \langle I, I', \langle \sigma_i : i \leq \eta \rangle \rangle$$

be a duplication of length $\eta + 1$. We turn this into a *potential duplication* D of length $\eta + 2$ by appointing a ν_ξ such that $\nu_\xi > \nu_i$ for $0 < i < \eta$.

By a *realization* of $\tilde{D}$ of length $\eta + 2$ by appointing a ν_η such that $\nu_\eta < \nu_i$ for $0 < i < \eta$. By a *realization* of $\tilde{D}$, we mean a duplication $\mathring{D} = \langle \mathring{I}, \mathring{J}, \langle \dot{\sigma}_i : i \leq \eta + 1 \rangle \rangle$ of length $\eta + 2$ such that $\mathring{D}|\eta + 1 = D$ and $\dot{\nu}_\eta = \nu_\eta$. It follows easily that $\tilde{D}$ has at most one realization.

Our analogue, Lemma 4.3.2, of Lemma 3.4.16 will continue to hold as stated if we enhance the definition of *exceptional point* as follows:

Definition 4.2.14. i is an *exceptional point* ($i \in EX$) iff either:

$$1 \leq_T i, (1, i]_T \text{ has no drop, and } \rho^1 \leq \lambda \text{ in } M$$

or there is an anomaly $h \leq_T i$ such that:

$$(0, i]_T \text{ has no drop, and } \rho^1 \leq \kappa \text{ in } N^*.$$

With this change Lemma 4.3.2 goes through exactly as before. As before, we derive this form Lemma 4.3.5. The proof is as before. As before the condition $i + 1 \notin EX$ guarantees that the map σ_i will always have sufficient preservation when we need it.

When we worked under the special assumption Lemma 4.3.3 was our analogue of Lemma 3.4.17. In the presence of anomalies the situation is somewhat more complex. We first note:

Lemma 4.2.17. *Let $\tilde{D} = \langle I, I', \langle \sigma_i : i \leq \eta \rangle \rangle$ be a potential duplication of length $\eta + 2$. If $\eta + 1$ is an anomaly, then $\tilde{D}$ is realizable.*

Proof. Form $N_{\eta+1}, \pi_{0,\eta+1} : N^* \longrightarrow N_{\eta+1}$ and $\sigma_{\eta+1}$ as in II(j). Set: $\tilde{N}_{\eta+1} = N_\eta$. The verification of I, II, III is straightforward.

QED(Lemma 4.2.17)

Now suppose that $\eta + 1$ is not an anomaly. Let $h = T(\eta + 1)$. Then η is an active point is any realization of $\hat{I}$, so we set: $\hat{\nu}_\eta = \nu'_\eta$. In order to realize $\tilde{D}$, we must apply $F = E^{M_\eta}_{\nu_\eta}$ to M^*_η, getting:

$$\pi_{h,\eta} : M^*_\eta \longrightarrow^*_F M_{\eta+1}.$$

Similarly we apply $F' = E^{N_\eta}_{\nu'_\eta}$ to N^*_η getting:

$$\pi'_{h,\eta} : N^*_\eta \longrightarrow^*_{F'} N_{\eta+1}.$$

We then set:

$$\sigma_{\eta+1}(\pi_{h\eta}(f)(\alpha)) = \pi'_{h\eta}\sigma_h(f)(\sigma_\eta(\alpha))$$

for $f \in \Gamma^*(\kappa_{\dot{\eta}}, M^*_{\dot{\eta}}), \alpha < \lambda_\eta$.

We must also extend $\hat{I}$. Since $\hat{\nu}_\eta = \nu_\eta$ and N_η is an initial segment of $\hat{N}_\eta$, we have:

$$F' = E^{\hat{N}_\eta}_{\hat{\nu}_\eta}.$$

Now let: $k = \hat{T}(\eta + 1)$. (k can be different from h!) III constrains us to set:

$$\hat{\pi}_{k,\eta+1} : \hat{N}^*_\eta \longrightarrow^*_F \hat{N}_{\eta+1}.$$

However, III also mandates that $\hat{N}_{\eta+1} = N_{\eta+1}$. Happily, we can prove:

Lemma 4.2.18. *Let $\tilde{D} = \langle I, I', \langle \sigma_i : i \leq \eta \rangle \rangle$ be as above, where $\eta + 1$ is not an anomaly. Then:*

(a) $N^*_\eta = \hat{N}^*_\eta$.

