Concentration inequalities

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what is concentration?

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We are interested in bounding random fluctuations of functions of many independent random variables.

$X_1, \dots, X_n$ are independent random variables taking values in some set $\mathcal{X}$. Let $f : \mathcal{X}^n \to \mathbb{R}$ and

$$Z = f(X_1, \dots, X_n) .$$

How large are "typical" deviations of Z from $\mathbb{E}Z$?
In particular, we seek upper bounds for

$$\mathbb{P}\{Z > \mathbb{E}Z + t\} \quad \text{and} \quad \mathbb{P}\{Z < \mathbb{E}Z - t\}$$

for $t > 0$.

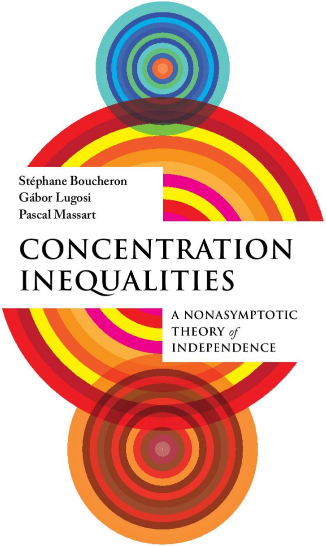
various approaches

- martingales (Yurinskii, 1974; Milman and Schechtman, 1986; Shamir and Spencer, 1987; McDiarmid, 1989,1998);

- information theoretic and transportation methods (Alhswede, Gács, and Körner, 1976; Marton 1986, 1996, 1997; Dembo 1997);

- Talagrand's induction method, 1996;

- logarithmic Sobolev inequalities (Ledoux 1996, Massart 1998, Boucheron, Lugosi, Massart 1999, 2001).



Stéphane Boucheron
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Pascal Massart

CONCENTRATION INEQUALITIES

A NONASYMPTOTIC THEORY *of* INDEPENDENCE

OXFORD

markov's inequality

If $\mathbf{Z} \geq \mathbf{0}$, then

$$\mathbb{P}\{\mathbf{Z} > \mathbf{t}\} \leq \frac{\mathbb{E}\mathbf{Z}}{\mathbf{t}} \; .$$

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This implies Chebyshev's inequality: if $\mathbf{Z}$ has a finite variance $\mathbf{Var}(\mathbf{Z}) = \mathbb{E}(\mathbf{Z} - \mathbb{E}\mathbf{Z})^2$, then

$$\mathbb{P}\{|\mathbf{Z} - \mathbb{E}\mathbf{Z}| > \mathbf{t}\} = \mathbb{P}\{(\mathbf{Z} - \mathbb{E}\mathbf{Z})^2 > \mathbf{t}^2\} \leq \frac{\mathbf{Var}(\mathbf{Z})}{\mathbf{t}^2} \, .$$

markov's inequality

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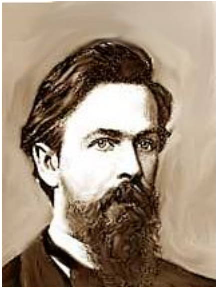

Andrey Markov (1856–1922)

sums of independent random variables

Let $X_1, \ldots, X_n$ be independent real-valued and let $Z = \sum_{i=1}^n X_i$.
By independence, $\mathrm{Var}(Z) = \sum_{i=1}^n \mathrm{Var}(X_i)$. If they are identically distributed, $\mathrm{Var}(Z) = n\mathrm{Var}(X_1)$, so

$$\mathbb{P}\left\{\left|\sum_{i=1}^n X_i - n\mathbb{E}X_1\right| > t\right\} \le \frac{n\mathrm{Var}(X_1)}{t^2} \ .$$

Equivalently,

$$\mathbb{P}\left\{\left|\sum_{i=1}^n X_i - n\mathbb{E}X_1\right| > t\sqrt{n}\right\} \le \frac{\mathrm{Var}(X_1)}{t^2} \ .$$

Typical deviations are at most of the order $\sqrt{n}$.

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Typical deviations are at most of the order $\sqrt{n}$.

Pafnuty Chebyshev (1821–1894)

chernoff bounds

By the central limit theorem,

$$\lim_{n\to\infty} \mathbb{P}\left\{\sum_{i=1}^{n} X_i - n\mathbb{E}X_1 > t\sqrt{n}\right\} = 1 - \Psi(t/\sqrt{\mathrm{Var}(X_1)})$$

$$\leq e^{-t^2/(2\mathrm{Var}(X_1))}$$

so we expect an exponential decrease in $t^2/\mathrm{Var}(X_1)$.

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so we expect an exponential decrease in $t^2/\mathrm{Var}(X_1)$.
Trick: use Markov's inequality in a more clever way: if $\lambda > 0$,

$$\mathbb{P}\{Z - \mathbb{E}Z > t\} = \mathbb{P}\left\{e^{\lambda(Z-\mathbb{E}Z)} > e^{\lambda t}\right\} \leq \frac{\mathbb{E}e^{\lambda(Z-\mathbb{E}Z)}}{e^{\lambda t}}$$

Now derive bounds for the moment generating function $\mathbb{E}e^{\lambda(Z-\mathbb{E}Z)}$ and optimize λ.

chernoff bounds

If $\mathbf{Z} = \sum_{i=1}^{n} \mathbf{X}_i$ is a sum of independent random variables,

$$\mathbb{E}e^{\lambda Z} = \mathbb{E}\prod_{i=1}^{n} e^{\lambda X_i} = \prod_{i=1}^{n} \mathbb{E}e^{\lambda X_i}$$

by independence. Now it suffices to find bounds for $\mathbb{E}e^{\lambda X_i}$.

chernoff bounds

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Serguei Bernstein (1880-1968)

Herman Chernoff (1923–)

hoeffding's inequality

If $X_1, \ldots, X_n \in [0,1]$, then

$$\mathbb{E} e^{\lambda(X_i - \mathbb{E} X_i)} \leq e^{\lambda^2/8} \,.$$

hoeffding's inequality

If $\mathbf{X_1, \ldots, X_n} \in [0,1]$, then

$$\mathbb{E}e^{\lambda(X_i - \mathbb{E}X_i)} \leq e^{\lambda^2/8} .$$

We obtain

$$\mathbb{P}\left\{ \left| \frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \right| > t \right\} \leq 2e^{-2nt^2}$$

Wassily Hoeffding (1914–1991)

bernstein's inequality

Hoeffding's inequality is distribution free. It does not take variance information into account.

Bernstein's inequality is an often useful variant:

Let $X_1, \ldots, X_n$ be independent such that $X_i \leq 1$. Let $v = \sum_{i=1}^n \mathbb{E}\left[X_i^2\right]$. Then

$$\mathbb{P}\left\{\sum_{i=1}^n (X_i - \mathbb{E}X_i) \geq t\right\} \leq \exp\left(-\frac{t^2}{2(v + t/3)}\right) .$$

a maximal inequality

Suppose $\mathbf{Y_1, \ldots, Y_N}$ are sub-Gaussian in the sense that

$$\mathbb{E} e^{\lambda Y_i} \le e^{\lambda^2 \sigma^2 / 2} \ .$$

Then

$$\mathbb{E} \max_{i=1,\ldots,N} Y_i \le \sigma \sqrt{2 \log N} \ .$$

a maximal inequality

Suppose $\mathbf{Y_1, \ldots, Y_N}$ are sub-Gaussian in the sense that

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Then

$$\mathbb{E} \max_{\mathbf{i=1,\ldots,N}} \mathbf{Y_i} \le \sigma \sqrt{\mathbf{2 \log N}} .$$

Proof:

$$\mathbf{e}^{\lambda \mathbb{E} \max_{\mathbf{i=1,\ldots,N}} \mathbf{Y_i}} \le \mathbb{E}\mathbf{e}^{\lambda \max_{\mathbf{i=1,\ldots,N}} \mathbf{Y_i}} \le \sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbb{E}\mathbf{e}^{\lambda \mathbf{Y_i}} \le \mathbf{N}\mathbf{e}^{\lambda^2\sigma^2/2}$$

Take logarithms, and optimize in λ.

an application

Let $A_1, \ldots, A_N \subset \mathcal{X}$ and let $X_1, \ldots, X_n$ be i.i.d. random points in $\mathcal{X}$. Let

$$P(A) = \mathbb{P}\{X_1 \in A\} \quad \text{and} \quad P_n(A) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X_i \in A}$$

By Hoeffding's inequality, for each A,

$$\begin{aligned}\mathbb{E}e^{\lambda(P(A)-P_n(A))} &= \mathbb{E}e^{(\lambda/n)\sum_{i=1}^{n}(P(A)-\mathbb{1}_{X_i\in A})} \\ &= \prod_{i=1}^{n}\mathbb{E}e^{(\lambda/n)(P(A)-\mathbb{1}_{X_i\in A})} \le e^{\lambda^2/(8n)} \,.\end{aligned}$$

By the maximal inequality,

$$\mathbb{E}\max_{j=1,\ldots,N}(P(A_j) - P_n(A_j)) \le \sqrt{\frac{\log N}{2n}} \,.$$

martingale representation

$X_1, \ldots, X_n$ are independent random variables taking values in some set $\mathcal{X}$. Let $f : \mathcal{X}^n \to \mathbb{R}$ and

$$Z = f(X_1, \ldots, X_n) \ .$$

Denote $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot | X_1, \ldots, X_i]$. Thus, $\mathbb{E}_0 Z = \mathbb{E}Z$ and $\mathbb{E}_n Z = Z$.

