APPLYING VOLATILITY ESTIMATORS BASED ON LIMIT ORDER BOOKS*

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This note reports simulation results for the discrete-time regression-type model and the estimator based on local minima from [1]. The application of estimator $\widehat{IV}_n^{h_n,r_n}$, (2.8) in [1], given by

$$(1.1)\quad \widehat{IV}_n^{h_n,r_n} = \sum_{l=1}^{r_n h_n^{-1}} \Psi_n^{-1}\Bigg(\sum_{k=(l-1)r_n^{-1}/2+1}^{l r_n^{-1}/2} \big(m_{n,2k} - m_{n,2k-1}\big)^2\, 2h_n^{-1} r_n\Bigg) h_n r_n^{-1}.$$

requires access to the functions Ψ_n. Though we have deduced theoretical properties of $\Psi = \lim_{n\to\infty} \Psi_n$ and (Ψ_n) in [1], closed forms of these functions are not accessible what makes the application more intricate. One way to apply the estimator in practice is to employ a Monte Carlo approximation of Ψ_n for a certain value of $\mathcal{K}$ in $h_n = \mathcal{K}^{2/3}(n\lambda)^{-2/3}$. Besides employing Monte Carlo approximations of Ψ_n, we also discuss a very simple first-order approximated version of our estimator here. In particular, this will help us to understand the behavior of the method for different choices of bin-widths h_n.

When $h_n \propto n^{-2/3}$ and the noise level is small compared to the volatility, i.e. $\lambda^{-1} \ll \sigma_t$ for all t, or when we choose $\mathcal{K}$ in $h_n = \mathcal{K}^{2/3}\lambda^{-2/3}n^{-2/3}$ sufficiently large, the local minima, (3.1) in [1], are predominantly determined by $\min_{i\in\mathcal{I}_k^n} X_{t_i^n}$. In accordance with high signal-to-noise ratios as found for high-frequency real data within the traditional centered noise model, see [3] and [2], and a realistic fit of real data examples, we are typically in situations where $\lambda^{-1} \ll \sigma_t$ for all t, and thus the approximation $m_{n,k} \approx \min_{i\in\mathcal{I}_k^n} X_{t_i^n}$ might be adequate to yield an efficient finite-sample estimator. Figure 1 illustrates simulated observations from a setup with large signal-to-noise ratio in which the bin-wise minima $m_{n,k}$, $k = 0,\ldots,h_n^{-1}-1$, are close to $\min_{i\in\mathcal{I}_k^n} X_{t_i^n}$. The limit law of $\min_{i\in\mathcal{I}_k^n} X_{t_i^n} - X_{kh_n}$ is explicit. The joint density of the endpoint of the process $Z = \int \sigma dW$ on $[0,t]$ and its minimum on the interval is

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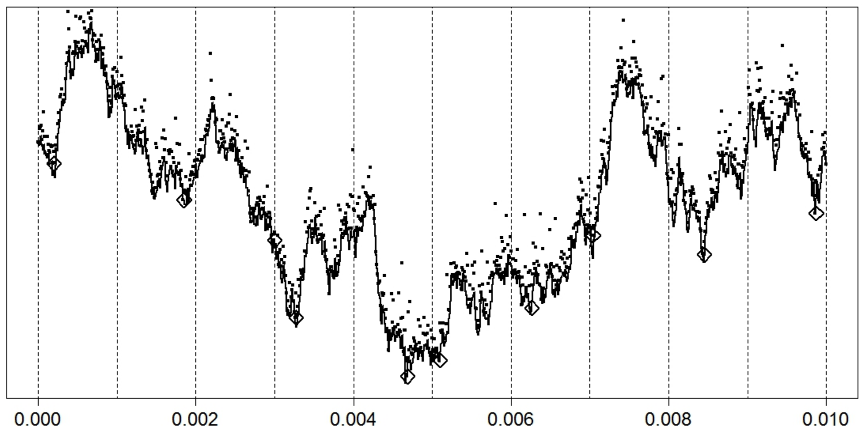

FIG 1. *Interval* $[0, 0.01]$ *with first 10 bins for one simulation of a Brownian motion* X *with exponential errors,* $n = 100000, h_n = 0.001, nh_n = 100$ *for* $\lambda = 200$*. Bin-wise minima are marked by diamonds.*

well-known and derived with

$$\mathbb{P}\Big(\min_{0\le s\le t} Z_s < m, Z_t \ge w\Big) = \int_{-\infty}^{2m-w} (2\pi\sigma^2)^{-1/2} \exp\{-1/(2\sigma^2)z^2\}dz$$

and thus given by

$$g(m,w) = \frac{2(w-2m)}{\sigma^3\sqrt{2\pi}} \exp\{-1/(2\sigma^2)(2m-w)^2\}, m \in (-\infty, 0], w \in [m, \infty)\,.$$