(b) *$\tilde{D}$ is realizable iff N^*_η is $*$-extendible by F'.*

Proof. We first prove (a). Let $h = T'(\eta + 1)$. Set:

$$l = l(h), r = r(h).$$

Then $h \in [l, r]$ where l is not an anomaly, $j + 1$ is an anomaly for $l \leq j < r$, and $r + 1$ is not an anomaly. h is least such that $\kappa'_\eta < \lambda'$ or $h = \eta$. $k = T'(\eta + 1)$ is least such that $k + 1$ is not an anomaly and $\kappa'_\eta < \lambda'_k$. Since j is not an anomaly for $l < j \leq r$, we conclude that $k = r$. Then $N_l = \hat{N}_j$ for $l \leq j \leq r$.

Case 1. $h = l$.

Then $\hat{N}_h = N_h$ and:

$$N^*_\eta = \hat{N}_\eta = N_h||\gamma$$

where γ is maximal such that τ'_η is a cardinal in $N_h||\gamma$. QED(Case 1)

Case 2. $l < h$.

Then $h = j+1$ where $l \leq j$. N_h is a proper segment of $\hat{N}_h$. We again have: $N^*_\eta = N_h||\gamma$ where $\gamma \leq \mathrm{On}_{N_h}$ is maximal such that τ'_η is a cardinal in $N_h||\gamma$. We have $r = \hat{T}(\eta+1)$ and $\hat{N}^*_\eta = \hat{N}_r||\hat{\gamma}$, where $\hat{\gamma} \leq \mathrm{On}_{\hat{N}_r}$ is maximal such that τ'_η is a cardinal in $\hat{N}_r||\hat{\gamma}$. But $\rho^\omega_{N_h} = \kappa_j$, where $h = j+1$ by Lemma 4.2.14 (4). Since $\lambda'_j \leq \kappa'_\eta < \tau'_\eta < \lambda'_h$ and N_h is a proper segment of $\hat{N}_h = \hat{N}_r$, we conclude that $\hat{\gamma} \leq \mathrm{On}_{N_h}$. Hence $\gamma = \hat{\gamma}$ and $N^*_\eta = \hat{N}^*_\eta$. QED(a)

We now prove (b). If $\hat{N}^*_\eta$ is not extendable by F', then no realization can exists, so assume otherwise. This gives us $N_{\eta+1}$ and $\pi'_{h,\eta+1}$, where $\hat{N}_{\eta+1} = N_{\eta+1}$ and $\hat{\pi}_{k,\eta+1} = \pi'_{h,\eta+1}$, where $k = T'(\eta+1)$. $\sigma_{\eta+1}$ is again defined by:

$$\sigma_{\eta+1}(\pi_{h,\eta+1}(f)(\alpha)) = \pi'_{h,\eta+1}\sigma_h(f)(\sigma_\eta(\alpha))$$

for $f \in \Gamma^*(\kappa_\eta, M^*_\eta), \alpha < \lambda_\eta$. The verification of I, II, III is much as before. However Case 2 splits into two subcases:

Case 2.1. $1 \leq_T \eta+1$.

This is exactly as before.

Case 2.2. $0 \leq_T \eta+1$.

Then there is $j \leq_T h$ such that j is an anomaly and $(0, \eta+1]_T$ has no drop. Moreover, $\rho^1 > \kappa$ in N^*. Then σ_h is a $\Sigma^{(m)}_0$-preserving where $m \leq \omega$ is maximal such that $\kappa < \rho^m$ in N^*. The rest of the proof is as before.

Case 3 also splits into two subcases:

Case 3.1. $1 \leq_T \eta+1$.

We argue as before.

Case 3.2. $0 \leq_T \eta+1$.

Then $j \leq_t h$, where j is an anomaly and $\rho^1 \leq \kappa$ in N^*. Hence $\rho^1 \leq \kappa_h$ in M_h and we argue as before. QED(Lemma 4.2.18)

Using Lemma 4.2.14 (9) we get:

Lemma 4.2.19. *Let* $D = \langle I, I', \langle \sigma_i \rangle \rangle$ *be a duplication of limit length* μ. *Let* $\hat{b}$ *be a cofinal well founded branch in* $\hat{I}$. *Let* X *be the set of* $i \in \hat{b}$ *which are not an anomaly. Let:*

$$b' = \{j : \bigvee i \in X j <_T i\}, b = \{j : \bigvee i \in X j <_T i\}.$$

Then D *has a unique extension to a* $\tilde{D}$ *of length* $\mu + 1$ *such that:*

$$\hat{T}\text{"}\{\mu\} = \hat{b}, T'\text{"}\{\mu\} = b', T\text{"}\{\mu\} = b.$$

The proof is left to the reader.

