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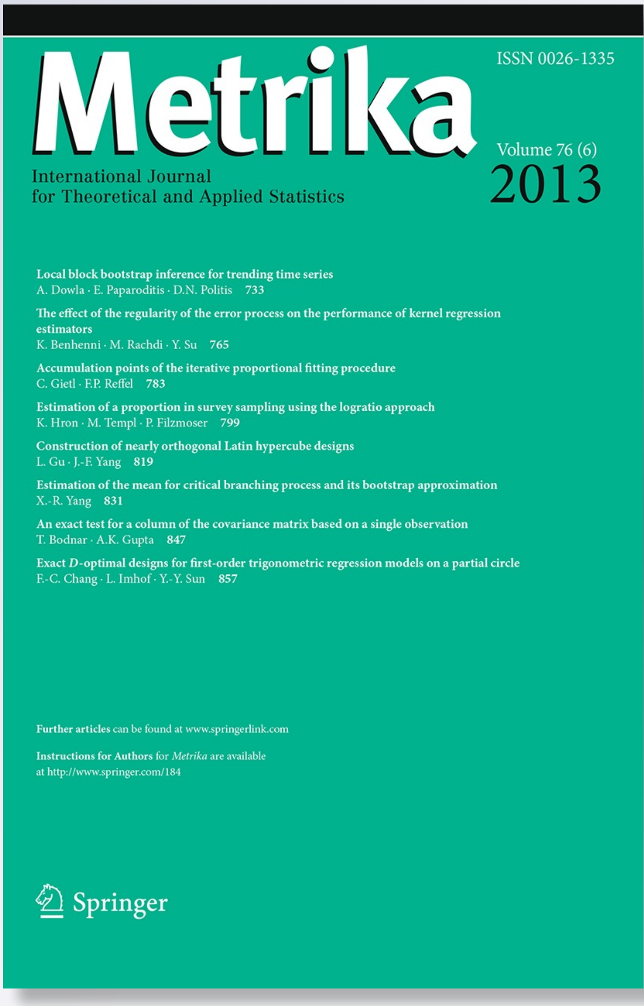

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An exact test for a column of the covariance matrix based on a single observation

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Abstract In this paper, we derive an exact test for a column of the covariance matrix. The test statistic is calculated by using a single observation. The exact distributions of the test statistic are derived under both the null and alternative hypotheses. We also obtain an analytical expression of the power function of the test for the equality of a column of the covariance matrix to a given vector. It is shown that the information contained in a single vector is large enough to ensure a good performance of the test. Moreover, the suggested test can be applied for time-dependent multivariate Gaussian processes.

Keywords Covariance matrix · Singular Wishart distribution · Skew normal distribution · Inference procedure

Mathematics Subject Classification (2000) 62H10 · 62H15 · 62E15 · 62F03

Dedicated to the memory of Sam Kotz.

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1 Introduction

In the classical test theory on the covariance matrix the sample covariance matrix is, usually, used for constructing a test statistic. Under the assumptions that the sample size is larger than the order of the covariance matrix and the observations are independently and identically normally distributed it holds that the sample covariance matrix has a Wishart distribution (see, e.g. Gupta and Nagar 2000, Theorem 3.3.6). Using this property different tests on the structure of the covariance matrix were suggested (cf., Anderson 2003; Muirhead 1982; Rencher 2002; Gupta and Xu 2006) and improved estimates of the covariance and the precision matrices were derived (see, e.g., Bodnar and Gupta 2009, 2011; Gupta et al. 2005; Sarr and Gupta 2009 and reference therein).

In some important practical situations we are not able to collect data of size larger than the process dimension which consists of independent observations. It might happen that the dimension of the stochastic process is too large or/and the data are dependent. The example of those data can be easily found in economics, especially in portfolio theory. In this case, we have to deal with data of a smaller frequency or in order to avoid the assumption of independence the estimation of the covariance matrix can be based on a single process realization.