In particular this yields with $k = \lfloor th_n^{-1}\rfloor$, when we act as if $m_{n,k} = \min_{i\in\mathcal{I}_k^n} X_{t_i^n}$:

$$\lim_{n\to\infty} h_n^{-1/2}\mathbb{E}[\mathcal{L}_{n,k}] = -\sqrt{(2/\pi)}\sigma_t\,, \tag{1.2a}$$

$$\lim_{n\to\infty} h_n^{-1}\mathbb{E}[\mathcal{L}_{n,k}^2] = \sigma_t^2\,, \tag{1.2b}$$

$$\lim_{n\to\infty} h_n^{-1}\mathbb{E}[\mathcal{L}_{n,k+1}\mathcal{R}_{n,k}] = (1/2)\sigma_t^2\,, \tag{1.2c}$$

$$\text{and that } \ \Psi(\sigma_t^2) = 2\,\sigma_t^2(\pi-2)/\pi\,. \tag{1.2d}$$

Figure 2 draws a comparison between $\sum_{k=1}^{h_n^{-1}/2}(m_{n,2k} - m_{n,2k-1})^2$ and the approximation $\sigma^2(\pi-2)/\pi$ for $\sigma \in [0,1]$, $n = 100000, nh_n = 100$ and $\lambda^{-1} = 0.005$ and $\lambda^{-1} = 0.05$, respectively. We conclude for $\mathcal{K}_\lambda = \mathcal{K}/\lambda$ in $h_n =$

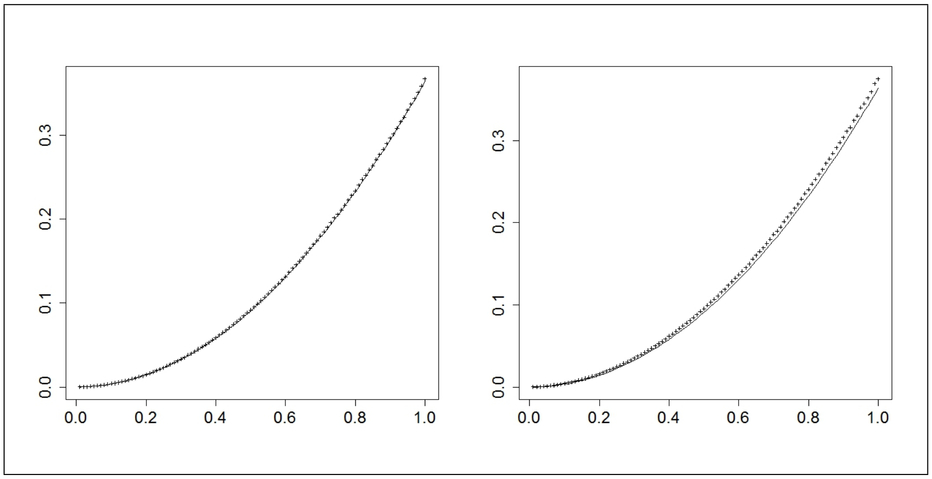

FIG 2. $\sum_k (m_{n,2k} - m_{n,2k-1})^2$ *based on highly precise Monte Carlo means (points),* $n = 100000, nh_n = 100$*, compared to* $\sigma^2(\pi-2)/\pi$ *(solid line) against* σ *for* $\sigma = u/100, u = 1,\ldots,100$*. Exponential error distribution with* $\lambda^{-1} = 0.005$ *(left),* $\lambda^{-1} = 0.05$ *(right).*

$\big(\mathcal{K}_\lambda\big)^{2/3} n^{-2/3}$ large enough the simplified version of estimator $\widehat{IV}_n^{h_n,r_n}$:

$$\widehat{IV}_{n,app}^{h_n} = \frac{\pi}{\pi-2} \sum_{k=1}^{h_n^{-1}/2} \big(m_{n,2k} - m_{n,2k-1}\big)^2 \,. \tag{1.3}$$

We analyse the estimators' finite-sample performances in the following simulation study. Observe that the main building block, a sum of squared differences of local minima, is the same for estimators (1.1) and (1.3). The only difference is that for (1.3) we approximate Ψ_n by the linear function from (1.2d).