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Writing

$$\Delta_i = \mathbb{E}_i Z - \mathbb{E}_{i-1} Z ,$$

we have

$$Z - \mathbb{E} Z = \sum_{i=1}^{n} \Delta_i$$

This is the Doob martingale representation of Z.

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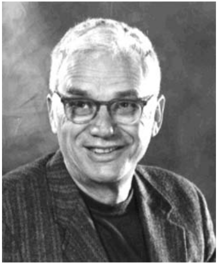

Joseph Leo Doob (1910–2004)

martingale representation: the variance

$$\mathbf{Var}\,(\mathbf{Z}) = \mathbb{E}\left[\left(\sum_{i=1}^{n}\boldsymbol{\Delta}_i\right)^2\right] = \sum_{i=1}^{n}\mathbb{E}\left[\boldsymbol{\Delta}_i^2\right] + 2\sum_{j>i}\mathbb{E}\boldsymbol{\Delta}_i\boldsymbol{\Delta}_j\ .$$

Now if $\mathbf{j} > \mathbf{i}$, $\mathbb{E}_i\boldsymbol{\Delta}_j = \mathbf{0}$, so

$$\mathbb{E}_i\boldsymbol{\Delta}_j\boldsymbol{\Delta}_i = \boldsymbol{\Delta}_i\mathbb{E}_i\boldsymbol{\Delta}_j = \mathbf{0}\ ,$$

We obtain

$$\mathbf{Var}\,(\mathbf{Z}) = \mathbb{E}\left[\left(\sum_{i=1}^{n}\boldsymbol{\Delta}_i\right)^2\right] = \sum_{i=1}^{n}\mathbb{E}\left[\boldsymbol{\Delta}_i^2\right]\ .$$

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$$\mathbf{Var}(\mathbf{Z}) = \mathbb{E}\left[\left(\sum_{i=1}^{n} \Delta_i\right)^2\right] = \sum_{i=1}^{n} \mathbb{E}\left[\Delta_i^2\right] + 2\sum_{j>i} \mathbb{E}\Delta_i\Delta_j \ .$$

Now if $j > i$, $\mathbb{E}_i\Delta_j = 0$, so

$$\mathbb{E}_i\Delta_j\Delta_i = \Delta_i\mathbb{E}_i\Delta_j = 0 \ ,$$

We obtain

$$\mathbf{Var}(\mathbf{Z}) = \mathbb{E}\left[\left(\sum_{i=1}^{n} \Delta_i\right)^2\right] = \sum_{i=1}^{n} \mathbb{E}\left[\Delta_i^2\right] \ .$$

From this, using independence, it is easy derive the Efron-Stein inequality.

efron-stein inequality (1981)

Let $X_1, \ldots, X_n$ be independent random variables taking values in $\mathcal{X}$. Let $f : \mathcal{X}^n \to \mathbb{R}$ and $Z = f(X_1, \ldots, X_n)$.
Then

$$\mathbf{Var}(Z) \leq \mathbb{E}\sum_{i=1}^{n}(Z - \mathbb{E}^{(i)}Z)^2 = \mathbb{E}\sum_{i=1}^{n}\mathbf{Var}^{(i)}(Z) \ .$$

where $\mathbb{E}^{(i)}Z$ is expectation with respect to the i-th variable X_i only.

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We obtain more useful forms by using that

$$\mathbf{Var}(X) = \frac{1}{2}\mathbb{E}(X - X')^2 \quad \text{and} \quad \mathbf{Var}(X) \leq \mathbb{E}(X - a)^2$$

for any constant a.

efron-stein inequality (1981)

If $X'_1, \ldots, X'_n$ are independent copies of $X_1, \ldots, X_n$, and

$$Z'_i = f(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n),$$

then

$$\mathbf{Var}(Z) \leq \frac{1}{2}\mathbb{E}\left[\sum_{i=1}^{n}(Z - Z'_i)^2\right]$$

Z is concentrated if it doesn't depend too much on any of its variables.

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Z is concentrated if it doesn't depend too much on any of its variables.
If $Z = \sum_{i=1}^{n} X_i$ then we have an equality. Sums are the "least concentrated" of all functions!

efron-stein inequality (1981)

If for some arbitrary functions f_i

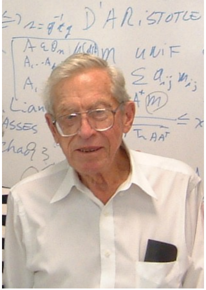
$$Z_i = f_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) ,$$

then

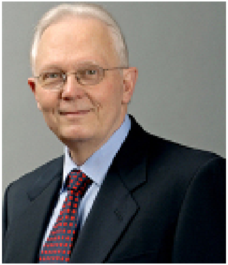
$$\mathbf{Var}(Z) \le \mathbb{E}\left[\sum_{i=1}^{n}(Z - Z_i)^2\right]$$

efron, stein, and steele

Bradley Efron



Charles Stein



Mike Steele

example: kernel density estimation

Let $X_1, \ldots, X_n$ be i.i.d. real samples drawn according to some density ϕ. The kernel density estimate is

$$\phi_n(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right),$$

where $h > 0$, and K is a nonnegative "kernel" $\int K = 1$. The L_1 error is

$$Z = f(X_1, \ldots, X_n) = \int |\phi(x) - \phi_n(x)| dx .$$

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It is easy to see that

$$\begin{aligned} & |f(x_1, \ldots, x_n) - f(x_1, \ldots, x_i', \ldots, x_n)| \\ \leq \; & \frac{1}{nh} \int \left| K\left(\frac{x - x_i}{h}\right) - K\left(\frac{x - x_i'}{h}\right) \right| dx \leq \frac{2}{n} \, , \end{aligned}$$

so we get $\quad \mathbf{Var}(Z) \leq \frac{2}{n}$.

example: uniform deviations

Let $\mathcal{A}$ be a collection of subsets of $\mathcal{X}$, and let $X_1, \ldots, X_n$ be n random points in $\mathcal{X}$ drawn i.i.d.
Let

$$P(A) = \mathbb{P}\{X_1 \in A\} \quad \text{and} \quad P_n(A) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{X_i \in A}$$

If $Z = \sup_{A \in \mathcal{A}} |P(A) - P_n(A)|$,

$$\mathrm{Var}(Z) \leq \frac{1}{2n}$$

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Let

$$P(A) = \mathbb{P}\{X_1 \in A\} \quad \text{and} \quad P_n(A) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X_i \in A}$$

If $Z = \sup_{A \in \mathcal{A}} |P(A) - P_n(A)|$,

$$\mathrm{Var}(Z) \leq \frac{1}{2n}$$

regardless of the distribution and the richness of $\mathcal{A}$.

bounding the expectation

Let $\mathbf{P}_n'(A) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{X_i' \in A}$ and let $\mathbb{E}'$ denote expectation only with respect to $X_1', \ldots, X_n'$.

$$\mathbb{E} \sup_{A \in \mathcal{A}} |P_n(A) - P(A)| = \mathbb{E} \sup_{A \in \mathcal{A}} |\mathbb{E}'[P_n(A) - P_n'(A)]|$$

$$\leq \mathbb{E} \sup_{A \in \mathcal{A}} |P_n(A) - P_n'(A)| = \frac{1}{n}\mathbb{E} \sup_{A \in \mathcal{A}} \left| \sum_{i=1}^n (\mathbb{1}_{X_i \in A} - \mathbb{1}_{X_i' \in A}) \right|$$

bounding the expectation

Let $P'_n(A) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X'_i \in A}$ and let $\mathbb{E}'$ denote expectation only with respect to $X'_1, \ldots, X'_n$.

$$\mathbb{E} \sup_{A \in \mathcal{A}} |P_n(A) - P(A)| = \mathbb{E} \sup_{A \in \mathcal{A}} |\mathbb{E}'[P_n(A) - P'_n(A)]|$$

$$\leq \mathbb{E} \sup_{A \in \mathcal{A}} |P_n(A) - P'_n(A)| = \frac{1}{n}\mathbb{E} \sup_{A \in \mathcal{A}} \left| \sum_{i=1}^{n} (\mathbb{1}_{X_i \in A} - \mathbb{1}_{X'_i \in A}) \right|$$