For such problems the properties of the singular Wishart distribution are applied in the test theory (see, e.g. Schott 2007; Srivastava 2005; Srivastava and Yanagihara 2010). The singular Wishart distribution appears to be the distribution of the sample covariance matrix when the sample size is smaller than the dimension of the process. The distribution theory for the singular Wishart distribution has recently been discussed in a number of papers (see, e.g., Díaz-García et al. 1997; Srivastava 2003; Bodnar and Okhrin 2008). Bodnar et al. (2009) applied the distributional properties of the singular Wishart distribution for deriving the sequential procedures for detecting changes in the covariance matrix of the Gaussian process. In the present paper, we derive an exact test for a column of the covariance matrix. One of the main advantage of the suggested approach is that it can be applied for time dependent stochastic processes.

The rest of the paper is structured as follows. In Sect. 2, main results are presented. Here, we introduce an estimator for the covariance matrix based on a single observation and derive a test for a column of the covariance matrix. The distribution of the test statistic is obtained under both the null and alternative hypotheses. We prove that under the null hypothesis the test statistic has a central χ^2-distribution, while under H_1 the density function depends only on the process dimension and a positive constant. The last result simplifies significantly the study of the test power. Final remarks are presented in Sect. 3, while all proofs are given in the "Appendix" (Sect. 3).

2 Main results

Let $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $p > 2$, with known mean vector $\boldsymbol{\mu}$. Without loss of generality we assume that $\boldsymbol{\mu} = \mathbf{0}_p$, where $\mathbf{0}_p$ stands for the p-dimensional vector of zeroes. If this assumption does not hold then the vector $\tilde{\mathbf{X}} = \mathbf{X} - \boldsymbol{\mu}$ should be considered instead of $\mathbf{X}$. Moreover, the derived results can also be applied if the mean vector $\boldsymbol{\mu}$ is unknown. In this case, we assume that $\mathbf{X}_1 \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\mathbf{X}_2 \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and

$\mathbf{X}_1, \mathbf{X}_2$ are independent. Then in the test statistic, the vector $\mathbf{X}$ should be replaced by $\check{\mathbf{X}} = \mathbf{X}_1 - \mathbf{X}_2$.

Before we present the main results of this section, an estimator of the covariance matrix has to be introduced. We apply the point estimator based on the single observation, i.e. the covariance matrix $\boldsymbol{\Sigma}$ is estimated by

$$\mathbf{V} = \mathbf{X}\mathbf{X}'. \tag{1}$$

It holds that $\mathbf{V}$ has a singular p-dimensional Wishart distribution with 1 degree of freedom and the covariance matrix $\boldsymbol{\Sigma}$ (see, e.g. Srivastava 2003). This assertion is denoted by $\mathbf{V} \sim W_p(1, \boldsymbol{\Sigma})$. Although the matrix $\mathbf{V}$ is singular (its rank is equal to 1), it provides us an unbiased estimator of $\boldsymbol{\Sigma}$.

We assume that $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$ (a known matrix) under H_0 and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_1$ under H_1. The matrices $\boldsymbol{\Sigma}_0$, $\boldsymbol{\Sigma}_1$, and $\mathbf{V}$ are partitioned as follows

$$\boldsymbol{\Sigma}_0 = \begin{bmatrix} \sigma_{0;11} & \boldsymbol{\Sigma}_{0;12} \\ \boldsymbol{\Sigma}_{0;21} & \boldsymbol{\Sigma}_{0;22} \end{bmatrix}, \quad \boldsymbol{\Sigma}_1 = \begin{bmatrix} \sigma_{1;11} & \boldsymbol{\Sigma}_{1;12} \\ \boldsymbol{\Sigma}_{1;21} & \boldsymbol{\Sigma}_{1;22} \end{bmatrix}, \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} v_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix} \tag{2}$$