1.1. *Simulations.* We simulate observations $Y_i = X_{i/n} + \varepsilon_i, i = 0,\ldots,n$, with X an Itô process with constant drift $a = 0.1$. First, we consider a model with a deterministic Lipschitz volatility function:

$$\sigma_t^2 = 0.1\big(1 - 0.4\sin\big(\tfrac{3}{4}\pi t\big)\big),\ t \in [0,1]. \tag{1.4}$$

A stochastic volatility model is also implemented:

$$\sigma_t^2 = \left(\int_0^t c\cdot\rho\, dW_s + \int_0^t \sqrt{1-\rho^2}\cdot c\, dW_s^{\perp}\right)\cdot\tilde\sigma_t\,, \tag{1.5a}$$

with $W^{\perp}$ a standard Brownian motion independent of W, $c = 0.05$, $\rho = 0.5$, and $\tilde\sigma_t, t \in [0,1]$, a deterministic seasonality function

$$\tilde\sigma_t = 0.1\big(1 - t^{\frac{1}{3}} + 0.5\cdot t^2\big)\,, \tag{1.5b}$$

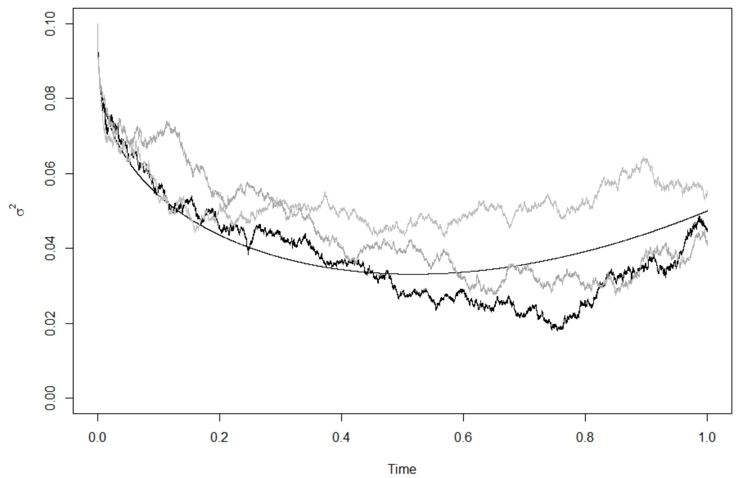

FIG 3. *Deterministic intra-day squared volatility shape function* $\tilde{\sigma}_t, t \in [0,1]$ *from* (1.5b), *with three simulated paths for* $\sigma_t^2, t \in [0,1]$ *from* (1.5a).

which mimics a typical intra-day shape of squared volatility. These volatility dynamics are illustrated in Figure 3. From our experience in extensive simulation experiments we can report that the estimators perform almost equally well in the deterministic and the random volatility case. Furthermore, we shall see that estimators (1.1) and (1.3) exhibit comparable variances. Thus, it will be informative to restrict to estimator (1.3) for the stochastic volatility model.

We simulate one-sided errors according to a parametric specification with exponential errors $\varepsilon_i \overset{iid}{\sim} \mathrm{Exp}(\lambda)$ or uniformly distributed errors $\varepsilon_i \overset{iid}{\sim} \mathrm{U}[0,\lambda^{-1}]$. We compare the estimators' precisions for different bin-widths h_n for $n = 100000$ and $n = 10000$. Sample sizes from the order flow of (only) best bid or ask prices are in practice in this range where $n = 100000$ is a rather usual sample size for one trading day and liquid stocks. We consider four different scenarios of noise dilution: $\varepsilon_i \overset{iid}{\sim} \mathrm{Exp}(\lambda)$ with $\lambda = 20, 200, 2000$ and $\varepsilon_i \overset{iid}{\sim} \mathrm{U}[0,\lambda^{-1}]$ with $\lambda = 200$. From empirical studies as [3] and [2] we consider $\lambda \in (200, 2000)$ a realistic range. The high noise level scenario for the complex volatility model is for illustration of the estimator's reaction. Since the estimator's variance hinges on the realised volatility path, we fix one path simulated from model (1.5a) for the iterations in Table 3, to assure good comparability. In this realisation of (1.5a) the true integrated squared volatility equals ca. 0.036. For (1.4) it equals ca. 0.072. In Table 2 results for

$n = 100000$		$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 2000)$			$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 200)$		
h_n^{-1}	nh_n	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
500	200	0.0015	0.1031	0.00	0.0332	0.1171	0.93
1000	100	0.0068	0.0542	0.08	0.0407	0.0518	3.10
10000	10	0.1393	0.0060	76.40	0.8688	0.0085	98.88
$n = 10000$		$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 2000)$			$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 200)$		
h_n^{-1}	nh_n	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
100	100	0.0020	0.1227	0.00	0.0179	0.1200	0.26
500	20	0.0183	0.0219	1.35	0.0555	0.0239	11.43
1000	10	0.0392	0.0116	11.75	0.1401	0.0138	58.72