Second symmetrization: if $\varepsilon_1, \ldots, \varepsilon_n$ are independent Rademacher variables, then

$$= \frac{1}{n}\mathbb{E} \sup_{A \in \mathcal{A}} \left| \sum_{i=1}^{n} \varepsilon_i (\mathbb{1}_{X_i \in A} - \mathbb{1}_{X'_i \in A}) \right| \leq \frac{2}{n}\mathbb{E} \sup_{A \in \mathcal{A}} \left| \sum_{i=1}^{n} \varepsilon_i \mathbb{1}_{X_i \in A} \right|$$

conditional rademacher average

If

$$R_n = \mathbb{E}_\varepsilon \sup_{A \in \mathcal{A}} \left| \sum_{i=1}^{n} \varepsilon_i \mathbb{1}_{X_i \in A} \right|$$

then

$$\mathbb{E} \sup_{A \in \mathcal{A}} |P_n(A) - P(A)| \le \frac{2}{n} \mathbb{E} R_n .$$

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then

$$\mathbb{E} \sup_{A \in \mathcal{A}} |P_n(A) - P(A)| \leq \frac{2}{n} \mathbb{E} R_n \ .$$

$\mathbf{R_n}$ is a data-dependent quantity!

concentration of conditional rademacher average

Define

$$\mathbf{R}_n^{(i)} = \mathbb{E}_\varepsilon \sup_{A \in \mathcal{A}} \left| \sum_{j \neq i} \varepsilon_j \mathbb{1}_{X_j \in A} \right|$$

One can show easily that

$$0 \leq \mathbf{R}_n - \mathbf{R}_n^{(i)} \leq 1 \quad \text{and} \quad \sum_{i=1}^{n} (\mathbf{R}_n - \mathbf{R}_n^{(i)}) \leq \mathbf{R}_n \ .$$

By the Efron-Stein inequality,

$$\mathbf{Var}(\mathbf{R}_n) \leq \mathbb{E} \sum_{i=1}^{n} (\mathbf{R}_n - \mathbf{R}_n^{(i)})^2 \leq \mathbb{E}\mathbf{R}_n \ .$$

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By the Efron-Stein inequality,

$$\mathbf{Var}(\mathbf{R_n}) \le \mathbb{E} \sum_{i=1}^{n} (\mathbf{R_n} - \mathbf{R_n^{(i)}})^2 \le \mathbb{E}\mathbf{R_n} \ .$$

Standard deviation is at most $\sqrt{\mathbb{E}\mathbf{R_n}}$!

concentration of conditional rademacher average

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One can show easily that

$$0 \leq R_n - R_n^{(i)} \leq 1 \quad \text{and} \quad \sum_{i=1}^{n} (R_n - R_n^{(i)}) \leq R_n \ .$$

By the Efron-Stein inequality,

$$\mathrm{Var}(R_n) \leq \mathbb{E} \sum_{i=1}^{n} (R_n - R_n^{(i)})^2 \leq \mathbb{E} R_n \ .$$

Standard deviation is at most $\sqrt{\mathbb{E} R_n}$!

Such functions are called self-bounding.

bounding the conditional rademacher average

If $S(X_1^n, \mathcal{A})$ is the number of different sets of form

$$\{X_1, \ldots, X_n\} \cap A : A \in \mathcal{A}$$

then R_n is the maximum of $S(X_1^n, \mathcal{A})$ sub-Gaussian random variables. By the maximal inequality,

$$\frac{1}{2} R_n \leq \sqrt{\frac{\log S(X_1^n, \mathcal{A})}{2n}} \; .$$

bounding the conditional rademacher average

If $\mathbf{S(X_1^n, \mathcal{A})}$ is the number of different sets of form

$$\{X_1, \ldots, X_n\} \cap A : A \in \mathcal{A}$$

then $\mathbf{R_n}$ is the maximum of $\mathbf{S(X_1^n, \mathcal{A})}$ sub-Gaussian random variables. By the maximal inequality,

$$\frac{1}{2} R_n \leq \sqrt{\frac{\log S(X_1^n, \mathcal{A})}{2n}} \ .$$

In particular,

$$\mathbb{E} \sup_{A \in \mathcal{A}} |P_n(A) - P(A)| \leq 2\mathbb{E}\sqrt{\frac{\log S(X_1^n, \mathcal{A})}{2n}} \ .$$

random VC dimension

Let $\mathbf{V} = \mathbf{V}(x_1^n, \mathcal{A})$ be the size of the largest subset of $\{x_1, \ldots, x_n\}$ shattered by $\mathcal{A}$.
By Sauer's lemma,

$$\log S(X_1^n, \mathcal{A}) \leq V(X_1^n, \mathcal{A}) \log(n + 1) \ .$$

random VC dimension

Let $\mathbf{V} = \mathbf{V}(\mathbf{x}_1^n, \mathcal{A})$ be the size of the largest subset of $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ shattered by $\mathcal{A}$.
By Sauer's lemma,

$$\log \mathbf{S}(\mathbf{X}_1^n, \mathcal{A}) \leq \mathbf{V}(\mathbf{X}_1^n, \mathcal{A}) \log(\mathbf{n}+1) \ .$$

$\mathbf{V}$ is also self-bounding:

$$\sum_{i=1}^{n} (\mathbf{V} - \mathbf{V}^{(i)})^2 \leq \mathbf{V}$$

so by Efron-Stein,

$$\mathbf{Var}(\mathbf{V}) \leq \mathbb{E}\mathbf{V}$$

vapnik and chervonenkis

Vladimir Vapnik

Alexey Chervonenkis

beyond the variance

$X_1, \dots, X_n$ are independent random variables taking values in some set $\mathcal{X}$. Let $f : \mathcal{X}^n \to \mathbb{R}$ and $Z = f(X_1, \dots, X_n)$. Recall the Doob martingale representation:

$$Z - \mathbb{E}Z = \sum_{i=1}^{n} \Delta_i \quad \text{where} \quad \Delta_i = \mathbb{E}_i Z - \mathbb{E}_{i-1} Z \ ,$$

with $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot | X_1, \dots, X_i]$.

To get exponential inequalities, we bound the moment generating function $\mathbb{E}e^{\lambda(Z - \mathbb{E}Z)}$.

azuma's inequality

Suppose that the martingale differences are bounded: $|\Delta_i| \leq c_i$. Then

$$\begin{aligned}
\mathbb{E}e^{\lambda(Z-\mathbb{E}Z)} &= \mathbb{E}e^{\lambda\left(\sum_{i=1}^{n}\Delta_i\right)} = \mathbb{E}\mathbb{E}_n e^{\lambda\left(\sum_{i=1}^{n-1}\Delta_i\right)+\lambda\Delta_n} \\
&= \mathbb{E}e^{\lambda\left(\sum_{i=1}^{n-1}\Delta_i\right)}\mathbb{E}_n e^{\lambda\Delta_n} \\
&\leq \mathbb{E}e^{\lambda\left(\sum_{i=1}^{n-1}\Delta_i\right)} e^{\lambda^2 c_n^2/2} \text{ (by Hoeffding)} \\
&\cdots \\
&\leq e^{\lambda^2\left(\sum_{i=1}^{n} c_i^2\right)/2} .
\end{aligned}$$

This is the Azuma-Hoeffding inequality for sums of bounded martingale differences.

bounded differences inequality

If $\mathbf{Z} = \mathbf{f}(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ and $\mathbf{f}$ is such that

$$|\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n) - \mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_i', \ldots, \mathbf{x}_n)| \leq \mathbf{c}_i$$

then the martingale differences are bounded.

bounded differences inequality

If $Z = f(X_1, \ldots, X_n)$ and f is such that

$$|f(x_1, \ldots, x_n) - f(x_1, \ldots, x_i', \ldots, x_n)| \leq c_i$$

then the martingale differences are bounded.

Bounded differences inequality: if $X_1, \ldots, X_n$ are independent, then

$$\mathbb{P}\{|Z - \mathbb{E}Z| > t\} \leq 2e^{-2t^2/\sum_{i=1}^n c_i^2} .$$

bounded differences inequality

If $\mathbf{Z} = \mathbf{f}(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ and $\mathbf{f}$ is such that

$$|f(x_1, \ldots, x_n) - f(x_1, \ldots, x_i', \ldots, x_n)| \le c_i$$

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Bounded differences inequality: if $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are independent, then

$$\mathbb{P}\{|Z - \mathbb{E}Z| > t\} \le 2e^{-2t^2/\sum_{i=1}^n c_i^2} .$$

McDiarmid's inequality.