Now let S be a successful normal iteration strategy for N. We define an iteration strategy S^* for $\langle N, M, \lambda \rangle$ as follows:

Let I be an iteration of $\langle N, M, \lambda \rangle$ of limit length μ. We ask whether there is a duplication $\langle I, I', \langle \sigma_0 \rangle \rangle$ induced by σ^*. If not, then $S^*(I)$ is undefined. Otherwise, we ask whether $S(\hat{I})$ is defined. If not, then $S^*(I)$ is undefined. If not, then $S^*(I)$ is undefined. If $\hat{b} = S(\hat{I})$, define b', b as above and set: $S^*(I) = b$. It is easily seen that if I is any S^*-conforming normal iteration of $\langle N, M, \lambda \rangle$, then the duplication $\langle I, I', \langle \sigma_i \rangle \rangle$ exists. Moreover $\hat{I}$ is S-conforming. In particular, if I is of limit length, then $S(I)$ is defined. Moreover, if I is of length $\eta + 1$, and $\nu > \nu_i$ for $i < \eta$, then by Lemma 4.2.18, we can extend I to an $\tilde{I}$ of length $\eta + 2$ by setting: $\nu_\eta = \nu$. Hence S is a successful iteration strategy.

This proves Lemma 4.2.7 at last!

We note however, that our strategy S^* is defined only for strict iteration of $\langle N, M, \lambda \rangle$. We can remedy this in the usual way. Let:

$$I = \langle \langle M_i \rangle, \langle \nu_i : i \in A \rangle, \langle \pi_{ij} \rangle, T \rangle$$

be a padded iteration of $\langle N, M, \lambda \rangle$, of length μ. Let h be the monotone enumeration of:

$$\{i : i = 0 \vee i \in A \vee i + 1 = \mu\}.$$

The *strict pullback* of I is then:

$$\dot{I} = \langle \langle \dot{M}_i \rangle, \langle \dot{\nu}_i \rangle, \langle \dot{\pi}_{ij} \rangle, \hat{T} \rangle$$

where:

$$\dot{M}_i = M_{h(i)}, \dot{\nu}_i = \nu_{h(i)}, \dot{\pi}_{ij} = \pi_{h(i),h(i)}$$

and:

$$i\hat{T}j \longleftrightarrow h(i)Th(j).$$

$\dot{I}$ is a strict iteration and contains all essential information about I. We extend S^* to a strategy on padded iteration as follows: Let I be a padded iteration of limit length μ. If A is cofinal in μ, we form $\dot{I}$, which is then also of limit length. We set:

$$S^*(I) = b, \text{ where } S^*(\dot{I}) = \dot{b},$$

and $b = \{i : \bigvee j(i \leq_T h(j))\}$. If A is not cofinal in μ, there is $j < \mu$ such that $A \cap [j, \mu) = \varnothing$. We set:

$$S^*(I) = \{i < \mu : iTj \vee j \leq i\}.$$

It follows that I is S^*-conforming iff $\dot{I}$ is S^*-conforming.

Since $\dot{I}$ is strict, we have $I', \hat{I}, \langle \sigma_i : i < \dot{\mu} \rangle$, (where $\dot{\mu}$ is the length of $\dot{I}$). We shall make use of this machinery in analyzing what happens when we coiterate N against $\langle N, M, \sigma \rangle$. This will yield the "simplicity lemma" stated below.

Note. We could, of course, have defined $I', \hat{I}$ and $\langle \sigma_i : i < \mu \rangle$ for arbitrary padded I, but this will not be necessary.

Building upon what we have done thus far, we prove the following "simplicity lemma", which will play a central role in our further deliberations:

Lemma 4.2.20. *Let N be a countable premouse which is presolid and fully ω_1+1 iterable. Let $\langle N, M, \sigma \rangle$ be witnessed by σ. Set $Q^0 = N, Q^1 = \langle N, M, \sigma \rangle$. There exist successful $\omega_1 + 1$ normal iteration strategies S^0, S^1 for Q^0, Q^1 respectively such that $\langle I^0, I^1 \rangle$ is the coiteration of Q^0, Q^1 by S^0, S^1 respectively with coiteration indices ν_i, then the coiteration terminates at $\mu < \omega_1$ with:*

$$I_0 = \langle \langle Q_i \rangle, \langle \nu_i \rangle, \langle \pi^0_{ij} \rangle, T^0 \rangle$$

$$I_1 = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi^1_{ij} \rangle, T^1 \rangle$$

such that:

(a) $M_\mu \triangleleft Q_\mu$.