Let $\boldsymbol{\Sigma}_{0;22\cdot 1} = \boldsymbol{\Sigma}_{0;22} - \boldsymbol{\Sigma}_{0;21}\boldsymbol{\Sigma}'_{0;21}/\sigma_{0;11}$, $\boldsymbol{\Sigma}_{1;22\cdot 1} = \boldsymbol{\Sigma}_{1;22} - \boldsymbol{\Sigma}_{1;21}\boldsymbol{\Sigma}'_{1;21}/\sigma_{1;11}$. Without loss of generality we now present a test for the first column of the covariance matrix $\boldsymbol{\Sigma}_0$. For the ith column the test statistic can be derived similarly. In this case instead of the partitions (2), we construct the partition for the (i, i)th element of the matrices $\boldsymbol{\Sigma}_0$, $\boldsymbol{\Sigma}_1$, and $\mathbf{V}$ as follows. Let $\sigma_{0;ii}$ denotes the (i, i)th element of the matrix $\boldsymbol{\Sigma}_0, i = 1, \ldots, p$. By $\boldsymbol{\Sigma}_{0;21,i}$ we denote the ith column of the matrix $\boldsymbol{\Sigma}_0$ without $\sigma_{0;ii}$. Let $\boldsymbol{\Sigma}_{0;22,i}$ denote a quadratic matrix of order $p - 1$, which is obtained from the matrix $\boldsymbol{\Sigma}_0$ by deleting the ith row and the ith column. Finally, $\boldsymbol{\Sigma}_{0;22\cdot 1,i} = \boldsymbol{\Sigma}_{0;22,i} - \boldsymbol{\Sigma}_{0;21,i}\boldsymbol{\Sigma}'_{0;21,i}/\sigma_{0;ii}$ is calculated. In the same way we define $\sigma_{1;ii}$, $\boldsymbol{\Sigma}_{1;21,i}$, $\boldsymbol{\Sigma}_{1;22,i}$, $\boldsymbol{\Sigma}_{1;22\cdot 1,i}$, v_{ii}, $\mathbf{V}_{21,i}$, $\mathbf{V}_{22,i}$, and $\mathbf{V}_{22\cdot 1,i}$ by splitting $\boldsymbol{\Sigma}_1$ and $\mathbf{V}$ correspondingly.

We are interested in deriving a test for the first column of the covariance matrix based on the single observation $\mathbf{X}$. The hypotheses to be tested are given by

$$H_0: \ \boldsymbol{\Sigma}_{12} = c\boldsymbol{\Sigma}_{0;12} \quad \text{against} \quad H_1: \ \boldsymbol{\Sigma}_{12} = \boldsymbol{\Sigma}_{1;12} \neq c\boldsymbol{\Sigma}_{0;12}, \tag{3}$$

where $c > 0$ denotes an arbitrary (un)known constant.

We define

$$\boldsymbol{\eta} = \boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}\left(\frac{\mathbf{V}_{21}}{v_{11}} - \frac{\boldsymbol{\Sigma}_{0;21}}{\sigma_{0;11}}\right)v_{11}^{1/2}. \tag{4}$$

Let $\Phi(.)$ denote the cumulative distribution function of the univariate standard normal distribution. Let $\phi_k(.; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ stand for the density function of the k-dimensional multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. In Theorem 1 we derive the distributions of the random vector $\boldsymbol{\eta}$ under both H_0 and H_1 hypotheses.

Theorem 1 *Let* $\mathbf{X} \sim \mathcal{N}_p(\mathbf{0}_p, \boldsymbol{\Sigma})$. *Then*

a) *the density function of* $\boldsymbol{\eta}$ *is given by*

$$f_{\boldsymbol{\eta}}(\mathbf{x}) = 2\phi_{p-1}\left(\mathbf{x}; \mathbf{0}_{p-1}, \boldsymbol{\Omega} + \sigma_{1;11}\boldsymbol{\Delta}\boldsymbol{\Delta}'\right) \Phi\left(\frac{\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\mathbf{x}}{\sqrt{\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}}\right) \tag{5}$$

where

$$\boldsymbol{\Delta} = \boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}\left(\frac{\boldsymbol{\Sigma}_{1;21}}{\sigma_{1;11}} - \frac{\boldsymbol{\Sigma}_{0;21}}{\sigma_{0;11}}\right) \quad \text{and} \quad \boldsymbol{\Omega} = \boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}\boldsymbol{\Sigma}_{1;22\cdot 1}\boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}.$$

b) *under* H_0, $\boldsymbol{\eta} \sim \mathcal{N}_{p-1}(\mathbf{0}_{p-1}, \mathbf{I}_{p-1})$, *where* $\mathbf{I}_k$ *denotes a* $k \times k$ *identity matrix.*

The part a) of Theorem 1 is proved in the "Appendix". The part b) follows directly from the part a) by noting that $\boldsymbol{\Delta} = \mathbf{0}_{p-1}$ in this case. The result of part b) is also given by Bodnar and Okhrin (2008, Corollary 1a). The results of Theorem 1 show that the random vector $\boldsymbol{\eta}$ has a multivariate skew-normal distribution (cf. Azzalini 2005; Domínguez-Molina et al. 2007), while it has a standard multivariate normal distribution under H_0.