Colin McDiarmid

hoeffding in a hilbert space

Let $\mathbf{X_1, \dots, X_n}$ be independent zero-mean random variables in a separable Hilbert space such that $\|\mathbf{X_i}\| \le \mathbf{c/2}$ and denote $\mathbf{v = nc^2/4}$. Then, for all $\mathbf{t} \ge \sqrt{\mathbf{v}}$,

$$\mathbb{P}\left\{\left\|\sum_{i=1}^{n} \mathbf{X_i}\right\| > \mathbf{t}\right\} \le \mathbf{e}^{-(\mathbf{t}-\sqrt{\mathbf{v}})^2/(2\mathbf{v})} .$$

hoeffding in a hilbert space

Let $X_1, \ldots, X_n$ be independent zero-mean random variables in a separable Hilbert space such that $\|X_i\| \le c/2$ and denote $v = nc^2/4$. Then, for all $t \ge \sqrt{v}$,

$$\mathbb{P}\left\{\left\|\sum_{i=1}^n X_i\right\| > t\right\} \le e^{-(t-\sqrt{v})^2/(2v)} .$$

Proof: By the triangle inequality, $\left\|\sum_{i=1}^n X_i\right\|$ has the bounded differences property with constants c, so

$$\mathbb{P}\left\{\left\|\sum_{i=1}^n X_i\right\| > t\right\} = \mathbb{P}\left\{\left\|\sum_{i=1}^n X_i\right\| - \mathbb{E}\left\|\sum_{i=1}^n X_i\right\| > t - \mathbb{E}\left\|\sum_{i=1}^n X_i\right\|\right\}$$

$$\le \exp\left(-\frac{\left(t - \mathbb{E}\left\|\sum_{i=1}^n X_i\right\|\right)^2}{2v}\right) .$$

Also,

$$\mathbb{E}\left\|\sum_{i=1}^n X_i\right\| \le \sqrt{\mathbb{E}\left\|\sum_{i=1}^n X_i\right\|^2} = \sqrt{\sum_{i=1}^n \mathbb{E}\|X_i\|^2} \le \sqrt{v} .$$

bounded differences inequality

✳Easy to use.

✳Distribution free.

✳Often close to optimal (e.g., $\mathbf{L_1}$ error of kernel density estimate).

✳Does not exploit “variance information.”

✳Often too rigid.

✳Other methods are necessary.

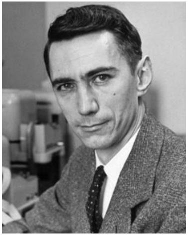
shannon entropy

If $\mathbf{X}, \mathbf{Y}$ are random variables taking values in a set of size $\mathbf{N}$,

$$\mathbf{H(X) = -\sum_{x} p(x) \log p(x)}$$

$$\begin{aligned}\mathbf{H(X|Y)} &= \mathbf{H(X, Y) - H(Y)} \\ &= \mathbf{-\sum_{x,y} p(x, y) \log p(x|y)}\end{aligned}$$

$\mathbf{H(X) \leq \log N}$ and $\mathbf{H(X|Y) \leq H(X)}$



Claude Shannon
(1916–2001)

han's inequality

Te Sun Han

If $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{X}^{(i)} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$, then

$$\sum_{i=1}^{n} \left(H(\mathbf{X}) - H(\mathbf{X}^{(i)}) \right) \leq H(\mathbf{X})$$

Proof:

$$\begin{aligned} H(\mathbf{X}) &= H(\mathbf{X}^{(i)}) + H(X_i | \mathbf{X}^{(i)}) \\ &\leq H(\mathbf{X}^{(i)}) + H(X_i | X_1, \ldots, X_{i-1}) \end{aligned}$$

Since $\sum_{i=1}^{n} H(X_i | X_1, \ldots, X_{i-1}) = H(\mathbf{X})$, summing the inequality, we get

$$(n-1) H(\mathbf{X}) \leq \sum_{i=1}^{n} H(\mathbf{X}^{(i)}) .$$

edge isoperimetric inequality on the hypercube

Let $\mathbf{A} \subset \{-1,1\}^n$. Let $\mathbf{E(A)}$ be the collection of pairs $\mathbf{x}, \mathbf{x}' \in \mathbf{A}$ such that $\mathbf{d_H(x, x') = 1}$. Then

$$|\mathbf{E(A)}| \leq \frac{|\mathbf{A}|}{2} \times \log_2 |\mathbf{A}| \ .$$

Proof: Let $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ be uniformly distributed over $\mathbf{A}$.
Then $\mathbf{p(x)} = \mathbb{1}_{\mathbf{x} \in \mathbf{A}} / |\mathbf{A}|$.
Clearly, $\mathbf{H(X)} = \log |\mathbf{A}|$. Also,

$$\mathbf{H(X)} - \mathbf{H(X^{(i)})} = \mathbf{H(X_i | X^{(i)})} = -\sum_{\mathbf{x} \in \mathbf{A}} \mathbf{p(x)} \log \mathbf{p(x_i | x^{(i)})} \ .$$

For $\mathbf{x} \in \mathbf{A}$,

$$\mathbf{p(x_i | x^{(i)})} = \begin{cases} 1/2 & \text{if } \overline{\mathbf{x}}^{(i)} \in \mathbf{A} \\ 1 & \text{otherwise} \end{cases}$$

where $\overline{\mathbf{x}}^{(i)} = (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, -\mathbf{x}_i, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n)$.

$$\mathbf{H(X) - H(X^{(i)}) = \frac{\log 2}{|A|} \sum_{x \in A} \mathbb{1}_{x, \bar{x}^{(i)} \in A}}$$

and therefore

$$\sum_{i=1}^{n} \left(\mathbf{H(X) - H(X^{(i)})} \right) = \frac{\log 2}{|A|} \sum_{x \in A} \sum_{i=1}^{n} \mathbb{1}_{x, \bar{x}^{(i)} \in A} = \frac{|E(A)|}{|A|} 2 \log 2 \, .$$

Thus, by Han's inequality,

$$\frac{|E(A)|}{|A|} 2 \log 2 = \sum_{i=1}^{n} \left(\mathbf{H(X) - H(X^{(i)})} \right) \leq \mathbf{H(X)} = \log |A| \, .$$

This is equivalent to the edge isoperimetric inequality on the hypercube: if

$$\partial_E(A) = \left\{(x, x') : x \in A, x' \in A^c, d_H(x, x') = 1\right\} .$$

is the edge boundary of A, then

$$|\partial_E(A)| \geq \log_2 \frac{2^n}{|A|} \times |A|$$

Equality is achieved for sub-cubes.

VC entropy is self-bounding

Let $\mathcal{A}$ is a class of subsets of $\mathbf{X}$ and $x = (x_1, \ldots, x_n) \in \mathcal{X}^n$.
Recall that $S(x, \mathcal{A})$ is the number of different sets of form

$$\{x_1, \ldots, x_n\} \cap A : A \in \mathcal{A}$$

Let $f_n(x) = \log_2 S(x, \mathcal{A})$ be the VC entropy.
Then $0 \leq f_n(x) - f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_n) \leq 1$ and

$$\sum_{i=1}^{n} (f_n(x) - f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_n)) \leq f_n(x) .$$

Proof: Put the uniform distribution on the class of sets $\{x_1, \ldots, x_n\} \cap A$ and use Han's inequality.

VC entropy is self-bounding

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Proof: Put the uniform distribution on the class of sets $\{x_1, \ldots, x_n\} \cap A$ and use Han's inequality.
Corollary: if $X_1, \ldots, X_n$ are independent, then

$$\mathrm{Var}(\log_2 S(X, \mathcal{A})) \leq \mathbb{E} \log_2 S(X, \mathcal{A}) \ .$$

subadditivity of entropy

The entropy of a random variable $\mathbf{Z} \geq \mathbf{0}$ is

$$\mathbf{Ent}(\mathbf{Z}) = \mathbb{E}\mathbf{\Phi}(\mathbf{Z}) - \mathbf{\Phi}(\mathbb{E}\mathbf{Z})$$

where $\mathbf{\Phi}(\mathbf{x}) = \mathbf{x} \log \mathbf{x}$. By Jensen's inequality, $\mathbf{Ent}(\mathbf{Z}) \geq \mathbf{0}$.

subadditivity of entropy

The entropy of a random variable $Z \geq 0$ is

$$\mathrm{Ent}(Z) = \mathbb{E}\Phi(Z) - \Phi(\mathbb{E}Z)$$

where $\Phi(x) = x \log x$. By Jensen's inequality, $\mathrm{Ent}(Z) \geq 0$.

Han's inequality implies the following sub-additivity property.
Let $X_1, \ldots, X_n$ be independent and let $Z = f(X_1, \ldots, X_n)$, where $f \geq 0$.
Denote

$$\mathrm{Ent}^{(i)}(Z) = \mathbb{E}^{(i)}\Phi(Z) - \Phi(\mathbb{E}^{(i)}Z)$$

Then

$$\mathrm{Ent}(Z) \leq \mathbb{E} \sum_{i=1}^{n} \mathrm{Ent}^{(i)}(Z) \ .$$

a logarithmic sobolev inequality on the hypercube

Let $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ be uniformly distributed over $\{-1, 1\}^n$. If $\mathbf{f} : \{-1, 1\}^n \to \mathbb{R}$ and $\mathbf{Z} = \mathbf{f}(\mathbf{X})$,

$$\mathbf{Ent}(\mathbf{Z}^2) \leq \frac{1}{2}\mathbb{E}\sum_{i=1}^{n}(\mathbf{Z} - \mathbf{Z}_i')^2$$

The proof uses subadditivity of the entropy and calculus for the case $\mathbf{n} = 1$.