(b) *$a \leq_{T^1} \mu$ in I^1.*

(c) *There is no drop point $i + 1 \leq_{T^1} \mu$ in I^1.*

In the next section we shall use this to derive the solidity lemma, which says that all mice are solid. We shall also us eit to derive a number of other structural facts about mice.

We now prove the simplicity lemma.

Let N be countable, presolid and fully $\omega_1 + 1$.iterable. Let $\langle N, M, \lambda\rangle$ be a phalanx witnessed by σ. (Recall that this entails $\lambda \in M$ and $\lambda = \text{crit}(\sigma)$. Moreover, σ is $\Sigma_0^{(n)}$-preserving whenever $\lambda < \rho_M^n$). Fix an enumeration $e = \langle e(n) : n < \omega\rangle$ of $\text{On}\cap N$. Suppose that $\sigma : N \longrightarrow_{\Sigma^*} N'$. We can define a sequence $e'_i \in N'(i < \omega)$ as follows. By induction on $i < \omega$ we define:

$$e'_i = \text{the least } \eta \in N' \text{ s.t. there is some } \sigma' : N \longrightarrow_{\Sigma^*} N' \text{ with } \sigma'(e_h) = e'_h \text{ for } h < i \text{ and } \eta = \sigma'(e_i).$$

It is not hard to see that there is exactly one $\sigma' : N \longrightarrow_{\Sigma^*} N$ such that $\sigma'(e_i) = e'_i$ for $i < \omega$. We then call σ' the *e-minimal* embedding of N into N'. The *Neeman-Steel Lemma* (Theorem 3.5.8) says that N has an *e-minimal normal iteration strategy* S with the following properties:

- S is a successul $\omega_1 + 1$ normal iteration strategy for N.
- Let N' be an iterate of N by an S-conforming iteration I. Let $\sigma : N \longrightarrow_{\Sigma^*} M \triangleleft N'$. Then I has no drop on its main branch $M = N'$ and the iteration map $\pi : N \longrightarrow N'$ is the e-minimal embedding.

Hence, in particular, if M is a proper segment of N' or the main branch of I has a drop, then there is no Σ^*-preserving embedding from N to M.

From now on let e be a fixed enumeration of On_N and let S be an e-minimal strategy for N. Let S^* be the induced strategy for $\langle N, M, \lambda\rangle$. Coiterate $Q_0 = N$ against $M_0 = \langle N, M, \lambda\rangle$ using the strategies S, S^* respectively. Let $\langle I^0, I^1\rangle$ be the coiteration with:

$$I^1 = \langle\langle M_i\rangle, \langle\nu_i^0\rangle, \langle\pi_{ij}^0\rangle, T^0\rangle$$

$$I^0 = \langle\langle Q_i\rangle, \langle\nu_i^1\rangle, \langle\pi_{ij}^1\rangle, T^1\rangle$$

and coiteration indices $\langle\nu_i : 1 \le i \le \mu\rangle$ where $\mu + 1 < \omega_1$ is the length of the coiteration.

We note some facts:

(A) If N' is any S-iterate of N (i.e. the result of an S-conforming iteration), then there is no Σ^*-preserving map of N into a proper segment of N'.

(B) Call N' a *truncating* S-iterate of N iff it results from an S-conforming iteration with a truncation on its main branch. If N' is a truncating S-iterate, then there is no Σ^*-preserving embedding of N into N'.

(C) If N' is a *non truncating* S-iterate of N, then the iteration map $\pi : N \longrightarrow N'$ is the unique e-minimal map.

Now form the *strict pullback* $\dot{I}$ of I^1 as before. Let I be of length $\mu + 1$. $\dot{I}$ will then be of length $\dot{\mu} + 1$. Let $I', \hat{I}, \langle \sigma_i : i \leq \dot{\mu} \rangle$ be defined as before. Set: $N' =: N'_{\dot{\mu}}$, $\hat{N} =: \hat{N}_{\dot{\mu}}$, $\sigma' = \sigma'_{\dot{\mu}}$. The following facts are easily established:

(D) $\hat{N}$ is an S-iterate of N. Moreover: $\sigma' : M_\mu \longrightarrow_{\Sigma_0} N'$ where $N' \lhd \hat{N}$.