TABLE 1. *Simulation results for integrated squared volatility estimator (1.3) in volatility specification (1.4) based on first-order approximation of Ψ. Bias rescaled with factor $n^{1/3}$, variance rescaled with factor $n^{2/3}$ and percentage of (systematic) squared bias in MSE.*

the original estimator (1.1) are presented. We use a Monte Carlo approximation of Ψ_n for each configuration and for the respective values of n and h_n. The coarse grid size $r_n^{-1}h_n$ is set by $r_n^{-1} = 100$ for most configurations which led to slightly smaller variances than setting $r_n^{-1} = \lfloor h_n^{-1/2} \rfloor$. Only in the case that $h_n^{-1} = 100$ we use $r_n^{-1} = 20$.

The simulation results presented in Tables 1, 2 and 3 show the following:

- Estimator (1.1) asks for a Monte Carlo approximation of Ψ_n first and thus is more elaborate than (1.3). The variances of both estimators are almost equal. Since estimator (1.1) shows no relevant bias, it performs well and allows to reach the smallest variances by choosing h_n small enough. Since the variances of (1.3) and (1.1) are very close to each other, we restrict to the explicit estimator (1.3) in Table 3.
- For an accurate choice of h_n, estimator (1.3) guarantees a highly precise estimation of integrated squared volatility. In Table 3, for $n = 100000, \lambda = 200$ the bin-width $h_n^{-1} = 2000, nh_n = 50$, yields the minimal MSE. In this case the MSE equals ca. $0.014n^{-2/3}$. The rescaled variances are quite small though the simple estimator (1.3) relies on a first-order approximation and uses only one half of the bins.
- Yet, the systematic bias of (1.3) by the first-order approximation $m_{n,k} \approx \min_{i \in \mathcal{I}_k^n} X_{t_i^n}$ prevents us from choosing h_n small enough to attain the lowest variance, see Tables 1 and 3. The best MSE in Table 3 for $n = 100000, \lambda = 200$, is achieved when the variance is still almost

$n = 100000$		$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 2000)$			$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 200)$		
h_n^{-1}	r_n^{-1}	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
500	100	-0.0069	0.1051	0.04	-0.0194	0.1049	0.36
1000	100	0.0095	0.0528	0.17	-0.0148	0.0571	0.38
10000	100	0.0064	0.0056	0.72	0.0066	0.0077	0.56
$n = 10000$		$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 2000)$			$\varepsilon_i \overset{iid}{\sim} \text{Exp}(\lambda = 200)$		
h_n^{-1}	r_n^{-1}	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
100	20	-0.0078	0.1168	0.05	0.0029	0.1199	0.00
500	100	-0.0136	0.0237	0.77	-0.0008	0.0257	0.00
1000	100	0.0086	0.0125	0.59	0.0109	0.0135	0.87

TABLE 2. *Simulation results for integrated squared volatility estimator* (1.1) *in volatility specification* (1.4). *Bias rescaled with factor $n^{1/3}$, variance rescaled with factor $n^{2/3}$ and percentage of (systematic) squared bias in MSE.*

3 times larger than its minimum.

- The estimators perform equally well also for semi-martingale random volatilities with leverage.

Altogether, the results are promising that relevant efficiency gains by exploiting limit order book price levels and estimation approaches stimulated from model (1.3) in [1] are attainable. Estimator (1.3) provides a simple and computationally fast estimation without Monte Carlo approximations. However, due to its bias it comes close but does not attain the same high efficiency.

A crucial and practically relevant question is if our method infers on the same integrated squared volatility as in the traditional model, i.e. if both hypothetical underlying objects correspond to each other. For our intra-day data example of the FB stock on 06/02/2014, used for Figure 2 of [1], we find with the asymptotically efficient locally parametric Fourier estimator by [4] applied to 37972 traded prices an integrated squared volatility estimate 0.00033. The tuning block-width for the approach is fixed to 38 blocks. When we apply method (1.3) with $h_n^{-1} = 1000$ bins to the 212463 available level 1 ask prices we obtain an estimate 0.00037 for integrated squared volatility. Based on 209175 level 1 bids and local maxima an analogous estimator yields 0.00038. Estimates are quite robust against different choices of h_n. Therefore, while estimation based on bids and asks is very close to each other, the gap to the estimate in the Gaussian noise model is a bit larger.

Still, this first impression indicates that the idea of the same underlying efficient price and its volatility is reasonable. A rigorous extensive empirical comparison of both approaches in practice remains open for consideration in further research.