Implies Efron-Stein.

herbst's argument: exponential concentration

If $\mathbf{f : \{-1,1\}^n \to \mathbb{R}}$, the log-Sobolev inequality may be used with

$$\mathbf{g(x) = e^{\lambda f(x)/2}} \quad \text{where} \quad \boldsymbol{\lambda} \in \mathbb{R} \ .$$

If $\mathbf{F(\lambda) = \mathbb{E}e^{\lambda Z}}$ is the moment generating function of $\mathbf{Z = f(X)}$,

$$\begin{aligned}\mathbf{Ent(g(X)^2)} &= \lambda \mathbb{E}\left[\mathbf{Ze^{\lambda Z}}\right] - \mathbb{E}\left[\mathbf{e^{\lambda Z}}\right] \log \mathbb{E}\left[\mathbf{Ze^{\lambda Z}}\right] \\ &= \lambda \mathbf{F'(\lambda) - F(\lambda) \log F(\lambda)} \ .\end{aligned}$$

Differential inequalities are obtained for $\mathbf{F(\lambda)}$.

herbst's argument

As an example, suppose $\mathbf{f}$ is such that $\sum_{i=1}^{n}(Z - Z_i')_+^2 \leq v$. Then by the log-Sobolev inequality,

$$\lambda F'(\lambda) - F(\lambda)\log F(\lambda) \leq \frac{v\lambda^2}{4} F(\lambda)$$

If $G(\lambda) = \log F(\lambda)$, this becomes

$$\left(\frac{G(\lambda)}{\lambda}\right)' \leq \frac{v}{4} \;.$$

This can be integrated: $G(\lambda) \leq \lambda \mathbb{E}Z + \lambda v/4$, so

$$F(\lambda) \leq e^{\lambda \mathbb{E}Z - \lambda^2 v/4}$$

This implies

$$\mathbb{P}\{Z > \mathbb{E}Z + t\} \leq e^{-t^2/v}$$

herbst's argument

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This implies

$$\mathbb{P}\{Z > \mathbb{E}Z + t\} \leq e^{-t^2/v}$$

Stronger than the bounded differences inequality!

gaussian log-sobolev inequality

Let $\mathbf{X} = (\mathbf{X_1}, \ldots, \mathbf{X_n})$ be a vector of i.i.d. standard normal If $\mathbf{f} : \mathbb{R}^\mathbf{n} \to \mathbb{R}$ and $\mathbf{Z} = \mathbf{f}(\mathbf{X})$,

$$\mathbf{Ent}(\mathbf{Z}^2) \leq 2\mathbb{E}\left[\|\nabla \mathbf{f}(\mathbf{X})\|^2\right]$$

(Gross, 1975).

gaussian log-sobolev inequality

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$$\mathbf{Ent}(\mathbf{Z}^2) \leq 2\mathbb{E}\left[\|\nabla \mathbf{f}(\mathbf{X})\|^2\right]$$

(Gross, 1975).

Proof sketch: By the subadditivity of entropy, it suffices to prove it for $\mathbf{n} = 1$.

Approximate $\mathbf{Z} = \mathbf{f}(\mathbf{X})$ by

$$\mathbf{f}\left(\frac{1}{\sqrt{m}} \sum_{i=1}^{m} \varepsilon_i\right)$$

where the ε_i are i.i.d. Rademacher random variables.

Use the log-Sobolev inequality of the hypercube and the central limit theorem.

gaussian concentration inequality

Herbst't argument may now be repeated:
Suppose $\mathbf{f}$ is Lipschitz: for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbf{n}}$,

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq \mathbf{L}\|\mathbf{x} - \mathbf{y}\| \ .$$

Then, for all $\mathbf{t} > \mathbf{0}$,

$$\mathbb{P}\left\{\mathbf{f}(\mathbf{X}) - \mathbb{E}\mathbf{f}(\mathbf{X}) \geq \mathbf{t}\right\} \leq \mathbf{e}^{-\mathbf{t}^2/(2\mathbf{L}^2)} \ .$$

(Tsirelson, Ibragimov, and Sudakov, 1976).

an application: supremum of a gaussian process

Let $(X_t)_{t \in \mathcal{T}}$ be an almost surely continuous centered Gaussian process. Let $Z = \sup_{t \in \mathcal{T}} X_t$. If

$$\sigma^2 = \sup_{t \in \mathcal{T}} \left(\mathbb{E}\left[X_t^2\right] \right) ,$$

then

$$\mathbb{P}\left\{|Z - \mathbb{E}Z| \geq u\right\} \leq 2e^{-u^2/(2\sigma^2)}$$

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then

$$\mathbb{P}\{|Z - \mathbb{E}Z| \geq u\} \leq 2e^{-u^2/(2\sigma^2)}$$

Proof: We may assume $\mathcal{T} = \{1, ..., n\}$. Let Γ be the covariance matrix of $X = (X_1, \ldots, X_n)$. Let $A = \Gamma^{1/2}$. If Y is a standard normal vector, then

$$f(Y) = \max_{i=1,\ldots,n} (AY)_i \overset{\text{distr.}}{=} \max_{i=1,\ldots,n} X_i$$

By Cauchy-Schwarz,

$$|(Au)_i - (Av)_i| = \left|\sum_j A_{i,j}(u_j - v_j)\right| \leq \left(\sum_j A_{i,j}^2\right)^{1/2} \|u - v\|$$

$$\leq \sigma\|u - v\|$$

beyond bernoulli and gaussian: the entropy method

For general distributions, logarithmic Sobolev inequalities are not available.

Solution: modified logarithmic Sobolev inequalities.
Suppose $X_1, \ldots, X_n$ are independent. Let $Z = f(X_1, \ldots, X_n)$ and $Z_i = f_i(X^{(i)}) = f_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$.

Let $\phi(x) = e^x - x - 1$. Then for all $\lambda \in \mathbb{R}$,

$$\lambda \mathbb{E}\left[Ze^{\lambda Z}\right] - \mathbb{E}\left[e^{\lambda Z}\right] \log \mathbb{E}\left[e^{\lambda Z}\right] \leq \sum_{i=1}^{n} \mathbb{E}\left[e^{\lambda Z} \phi\left(-\lambda(Z - Z_i)\right)\right].$$

Michel Ledoux

the entropy method

Define $\mathbf{Z_i = \inf_{x_i'} f(X_1, \ldots, x_i', \ldots, X_n)}$ and suppose

$$\sum_{i=1}^{n} (Z - Z_i)^2 \leq v \, .$$

Then for all $t > 0$,

$$\mathbb{P}\{Z - \mathbb{E}Z > t\} \leq e^{-t^2/(2v)} \, .$$

the entropy method

Define $\mathbf{Z_i = \inf_{x_i'} f(X_1, \ldots, x_i', \ldots, X_n)}$ and suppose

$$\sum_{i=1}^{n} (Z - Z_i)^2 \leq v \ .$$

Then for all $t > 0$,

$$\mathbb{P}\{Z - \mathbb{E}Z > t\} \leq e^{-t^2/(2v)} \ .$$

This implies the bounded differences inequality and much more.

example: the largest eigenvalue of a symmetric matrix

Let $A = (X_{i,j})_{n\times n}$ be symmetric, the $X_{i,j}$ independent ($i \le j$) with $|X_{i,j}| \le 1$. Let

$$Z = \lambda_1 = \sup_{u:\|u\|=1} u^T A u .$$

and suppose v is such that $Z = v^T A v$.
$A'_{i,j}$ is obtained by replacing $X_{i,j}$ by $x'_{i,j}$. Then

$$\begin{aligned}(Z - Z_{i,j})_+ &\le \left(v^T A v - v^T A'_{i,j} v\right) \mathbb{1}_{Z > Z_{i,j}} \\ &= \left(v^T (A - A'_{i,j}) v\right) \mathbb{1}_{Z > Z_{i,j}} \le 2\left(v_i v_j (X_{i,j} - X'_{i,j})\right)_+ \\ &\le 4|v_i v_j| .\end{aligned}$$

Therefore,

$$\sum_{1\le i\le j\le n} (Z - Z'_{i,j})_+^2 \le \sum_{1\le i\le j\le n} 16|v_i v_j|^2 \le 16\left(\sum_{i=1}^n v_i^2\right)^2 = 16 .$$

example: convex lipschitz functions

Let $\mathbf{f} : [0,1]^n \to \mathbb{R}$ be a convex function. Let $Z_i = \inf_{x_i'} f(X_1, \ldots, x_i', \ldots, X_n)$ and let X_i' be the value of x_i' for which the minimum is achieved. Then, writing $\overline{X}^{(i)} = (X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)$,

$$\sum_{i=1}^n (Z - Z_i)^2 = \sum_{i=1}^n (f(X) - f(\overline{X}^{(i)}))^2$$

$$\leq \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}(X) \right)^2 (X_i - X_i')^2$$