(E) If there is a drop point $i + 1 \leq_{T^1} \mu$ which is not an anomaly in I^1, then there is $i + 1 \leq_{T^0} \dot{\mu}$ which is not an anomaly in $\dot{I}$. Hence $\hat{N}$ is a truncating iterate of N and $\sigma' : M_\mu \longrightarrow_{\Sigma^*} \hat{N}$.

(F) If there is no anomaly $i + 1 \leq_{T^1} \mu$ in I, then there is no anomaly $i + 1 \leq_{\dot{T}} \dot{\mu}$ in $\dot{I}$.

(G) Suppose $0 \leq_{T^1} \mu$ and no $i + 1 \leq \mu$ is an anomaly. Hence the same situation holds in $\dot{I}$. Then $\hat{N}$ is an S-iterate of N by the iteration map $\sigma' \pi'_{0,\mu}$ (since $\dot{\sigma}_{\dot{\mu}} \dot{\pi}_{0,\dot{\mu}} = \hat{\pi}_{0,\dot{\mu}}$).

We now prove the simplicity lemma. We do this by eliminating all other possibilities.

Claim 1. Q_μ is not a proper segment of M_μ.

Proof. Suppose not. Then Q_μ is a non-truncating iterate of N with iteration map $\pi^0_{0,\mu}$. Hence $\sigma' \pi^0_{0,\mu} : N \longrightarrow_{\Sigma^*} \sigma_\mu(Q_\mu)$, where $\sigma_\mu(Q_\mu)$ is a proper segment of $\hat{N}$ and $\hat{N}$ is an S-iterate of N. Contradiction!

QED(Claim 1)

Claim 2. There is no truncation point $i + 1 \leq_{T^1} \mu$ such that $i + 1$ is not an anomaly in I^1.

Proof. Suppose not. Then $\sigma' : M_\mu \longrightarrow_{\Sigma^*} \hat{N}$, where $\hat{N}$ is a truncating S-iterate of N. I^0 is truncation free on its main branch, since I^1 is not. Hence $Q^0_\mu \lhd M_\mu$. Hence, $Q^0_\mu \lhd M'_\mu$ by Claim 1. Hence:

$$\sigma' \pi^0_{0,1} : N \longrightarrow_{\Sigma^*} \hat{N},$$

where $\hat{N}$ is a truncating iterate of N. Contradiction!

QED(Claim 2)

Claim 3. No $i + 1 \leq_{T^1} \mu$ is an anomaly in I^1.

Proof. Suppose not. Then $\kappa_i = \kappa$ and $\tau_i = \lambda$. Hence $\tau_i < \sigma(\lambda) = \kappa^{+N}$. Thus $M^*_i = N^*$, where $N^* = N||\eta$, η being maximal such that λ is a cardinal in $N||\eta$. By Claim 2, there is no drop point $j + 1 \leq_{T^1} \mu$ such that $i < j$. Hence:

$$\pi'_{0,\mu} : N^* \longrightarrow_{\Sigma^*} M_\mu.$$

$\kappa = \rho^\omega$ in N^*, since $\rho^\omega \leq \kappa$ by the definition of N^*, but $\rho^\omega \geq \kappa$ since $N^* \in N$ and κ is a cardinal in N. But $\kappa_i = \text{crit}(\pi^1_{0,\mu})$. Hence $\kappa = \rho^\omega$ in M_μ.

$Q_\mu = M_\mu$ as above. Moreover the iteration I^0 is truncation free on its main branch, since I^1 is not. Thus:

$$\pi^0_{0,\mu} : N \longrightarrow_{\Sigma^*} M_\mu$$

Hence $\kappa^0_i \geq \rho^\omega_N$ for $i + 1 \leq_{T^0} \mu$, since otherwise $\rho^\omega_{M_\mu} \geq \lambda_i > \kappa$. Hence:

$$\rho^\omega_N = \rho^\omega_{Q_\mu} = \kappa$$

and:

$$\mathbb{P}(\kappa) \cap N = \mathbb{P}(\kappa) \cap Q_\mu = \mathbb{P}(\kappa) \cap M_\mu = \mathbb{P}(\kappa) \cap N^*.$$

This is clearly a contradiction, since $N^* \in N$ and $\text{card}(N^*) = \kappa$ in N. Hence by a diagonal argument there is $A \in \mathbb{P}(\kappa) \cap N$ such that $A \notin N^*$.