Next, we introduce the test statistic given by

$$T = \boldsymbol{\eta}'\boldsymbol{\eta}, \tag{6}$$

which is motivated by the distributional properties of $\boldsymbol{\eta}$.

In the following we also use the generalized hypergeometric function (cf. Muirhead 1982, p. Ch. 1.3), i.e.

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x) = \frac{\prod_{l=1}^{q}\Gamma(b_l)}{\prod_{j=1}^{p}\Gamma(a_j)} \sum_{i=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_j + i)}{\prod_{l=1}^{q}\Gamma(b_l + i)} \frac{z^i}{i!}.$$

The technical computation of a hypergeometric function is a standard routine within many mathematical software packages like, e.g., in Mathematica.

The distribution of T is derived in Theorem 2. In the statement of the theorem we make use of ${}_1F_1(.; .; .)$, while in the proof of Theorem 2 ${}_0F_1(.; .)$ is used.

Theorem 2 *Let* $\mathbf{X} \sim \mathcal{N}_p(\mathbf{0}_p, \boldsymbol{\Sigma})$. *Then*

a) *the density function of* T *is given by*

$$f_T(x) = \frac{1}{(1+\lambda)^{1/2}} f_{p-1}(x)\ {}_1F_1\left(\frac{1}{2}; \frac{p-1}{2}; \frac{\lambda x}{2(1+\lambda)}\right) \tag{7}$$

where f_k *denotes the density of the* χ_k^2*-distribution with* k *degrees of freedom and*

$$\lambda = \sigma_{1;11}\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta} = \sigma_{1;11}\left(\frac{\boldsymbol{\Sigma}_{1;21}}{\sigma_{1;11}} - \frac{\boldsymbol{\Sigma}_{0;21}}{\sigma_{0;11}}\right)' \boldsymbol{\Sigma}_{1;22\cdot 1}^{-1}\left(\frac{\boldsymbol{\Sigma}_{1;21}}{\sigma_{1;11}} - \frac{\boldsymbol{\Sigma}_{0;21}}{\sigma_{0;11}}\right). \tag{8}$$

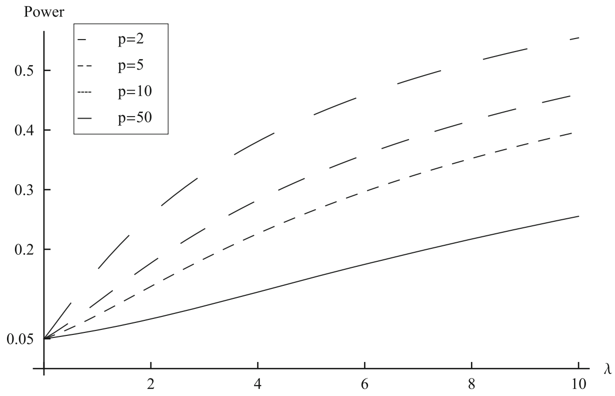

Fig. 1 Probability of rejection of the null hypothesis as a function of λ ($\alpha = 0.05$)

b) *under* H_0, $T \sim \chi^2_{p-1}$.

The proof of Theorem 2a) is given in the "Appendix". The part b) follows directly from part a) by noting that $\lambda = 0$ and ${}_1F_1\left(\frac{1}{2}; \frac{p-1}{2}; 0\right) = 1$ under H_0. The symbol λ possesses an interesting interpretation. It measures the distance between $\boldsymbol{\Sigma}_{1;21}$ and $\boldsymbol{\Sigma}_{0;21}$.