(by convexity)

$$\leq \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}(X) \right)^2$$

$$= \|\nabla f(X)\|^2 \leq L^2 \ .$$

convex lipschitz functions

If $\mathbf{f : [0,1]^n \to \mathbb{R}}$ is a convex Lipschitz function and $\mathbf{X_1, \ldots, X_n}$ are independent taking values in $\mathbf{[0,1]}$, $\mathbf{Z = f(X_1, \ldots, X_n)}$ satisfies

$$\mathbb{P}\{\mathbf{Z > \mathbb{E}Z + t}\} \leq \mathbf{e^{-t^2/(2L^2)}} \; .$$

convex lipschitz functions

If $\mathbf{f : [0,1]^n \to \mathbb{R}}$ is a convex Lipschitz function and $\mathbf{X_1, \ldots, X_n}$ are independent taking values in $\mathbf{[0,1]}$, $\mathbf{Z = f(X_1, \ldots, X_n)}$ satisfies

$$\mathbb{P}\{Z > \mathbb{E}Z + t\} \le e^{-t^2/(2L^2)} .$$

A similar lower tail bound also holds.

self-bounding functions

Suppose $\mathbf{Z}$ satisfies

$$0 \leq Z - Z_i \leq 1 \quad \text{and} \quad \sum_{i=1}^{n} (Z - Z_i) \leq Z .$$

Recall that $\mathbf{Var}(Z) \leq \mathbb{E}Z$. We have much more:

$$\mathbb{P}\{Z > \mathbb{E}Z + t\} \leq e^{-t^2/(2\mathbb{E}Z + 2t/3)}$$

and

$$\mathbb{P}\{Z < \mathbb{E}Z - t\} \leq e^{-t^2/(2\mathbb{E}Z)}$$

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$$\mathbb{P}\{Z < \mathbb{E}Z - t\} \leq e^{-t^2/(2\mathbb{E}Z)}$$

Rademacher averages, random VC dimension, random VC entropy, longest increasing subsequence in a random permutation, are all examples of self bounding functions.

self-bounding functions

Suppose $\mathbf{Z}$ satisfies

$$\mathbf{0 \le Z - Z_i \le 1} \quad \text{and} \quad \sum_{i=1}^{n} (\mathbf{Z} - \mathbf{Z_i}) \le \mathbf{Z} \; .$$

Recall that $\mathbf{Var(Z)} \le \mathbb{E}\mathbf{Z}$. We have much more:

$$\mathbb{P}\{\mathbf{Z} > \mathbb{E}\mathbf{Z} + \mathbf{t}\} \le \mathbf{e}^{-\mathbf{t}^2/(2\mathbb{E}\mathbf{Z} + 2\mathbf{t}/3)}$$

and

$$\mathbb{P}\{\mathbf{Z} < \mathbb{E}\mathbf{Z} - \mathbf{t}\} \le \mathbf{e}^{-\mathbf{t}^2/(2\mathbb{E}\mathbf{Z})}$$

Rademacher averages, random VC dimension, random VC entropy, longest increasing subsequence in a random permutation, are all examples of self bounding functions.

Configuration functions.

exponential efron-stein inequality

Define

$$V^+ = \sum_{i=1}^{n} \mathbb{E}' \left[(Z - Z_i')_+^2 \right]$$

and

$$V^- = \sum_{i=1}^{n} \mathbb{E}' \left[(Z - Z_i')_-^2 \right] .$$

By Efron-Stein,

$$\mathbf{Var}(Z) \leq \mathbb{E}V^+ \quad \text{and} \quad \mathbf{Var}(Z) \leq \mathbb{E}V^- .$$

exponential efron-stein inequality

Define

$$\mathbf{V}^+ = \sum_{i=1}^{n} \mathbb{E}' \left[(\mathbf{Z} - \mathbf{Z}_i')_+^2 \right]$$

and

$$\mathbf{V}^- = \sum_{i=1}^{n} \mathbb{E}' \left[(\mathbf{Z} - \mathbf{Z}_i')_-^2 \right] .$$

By Efron-Stein,

$$\mathrm{Var}(\mathbf{Z}) \le \mathbb{E}\mathbf{V}^+ \quad \text{and} \quad \mathrm{Var}(\mathbf{Z}) \le \mathbb{E}\mathbf{V}^- .$$

The following exponential versions hold for all $\lambda, \theta > 0$ with $\lambda\theta < 1$:

$$\log \mathbb{E}e^{\lambda(\mathbf{Z}-\mathbb{E}\mathbf{Z})} \le \frac{\lambda\theta}{1-\lambda\theta} \log \mathbb{E}e^{\lambda \mathbf{V}^+/\theta} .$$

If also $\mathbf{Z}_i' - \mathbf{Z} \le 1$ for every $\mathbf{i}$, fhen for all $\lambda \in (0, 1/2)$,

$$\log \mathbb{E}e^{\lambda(\mathbf{Z}-\mathbb{E}\mathbf{Z})} \le \frac{2\lambda}{1-2\lambda} \log \mathbb{E}e^{\lambda \mathbf{V}^-} .$$

weakly self-bounding functions

$f : \mathcal{X}^n \to [0, \infty)$ is weakly (a, b)-self-bounding if there exist $f_i : \mathcal{X}^{n-1} \to [0, \infty)$ such that for all $x \in \mathcal{X}^n$,

$$\sum_{i=1}^{n} \left(f(x) - f_i(x^{(i)})\right)^2 \leq af(x) + b\,.$$

weakly self-bounding functions

$f : \mathcal{X}^n \to [0, \infty)$ is weakly (a, b)-self-bounding if there exist $f_i : \mathcal{X}^{n-1} \to [0, \infty)$ such that for all $x \in \mathcal{X}^n$,

$$\sum_{i=1}^{n} \left(f(x) - f_i(x^{(i)}) \right)^2 \leq af(x) + b \, .$$

Then

$$\mathbb{P}\{Z \geq \mathbb{E}Z + t\} \leq \exp\left(-\frac{t^2}{2\left(a\mathbb{E}Z + b + at/2\right)} \right) \, .$$

weakly self-bounding functions

$\mathbf{f} : \mathcal{X}^n \to [0, \infty)$ is weakly $(\mathbf{a}, \mathbf{b})$-self-bounding if there exist
$\mathbf{f_i} : \mathcal{X}^{n-1} \to [0, \infty)$ such that for all $\mathbf{x} \in \mathcal{X}^n$,

$$\sum_{i=1}^{n} \left(f(x) - f_i(x^{(i)}) \right)^2 \leq a f(x) + b \, .$$

Then

$$\mathbb{P}\{Z \geq \mathbb{E}Z + t\} \leq \exp\left(-\frac{t^2}{2\left(a\mathbb{E}Z + b + at/2\right)} \right) \; .$$

If, in addition, $f(x) - f_i(x^{(i)}) \leq 1$, then for $0 < t \leq \mathbb{E}Z$,

$$\mathbb{P}\{Z \leq \mathbb{E}Z - t\} \leq \exp\left(-\frac{t^2}{2\left(a\mathbb{E}Z + b + c\_t\right)} \right) \; .$$

where $c = (3a - 1)/6$.

the isoperimetric view

Let $X = (X_1, \ldots, X_n)$ have independent components, taking values in $\mathcal{X}^n$. Let $A \subset \mathcal{X}^n$.

The Hamming distance of X to A is

$$d(X, A) = \min_{y \in A} d(X, y) = \min_{y \in A} \sum_{i=1}^{n} \mathbb{1}_{X_i \neq y_i} .$$

Michel Talagrand

the isoperimetric view

Let $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ have independent components, taking values in $\mathcal{X}^n$. Let $\mathbf{A} \subset \mathcal{X}^n$.

The Hamming distance of $\mathbf{X}$ to $\mathbf{A}$ is

$$\mathbf{d}(\mathbf{X}, \mathbf{A}) = \min_{\mathbf{y} \in \mathbf{A}} \mathbf{d}(\mathbf{X}, \mathbf{y}) = \min_{\mathbf{y} \in \mathbf{A}} \sum_{i=1}^{n} \mathbb{1}_{X_i \neq y_i} .$$

Michel Talagrand

$$\mathbb{P}\left\{ \mathbf{d}(\mathbf{X}, \mathbf{A}) \geq t + \sqrt{\frac{n}{2} \log \frac{1}{\mathbb{P}[\mathbf{A}]}} \right\} \leq e^{-2t^2/n} .$$

the isoperimetric view

Let $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ have independent components, taking values in $\mathcal{X}^n$. Let $\mathbf{A} \subset \mathcal{X}^n$.