QED(Claim 3)

It remain only to show:

Claim 4. $1 \leq_{T^1} \mu$.

Proof. Suppose not. Then $o <_{T^1} \mu$. By Claim 3 there is no anomaly on the main branch of I^1. Hence, if $\kappa_i < \lambda$ and $i + 1 \leq_{T^1} \mu$, we have $\tau_i < \lambda$. But then $M^*_{\nu^1_i} = N$. By claim 2 there is no drop on the main branch of I^1. Hence:

$$\pi^1_{0,\mu} : N \longrightarrow_{\Sigma^*} M_\mu.$$

$M_\mu \triangleleft Q_\mu$ by Claim 1. Hence $M_\mu = Q_\mu$, since otherwise $\pi^1_{0,\mu}$ would map N into a proper segment of an S-iterator of N. Thus we have:

$$\pi^0_{0,\mu}; N \longrightarrow_{\Sigma^*} M_\mu.$$

Set: $\pi^0 = \pi^0_{0,\mu}, \pi^1 = \pi^1_{0,\mu}$. We claim:

Claim. $\pi^0 = \pi^1$.

Proof. Suppose not. Let i be least such that $\pi^0(e_i) \neq \pi^1(e_i)$. Then $\pi^1(e_i) > \pi^0(e_i)$ since the map π^0, being an S-iteration map, is e-minimal. But $\sigma'\pi^1$

is the S-iteration map from N to $\hat{N}$. Hence $\sigma'\pi^1(e_i) < \sigma'\pi^0(e_i)$, since $\sigma'\pi^0 : N \longrightarrow_{\Sigma^*} \hat{N}$. Hence $\pi^1(e_i) < \pi^0(e_i)$. Contradiction!

QED(Claim)

Let $i_h + 1 \leq_{T^h} \mu$ with $o = T^h(i_h + 1)$ for $h = 0, 1$. Then $\kappa_{i_0} = \kappa_{i_1} = \text{crit}(\pi)$, where $\pi = \pi^0_{0,\mu} = \pi^1_{0,\mu}$. Set:

$$F^0 = E^{Q_0}_{\nu_{i_0}}, F^1 = E^{M_0}_{\nu_{i_1}}.$$

Then:

$$F^h(X) = \pi^h_{0,i_h+1}(X) \text{ for } X \in \mathbb{P}(\kappa_{i_h}) \cap N.$$

Thus:

$$\alpha \in F^h(X) \longleftrightarrow \alpha \in \pi(X) \text{ for } \alpha < \lambda_{i_h},$$

since $\pi = \pi^h_{i_h+1,\mu} \circ \pi^h_{0,i_h+1}$. But then $\nu_{i_0} \not< \nu_{i1}$, since otherwise $F^0 \in J^{E^{M_{i_1}}}_{\nu_{i_1}}$ by the initial segment condition, whereas ν_{i_0} is a cardinal in $J^{E^{M_{i_1}}}_{\nu_{i_1}}$. Contradiction! Similarly $\nu_{i_1} \not< \nu_{i_0}$. Thus $i_0 = i_1 = i$ and $F^0 = F^1$. But then ν_i is not a coiteration index! Contradiction.

QED(Claim 4)

This proves the simplicity lemma.

4.3 Solidity and Condensation

In this section we employ the simplicity lemma to establish some deep structural properties of mice. In §4.3.1 we prove the **Solidity Lemma** which says that every mouse is solid. In §4.3.2 we expand upon this showing that any mouse N has a unique core $\overline{N}$ and core map σ defined by the properties:

- $\overline{N}$ is sound.
- $\sigma :\longrightarrow_{\Sigma^*} N$.
- $\rho^\omega_{\overline{N}} = \rho^\omega_N$ and $\sigma \upharpoonright \rho^\omega_N := \text{id}$.
- $\sigma(p^i_{\overline{N}}) = p^i_N$ for all i.

In §4.3.3 we consider the *c*ondensation properties of mice. The condensation lemma for L says that if $\pi : M \longrightarrow_{\Sigma_1} J_\alpha$ and M is transitive, then $M \lhd J_\alpha$. Could the same hold for an arbitrary sound mouse in place of J_α? In