Conclusion. The simulation studies demonstrate the high precision of our estimation approach in realistic finite-sample settings. It is shown that for applications one may use a very simple and explicit estimator derived by a first-order approximation when involving accurate bin-widths. Whenever $(nh_n\lambda)^{-1} \ll \sqrt{h_n}\sigma_t, t \in [0,1]$, already the first-order term of Ψ_n and the simple estimator (1.3) guarantee a precise estimation of integrated squared volatility. However, refinements of the estimation method stimulated by the model with one-sided errors from [1] should further improve upon the practical performance. Computing the second-order term in the expansion as $\mathcal{K} \to \infty$ is one task to improve the efficiency and avoid a Monte Carlo approximation. While building upon the methodology of [1], this point poses a non-trivial extension which goes beyond the scope of the current work and is left for future research.

$n=100000$		$\varepsilon_i \overset{iid}{\sim} \mathrm{Exp}(\lambda=2000)$			$\varepsilon_i \overset{iid}{\sim} \mathrm{Exp}(\lambda=200)$		
h_n^{-1}	nh_n	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
100	1000	0.0003	0.1841	0.00	0.0047	0.1614	0.01
500	200	0.0004	0.0389	0.00	0.0128	0.0351	0.47
800	125	0.0028	0.0236	0.00	0.0302	0.0239	3.67
1000	100	0.0034	0.0183	0.06	0.0298	0.0185	4.60
1250	80	0.0034	0.0121	0.09	0.0427	0.0156	10.44
2000	50	0.0153	0.0082	2.76	0.0713	0.0094	35.00
5000	20	0.0398	0.0031	33.75	0.2261	0.0041	92.61
10000	10	0.0857	0.0016	81.73	0.6698	0.0032	99.29
20000	5	0.2036	0.0009	97.75	3.0679	0.0068	99.92
$n=100000$		$\varepsilon_i \overset{iid}{\sim} \mathrm{Exp}(\lambda=20)$			$\varepsilon_i \overset{iid}{\sim} \mathrm{U}[0,1/200]$		
h_n^{-1}	nh_n	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
100	1000	0.0164	0.1610	0.17	-0.0068	0.1408	0.03
500	200	0.0806	0.0315	17.08	0.0119	0.0292	0.48
800	125	0.1499	0.0218	50.73	0.0215	0.0211	2.14
1000	100	0.1885	0.0175	66.99	0.0237	0.0128	4.21
1250	80	0.2565	0.0128	83.69	0.0387	0.0110	7.49
2000	50	0.5491	0.0114	96.35	0.0592	0.0089	28.25
5000	20	4.4579	0.0490	99.75	0.1782	0.0040	88.86
10000	10	32.426	1.0599	99.89	0.4804	0.0026	98.89
20000	5	256.13	32.935	99.95	1.6122	0.0030	99.90
$n=10000$		$\varepsilon_i \overset{iid}{\sim} \mathrm{Exp}(\lambda=2000)$			$\varepsilon_i \overset{iid}{\sim} \mathrm{Exp}(\lambda=200)$		
h_n^{-1}	nh_n	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
100	100	0.0009	0.0301	0.00	0.0000	0.0285	0.00
200	50	0.0071	0.0142	0.35	0.0112	0.0152	0.83
500	20	0.0127	0.0058	2.71	0.0318	0.0060	14.50
1000	10	0.0201	0.0028	12.26	0.0883	0.0033	69.97
2000	5	0.0491	0.0015	61.86	0.2530	0.0023	96.51
$n=10000$		$\varepsilon_i \overset{iid}{\sim} \mathrm{Exp}(\lambda=20)$			$\varepsilon_i \overset{iid}{\sim} \mathrm{U}[0,1/200]$		
h_n^{-1}	nh_n	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE	Bias $n^{1/3}$	Var $n^{2/3}$	Bias^2%MSE
100	100	0.0347	0.0302	3.83	0.0067	0.0293	0.15
200	50	0.0715	0.0160	24.24	0.0066	0.0143	0.30
500	20	0.3395	0.0111	91.19	0.0242	0.0060	8.86
1000	10	1.6902	0.0385	98.67	0.0557	0.0031	49.73
2000	5	12.088	0.7573	99.48	0.1359	0.0019	90.79

TABLE 3. *Simulation results for integrated squared volatility estimator* (1.3) *for* (1.5a) *based on first-order approximation of* Ψ. *Bias rescaled with factor* $n^{1/3}$, *variance rescaled with factor* $n^{2/3}$ *and percentage of (systematic) squared bias in MSE.*

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