The Hamming distance of $\mathbf{X}$ to $\mathbf{A}$ is

$$\mathbf{d}(\mathbf{X}, \mathbf{A}) = \min_{\mathbf{y} \in \mathbf{A}} \mathbf{d}(\mathbf{X}, \mathbf{y}) = \min_{\mathbf{y} \in \mathbf{A}} \sum_{i=1}^{n} \mathbb{1}_{\mathbf{X}_i \neq \mathbf{y}_i} .$$

Michel Talagrand

$$\mathbb{P}\left\{ \mathbf{d}(\mathbf{X}, \mathbf{A}) \geq \mathbf{t} + \sqrt{\frac{\mathbf{n}}{2} \log \frac{1}{\mathbb{P}[\mathbf{A}]}} \right\} \leq \mathbf{e}^{-2\mathbf{t}^2/\mathbf{n}} .$$

Concentration of measure!

the isoperimetric view

Proof: By the bounded differences inequality,

$$\mathbb{P}\{\mathbb{E}d(X,A) - d(X,A) \geq t\} \leq e^{-2t^2/n}.$$

Taking $t = \mathbb{E}d(X,A)$, we get

$$\mathbb{E}d(X,A) \leq \sqrt{\frac{n}{2}\log\frac{1}{\mathbb{P}\{A\}}}.$$

By the bounded differences inequality again,

$$\mathbb{P}\left\{d(X,A) \geq t + \sqrt{\frac{n}{2}\log\frac{1}{\mathbb{P}\{A\}}}\right\} \leq e^{-2t^2/n}$$

talagrand's convex distance

The weighted Hamming distance is

$$d_\alpha(x, A) = \inf_{y \in A} d_\alpha(x, y) = \inf_{y \in A} \sum_{i: x_i \neq y_i} |\alpha_i|$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$. The same argument as before gives

$$\mathbb{P}\left\{ d_\alpha(X, A) \geq t + \sqrt{\frac{\|\alpha\|^2}{2} \log \frac{1}{\mathbb{P}\{A\}}} \right\} \leq e^{-2t^2/\|\alpha\|^2} ,$$

This implies

$$\sup_{\alpha: \|\alpha\|=1} \min\left(\mathbb{P}\{A\}, \mathbb{P}\left\{d_\alpha(X, A) \geq t\right\}\right) \leq e^{-t^2/2} .$$

convex distance inequality

convex distance:

$$\mathbf{d_T(x, A) = \sup_{\alpha \in [0,\infty)^n : \|\alpha\| = 1} d_\alpha(x, A)} \ .$$

convex distance inequality

convex distance:

$$\mathbf{d_T(x, A) = \sup_{\alpha \in [0,\infty)^n : \|\alpha\| = 1} d_\alpha(x, A)} \ .$$

Talagrand's convex distance inequality:

$$\mathbb{P}\{\mathbf{A}\}\mathbb{P}\{\mathbf{d_T(X, A) \geq t}\} \leq \mathbf{e^{-t^2/4}} \ .$$

convex distance inequality

convex distance:

$$\mathbf{d_T(x, A) = \sup_{\alpha \in [0,\infty)^n : \|\alpha\|=1} d_\alpha(x, A)} \ .$$

Talagrand's convex distance inequality:

$$\mathbb{P}\{\mathbf{A}\}\mathbb{P}\left\{\mathbf{d_T(X, A) \geq t}\right\} \leq \mathbf{e^{-t^2/4}} \ .$$

Follows from the fact that $\mathbf{d_T(X, A)^2}$ is $\mathbf{(4, 0)}$ weakly self bounding (by a saddle point representation of $\mathbf{d_T}$).

Talagrand's original proof was different.

convex lipschitz functions

For $\mathbf{A} \subset [0,1]^n$ and $\mathbf{x} \in [0,1]^n$, define

$$D(x, A) = \inf_{y \in A} \|x - y\| \ .$$

If $\mathbf{A}$ is convex, then

$$D(x, A) \leq d_T(x, A) \ .$$

convex lipschitz functions

For $\mathbf{A} \subset [0,1]^n$ and $\mathbf{x} \in [0,1]^n$, define

$$\mathbf{D}(\mathbf{x}, \mathbf{A}) = \inf_{\mathbf{y} \in \mathbf{A}} \|\mathbf{x} - \mathbf{y}\| .$$

If $\mathbf{A}$ is convex, then

$$\mathbf{D}(\mathbf{x}, \mathbf{A}) \leq \mathbf{d_T}(\mathbf{x}, \mathbf{A}) .$$

Proof:

$$\begin{aligned}
\mathbf{D}(\mathbf{x}, \mathbf{A}) &= \inf_{\nu \in \mathcal{M}(\mathbf{A})} \|\mathbf{x} - \mathbb{E}_\nu \mathbf{Y}\| \quad \text{(since } \mathbf{A} \text{ is convex)} \\
&\leq \inf_{\nu \in \mathcal{M}(\mathbf{A})} \sqrt{\sum_{j=1}^{n} \left(\mathbb{E}_\nu \mathbb{1}_{x_j \neq Y_j}\right)^2} \quad \text{(since } \mathbf{x_j}, \mathbf{Y_j} \in [0,1]) \\
&= \inf_{\nu \in \mathcal{M}(\mathbf{A})} \sup_{\alpha : \|\alpha\| \leq 1} \sum_{j=1}^{n} \alpha_j \mathbb{E}_\nu \mathbb{1}_{x_j \neq Y_j} \quad \text{(by Cauchy-Schwarz)} \\
&= \mathbf{d_T}(\mathbf{x}, \mathbf{A}) \quad \text{(by minimax theorem)} .
\end{aligned}$$

John von Neumann (1903–1957)

Sergei Lvovich Sobolev
(1908–1989)

convex lipschitz functions

Let $\mathbf{X} = (\mathbf{X_1}, \ldots, \mathbf{X_n})$ have independent components taking values in $[0, 1]$. Let $\mathbf{f} : [0, 1]^n \to \mathbb{R}$ be quasi-convex such that $|\mathbf{f(x)} - \mathbf{f(y)}| \leq \|\mathbf{x} - \mathbf{y}\|$. Then

$$\mathbb{P}\{\mathbf{f(X)} > \mathbb{M}\mathbf{f(X)} + \mathbf{t}\} \leq 2e^{-t^2/4}$$

and

$$\mathbb{P}\{\mathbf{f(X)} < \mathbb{M}\mathbf{f(X)} - \mathbf{t}\} \leq 2e^{-t^2/4} .$$

convex lipschitz functions

Let $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ have independent components taking values in $[0,1]$. Let $\mathbf{f} : [0,1]^n \to \mathbb{R}$ be quasi-convex such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq \|\mathbf{x} - \mathbf{y}\|$. Then

$$\mathbb{P}\{\mathbf{f}(\mathbf{X}) > \mathbb{M}\mathbf{f}(\mathbf{X}) + \mathbf{t}\} \leq 2e^{-t^2/4}$$

and

$$\mathbb{P}\{\mathbf{f}(\mathbf{X}) < \mathbb{M}\mathbf{f}(\mathbf{X}) - \mathbf{t}\} \leq 2e^{-t^2/4} .$$

Proof: Let $\mathbf{A}_s = \{\mathbf{x} : \mathbf{f}(\mathbf{x}) \leq \mathbf{s}\} \subset [0,1]^n$. $\mathbf{A}_s$ is convex. Since $\mathbf{f}$ is Lipschitz,

$$\mathbf{f}(\mathbf{x}) \leq \mathbf{s} + \mathbf{D}(\mathbf{x}, \mathbf{A}_s) \leq \mathbf{s} + \mathbf{d}_T(\mathbf{x}, \mathbf{A}_s) ,$$

By the convex distance inequality,

$$\mathbb{P}\{\mathbf{f}(\mathbf{X}) \geq \mathbf{s} + \mathbf{t}\}\mathbb{P}\{\mathbf{f}(\mathbf{X}) \leq \mathbf{s}\} \leq e^{-t^2/4} .$$

Take $\mathbf{s} = \mathbb{M}\mathbf{f}(\mathbf{X})$ for the upper tail and $\mathbf{s} = \mathbb{M}\mathbf{f}(\mathbf{X}) - \mathbf{t}$ for the lower tail.

empirical processes

Let $\mathcal{T}$ be a countable index set.
For $i = 1, \ldots, n$, let $X_i = (X_{i,s})_{s \in \mathcal{T}}$ be vectors of real-valued random variables. Assume that $X_1, \ldots, X_n$ are independent.

The empirical process is $\sum_{i=1}^n X_{i,s}$, $s \in \mathcal{T}$.

We study concentration of the supremum:

$$Z = \sup_{s \in \mathcal{T}} \sum_{i=1}^n X_{i,s} .$$

empirical processes–the variance

We may use Efron-Stein: let

$$Z_i = \sup_{s \in \mathcal{T}} \sum_{j:j \neq i} X_{j,s}$$

and $\widehat{s} \in \mathcal{T}$ be such that $Z = \sum_{i=1}^{n} X_{i,\widehat{s}}$. Then

$$(Z - Z_i)_+ \leq (X_{i,\widehat{s}})_+ \leq \sup_{s \in \mathcal{T}} |X_{i,s}|$$

so

$$\mathbf{Var}(Z) \leq \mathbb{E} \sum_{i=1}^{n} (Z - Z_i)^2 \leq \mathbb{E} \sum_{i=1}^{n} \sup_{s \in \mathcal{T}} X_{i,s}^2 \, .$$

empirical processes–the variance

A more clever use of Efron-Stein: suppose $\mathbb{E}X_{i,s} = 0$.