The suggested test on the covariance matrix possesses several advantages. First, only one observation of the process is used for constructing the test statistic. As a result, the test can also be applied to correlated data for checking if the ith column of the covariance matrix for each observation of the sample is equal to a preselected vector. Second, the test statistic (6) possesses the classical distribution under H_0. Hence, the test is easy to perform by comparing the values of the test statistic with the quantile of the χ^2-distribution. Third, the result of Theorem 2a) allows us to study the power of the suggested test. The power function is a function of only one parameter λ.

In Fig. 1, we plot the power function of the test as a function of λ for different values of $p \in \{2, 5, 10, 50\}$. Note that the power function is a decreasing function of p for a fixed value of λ. It is quite large in the case $p = 2$, while for larger values of p the power becomes small. On the other hand we note that the parameter λ is, usually, larger for larger values of p. Hence, the suggested test is powerful enough to reject the null hypothesis for moderate and larger values of λ.

3 Summary

The covariance matrix is, usually, used as a risk measure for multivariate processes. As a result, testing for the structure of the covariance matrix is a very important problem which has a lot of applications in practice.

In the present paper we used a single observation of the multivariate Gaussian process for constructing an estimator of the covariance matrix. Although, this estimator does not possess one of the main properties of the covariance matrix, namely it is not positive definite, it appears to be unbiased and has sufficient amount of information for deriving a test on the covariance matrix. We suggest an exact test on a column of the covariance matrix and derive the distribution of the test statistic under the null and the alternative hypothesis. Under the null hypothesis the test statistic is χ^2_{p-1}-distributed. Using the distributional results obtained under H_1 we calculate the power function of the test, which appears to be a function only of the process dimension and a positive constant.

One of the main advantages of the approach suggested in this paper is that it can be applied for time-dependent multivariate data because only a single observation vector, for example the most recent one, from the multivariate stochastic process is used. Consequently, the distribution of the test statistic under the null hypothesis as well as under the alternative hypothesis does not depend on the time dependent structure of the stochastic process. The only assumption needed for the application of the suggested approach is that the components of the multivariate stochastic process are multivariate normally distributed at each time point. However, no assumption is assumed on the dependence structure between two observation vectors from the stochastic process. The approach can be applied to the multivariate Gaussian processes as well as to their extensions for which the elements of the stochastic process at each time point are multivariate normally distributed but not obviously the joint distribution calculated for elements from different time points is normal.

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Appendix

In this section the proofs of Theorems 1a and 2a are given.

Proof of Theorem 1a Application of Lemma 1b by Bodnar and Okhrin (2008) leads to

$$\mathbf{V}_{21}|v_{11} \sim \mathcal{N}_{p-1}(\boldsymbol{\Sigma}_{1;21}\sigma_{1;11}^{-1}v_{11}, \boldsymbol{\Sigma}_{1;22\cdot 1}v_{11}).$$

Thus,

$$\frac{\mathbf{V}_{21}}{v_{11}}|v_{11} \sim \mathcal{N}_{p-1}\left(\frac{\boldsymbol{\Sigma}_{1;21}}{\sigma_{1;11}}, \frac{\boldsymbol{\Sigma}_{1;22\cdot 1}}{v_{11}}\right)$$

and, hence,

$$\boldsymbol{\eta}|v_{11} \sim \mathcal{N}_{p-1}\left(\boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}\left(\frac{\boldsymbol{\Sigma}_{1;21}}{\sigma_{1;11}} - \frac{\boldsymbol{\Sigma}_{0;21}}{\sigma_{0;11}}\right)\sqrt{v_{11}}, \boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}\boldsymbol{\Sigma}_{1;22\cdot 1}\boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}\right). \quad (9)$$

Let

$$\boldsymbol{\Delta} = \boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}\left(\frac{\boldsymbol{\Sigma}_{1;21}}{\sigma_{1;11}} - \frac{\boldsymbol{\Sigma}_{0;21}}{\sigma_{0;11}}\right) \quad \text{and} \quad \boldsymbol{\Omega} = \boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}\boldsymbol{\Sigma}_{1;22\cdot 1}\boldsymbol{\Sigma}_{0;22\cdot 1}^{-1/2}.$$

Because $v_{11}/\sigma_{1;11} \sim \chi^2_1$ (see, e.g. Srivastava 2003, Corollary 3.4) the unconditional density of $\boldsymbol{\eta}$ is given by

$$f_{\boldsymbol{\eta}}(\mathbf{x}) = \frac{\pi^{-p/2}2^{-p/2}}{\sigma_{1;11}^{1/2}|\boldsymbol{\Omega}|^{1/2}} \times \int_0^\infty y^{-1/2} exp\left(-\frac{1}{2}\left(\frac{y}{\sigma_{1;11}} + (\mathbf{x}-\boldsymbol{\Delta}\sqrt{y})'\boldsymbol{\Omega}^{-1}(\mathbf{x}-\boldsymbol{\Delta}\sqrt{y})\right)\right)dy$$

The transformation $y = t^2$ yields

$$\begin{aligned} f_{\boldsymbol{\eta}}(\mathbf{x}) &= \frac{\pi^{-p/2}2^{-p/2}}{\sigma_{1;11}^{1/2}|\boldsymbol{\Omega}|^{1/2}} 2\int_0^\infty exp\left(-\frac{1}{2}\left(\frac{t^2}{\sigma_{1;11}} + (\mathbf{x}-\boldsymbol{\Delta}t)'\boldsymbol{\Omega}^{-1}(\mathbf{x}-\boldsymbol{\Delta}t)\right)\right)dt \\ &= \frac{\pi^{-p/2}2^{-p/2}}{\sigma_{1;11}^{1/2}|\boldsymbol{\Omega}|^{1/2}} 2exp\left(-\frac{1}{2}\left(\mathbf{x}'\left(\boldsymbol{\Omega}^{-1} - \frac{\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}}{\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}\right)\mathbf{x}\right)\right) \\ &\quad \times \int_0^\infty exp\left(-\frac{\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}{2}\left(t - \frac{\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\mathbf{x}}{\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}\right)^2\right)dt\,. \end{aligned}$$

The last integral is evaluated as

$$\begin{aligned} f_{\boldsymbol{\eta}}(\mathbf{x}) &= \frac{1}{\sigma_{1;11}^{1/2}|\boldsymbol{\Omega}|^{1/2}} \frac{(\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta})^{-1/2}}{\left|\boldsymbol{\Omega}^{-1} - \frac{\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}}{\sigma_{1;11}^{-1}+\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}\right|^{1/2}} \\ &\quad \times \phi_{p-1}\left(\mathbf{x}; \mathbf{0}_{p-1}, \left(\boldsymbol{\Omega}^{-1} - \frac{\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}}{\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}\right)^{-1}\right) \\ &\quad \times 2\int_0^\infty \phi\left(t; \frac{\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\mathbf{x}}{\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}, (\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta})^{-1}\right)dt\,. \end{aligned}$$

The applications of Theorem 18.1.1 and Theorem 18.2.8 of Harville (1997) leads to

$$\begin{aligned} \left|\boldsymbol{\Omega}^{-1} - \frac{\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}}{\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}\right| &= |\boldsymbol{\Omega}^{-1}|(\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta})^{-1} \\ &\quad \times (\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta} - \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}) \\ &= |\boldsymbol{\Omega}|^{-1}(\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta})^{-1}\sigma_{1;11}^{-1}, \end{aligned}$$

$$\left(\boldsymbol{\Omega}^{-1} - \frac{\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}}{\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}\right)^{-1} = \boldsymbol{\Omega} + \frac{\frac{\boldsymbol{\Delta}\boldsymbol{\Delta}'}{\sigma_{1;11}^{-1}+\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}}{1 - \frac{\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}{\sigma_{1;11}^{-1}+\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}} = \boldsymbol{\Omega} + \sigma_{1;11}\boldsymbol{\Delta}\boldsymbol{\Delta}'.$$

Hence,

$$f_{\boldsymbol{\eta}}(\mathbf{x}) = \phi_{p-1}\left(\mathbf{x}; \mathbf{0}_{p-1}, \boldsymbol{\Omega} + \sigma_{1;11}\boldsymbol{\Delta}\boldsymbol{\Delta}'\right) \times 2\left(1 - \Phi\left(0; \frac{\boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\mathbf{x}}{\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta}}, (\sigma_{1;11}^{-1} + \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta})^{-1}\right)\right),$$

where the symbol $\Phi(.; \mu, \sigma^2)$ denotes the cumulative distribution function of the normal distribution with mean μ and variance σ^2. The statement of Theorem 1a follows from the identity $\Phi(x; \mu, \sigma^2) = \Phi\left((x-\mu)/\sigma\right)$. The theorem is proved. □

Proof of Theorem 2a From the proof of Theorem 1a we get

$$\boldsymbol{\eta}|v_{11} \sim \mathcal{N}_{p-1}\left(\boldsymbol{\Delta}\sqrt{v_{11}}, \boldsymbol{\Omega}\right).$$

Thus,

$$T|v_{11} = \boldsymbol{\eta}'\boldsymbol{\eta}|v_{11} \sim \chi^2_{p-1}(\tilde{\lambda}v_{11})$$

with

$$\tilde{\lambda} = \boldsymbol{\Delta}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Delta} = \left(\frac{\boldsymbol{\Sigma}_{1;21}}{\sigma_{1;11}} - \frac{\boldsymbol{\Sigma}_{0;21}}{\sigma_{0;11}}\right)' \boldsymbol{\Sigma}^{-1}_{1;22\cdot 1}\left(\frac{\boldsymbol{\Sigma}_{1;21}}{\sigma_{1;11}} - \frac{\boldsymbol{\Sigma}_{0;21}}{\sigma_{0;11}}\right).$$

Using the fact that $v_{11}/\sigma_{1;11} \sim \chi^2_1$ (see, e.g. Srivastava 2003, Corollary 3.4) the unconditional density of T is given by

$$f_T(x) = \frac{\pi^{-1/2}2^{-1/2}}{\sigma_{1;11}^{1/2}} \int_0^\infty y^{-1/2} exp\left(-\frac{1}{2}\left(\frac{y}{\sigma_{1;11}}\right)\right) f_{\chi^2_{p-1}(\tilde{\lambda}v_{11})}(x)dy.$$

Let f_{p-1} denote the density of the χ^2_{p-1}-distribution. The application of the identity (Muirhead 1982, Theorem 1.3.4)

$$f_{\chi^2_{p-1}(\tilde{\lambda}y)}(x) = exp\left(-\frac{1}{2}\tilde{\lambda}y\right) {}_0F_1\left(\frac{p-1}{2}; \frac{1}{4}\tilde{\lambda}yx\right) f_{p-1}(x),$$

leads to

$$f_T(x) = \frac{\pi^{-1/2}2^{-1/2}}{\sigma_{1;11}^{1/2}} f_{p-1}(x) \times \int_0^\infty y^{-1/2} exp\left(-\frac{1}{2}(\sigma_{1;11}^{-1} + \tilde{\lambda})y\right) {}_0F_1\left(\frac{p-1}{2}; \frac{1}{4}\tilde{\lambda}yx\right) dy.$$

The last integral is calculated by using Lemma 1.3.3 of Muirhead (1982) and finally we get,

$$f_T(x) = \frac{\pi^{-1/2}2^{-1/2}}{\sigma_{1;11}^{1/2}} f_{p-1}(x)\Gamma(1/2)(\sigma_{1;11}^{-1} + \tilde{\lambda})^{-1/2}2^{1/2}$$
$$\times {}_1F_1\left(\frac{1}{2}; \frac{p-1}{2}; \frac{\tilde{\lambda}x}{2(\sigma_{1;11}^{-1} + \tilde{\lambda})}\right)$$
$$= \frac{1}{(1+\sigma_{1;11}\tilde{\lambda})^{1/2}} f_{p-1}(x)\ {}_1F_1\left(\frac{1}{2}; \frac{p-1}{2}; \frac{\tilde{\lambda}x}{2(\sigma_{1;11}^{-1} + \tilde{\lambda})}\right).$$

Noting that $\lambda = \sigma_{1;11}^{-1}\tilde{\lambda}$ completes the proof. The theorem is proved. □

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