Let $Z_i' = \sup_{s \in \mathcal{T}} \left(\sum_{j \neq i} X_{j,s} + X_{i,s}' \right)$. Note that

$$(Z - Z_i')_+^2 \le \left(X_{i,\widehat{s}} - X_{i,\widehat{s}}' \right)^2 .$$

By Efron-Stein,

$$\begin{aligned}
\mathbf{Var}(Z) &\le \mathbb{E} \sum_{i=1}^{n} (Z - Z_i')_+^2 \\
&\le \mathbb{E} \sum_{i=1}^{n} \mathbb{E}' \left[\left(X_{i,\widehat{s}} - X_{i,\widehat{s}}' \right)^2 \right] \\
&\le \mathbb{E} \sum_{i=1}^{n} \left(X_{i,\widehat{s}}^2 + \mathbb{E}' \left[X_{i,\widehat{s}}'^2 \right] \right) \\
&\le \mathbb{E} \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} X_{i,s}^2 + \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} \mathbb{E} X_{i,s}^2 .
\end{aligned}$$

weak and strong variance

We have proved that

$$\mathbf{Var}(\mathbf{Z}) \leq \mathbf{V} \quad \text{and} \quad \mathbf{Var}(\mathbf{Z}) \leq \mathbf{\Sigma}^2 + \boldsymbol{\sigma}^2$$

where

weak and strong variance

We have proved that

$$\mathbf{Var(Z)} \leq \mathbf{V} \quad \text{and} \quad \mathbf{Var(Z)} \leq \boldsymbol{\Sigma}^2 + \boldsymbol{\sigma}^2$$

where

$$\mathbf{V} = \sum_{i=1}^{n} \mathbb{E} \sup_{s \in \mathcal{T}} X_{i,s}^2 \qquad \text{strong variance}$$

weak and strong variance

We have proved that

$$\mathbf{Var(Z)} \leq \mathbf{V} \quad \text{and} \quad \mathbf{Var(Z)} \leq \mathbf{\Sigma}^2 + \boldsymbol{\sigma}^2$$

where

$$\mathbf{V} = \sum_{i=1}^{n} \mathbb{E} \sup_{s \in \mathcal{T}} X_{i,s}^2 \qquad \text{strong variance}$$

$$\mathbf{\Sigma}^2 = \mathbb{E} \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} X_{i,s}^2 \qquad \text{weak variance}$$

weak and strong variance

We have proved that

$$\mathrm{Var}(Z) \leq V \quad \text{and} \quad \mathrm{Var}(Z) \leq \Sigma^2 + \sigma^2$$

where

$$V = \sum_{i=1}^{n} \mathbb{E} \sup_{s \in \mathcal{T}} X_{i,s}^2 \qquad \text{strong variance}$$

$$\Sigma^2 = \mathbb{E} \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} X_{i,s}^2 \qquad \text{weak variance}$$

$$\sigma^2 = \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} \mathbb{E} X_{i,s}^2 \qquad \text{wimpy variance}$$

weak and strong variance

We have proved that

$$\mathrm{Var}(Z) \leq V \quad \text{and} \quad \mathrm{Var}(Z) \leq \Sigma^2 + \sigma^2$$

where

$$V = \sum_{i=1}^{n} \mathbb{E} \sup_{s \in \mathcal{T}} X_{i,s}^2 \qquad \text{strong variance}$$

$$\Sigma^2 = \mathbb{E} \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} X_{i,s}^2 \qquad \text{weak variance}$$

$$\sigma^2 = \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} \mathbb{E} X_{i,s}^2 \qquad \text{wimpy variance}$$

$$\sigma^2 \leq \Sigma^2 \leq V .$$

weak and strong variance

If $\mathbb{E}X_{i,s} = 0$ and $|X_{i,s}| \leq 1$, we also have, by symmetrization and contraction arguments,

$$\Sigma^2 \leq 8\mathbb{E}Z + \sigma^2$$

and therefore

$$\mathrm{Var}(Z) \leq 8\mathbb{E}Z + 2\sigma^2 \ .$$

weak and strong variance

If $\mathbb{E}X_{i,s} = 0$ and $|X_{i,s}| \leq 1$, we also have, by symmetrization and contraction arguments,

$$\Sigma^2 \leq 8\mathbb{E}Z + \sigma^2$$

and therefore

$$\mathrm{Var}(Z) \leq 8\mathbb{E}Z + 2\sigma^2 \ .$$

If the X_i are also identicaly distributed, then

$$\mathrm{Var}(Z) \leq 2\mathbb{E}Z + \sigma^2 \ .$$

empirical processes–exponential inequalities

A Bernstein type inequality. "Talagrand's inequality".

empirical processes–exponential inequalities

A Bernstein type inequality. "Talagrand's inequality".

Assume $\mathbb{E}\mathbf{X}_{i,s} = \mathbf{0}$, and $|\mathbf{X}_{i,s}| \leq \mathbf{1}$. For $\mathbf{t} \geq \mathbf{0}$,

$$\mathbb{P}\left\{\mathbf{Z} \geq \mathbb{E}\mathbf{Z} + \mathbf{t}\right\} \leq \exp\left(-\frac{\mathbf{t}^2}{2\left(2(\boldsymbol{\Sigma}^2 + \sigma^2) + \mathbf{t}\right)}\right) .$$

proof.

For each $i = 1, \ldots, n$, let $Z'_i = \sup_{s \in \mathcal{T}} (X'_{i,s} + \sum_{j \neq i} X_{j,s})$.
We already proved that

$$\sum_{i=1}^{n} \mathbb{E}'(Z - Z'_i)_+^2 \leq \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} X_{i,s}^2 + \sigma^2 \stackrel{\text{def.}}{=} W + \sigma^2 \ .$$

By the exponential Efron-Stein inequality, for $\lambda \in [0, 1)$,

$$\log \mathbb{E} e^{\lambda(Z - \mathbb{E}Z)} \leq \frac{\lambda}{1 - \lambda} \log \mathbb{E} e^{\lambda(W + \sigma^2)} \ .$$

proof.

For each $\mathbf{i = 1, \ldots, n}$, let $\mathbf{Z'_i = \sup_{s \in \mathcal{T}} (X'_{i,s} + \sum_{j \neq i} X_{j,s})}$.
We already proved that

$$\sum_{i=1}^{n} \mathbb{E}'(Z - Z'_i)_+^2 \leq \sup_{s \in \mathcal{T}} \sum_{i=1}^{n} X_{i,s}^2 + \sigma^2 \stackrel{\text{def.}}{=} W + \sigma^2 \; .$$

By the exponential Efron-Stein inequality, for $\lambda \in [0, 1)$,

$$\log \mathbb{E} e^{\lambda(Z - \mathbb{E}Z)} \leq \frac{\lambda}{1 - \lambda} \log \mathbb{E} e^{\lambda(W + \sigma^2)} \; .$$

$\mathbf{W}$ is a self-bounding function, so

$$\log \mathbb{E} e^{\lambda W} \leq \Sigma^2 \left(e^{\lambda} - 1 \right) \; .$$

Putting things together implies the inequality.

bousquet's inequality

A Bennett type inequality with the right constant.
Assume $\mathbf{X_1, \ldots, X_n}$ are i.i.d. with $\mathbb{E}\mathbf{X_{i,s} = 0}$ and $\mathbf{X_{i,s} \leq 1}$.
For all $\mathbf{t \geq 0}$,

$$\mathbb{P}\{\mathbf{Z} \geq \mathbb{E}\mathbf{Z} + \mathbf{t}\} \leq \mathbf{e}^{-\mathbf{v}\mathbf{h}(\mathbf{t}/\mathbf{v})} .$$

where $\mathbf{v} = 2\mathbb{E}\mathbf{Z} + \sigma^2$ and $\mathbf{h(u) = (1+u)\log(1+u) - u}$.
In particular,

$$\mathbb{P}\{\mathbf{Z} \geq \mathbb{E}\mathbf{Z} + \mathbf{t}\} \leq \exp\left(-\frac{\mathbf{t}^2}{2(\mathbf{v} + \mathbf{t}/3)}\right) .$$

ϕ entropies

For a convex function ϕ on $[0, \infty)$, the ϕ-entropy of $Z \geq 0$ is

$$H_\phi(Z) = \mathbb{E}[\phi(Z)] - \phi(\mathbb{E}[Z]) \ .$$

H_ϕ is subadditive:

$$H_\phi(Z) \leq \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}\left[\phi(Z) \mid X^{(i)}\right] - \phi\left(\mathbb{E}\left[Z \mid X^{(i)}\right]\right)\right]$$

if (and only if) ϕ is twice differentiable on $(0, \infty)$, and either ϕ is affine strictly positive and $1/\phi''$ is concave.

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if (and only if) ϕ is twice differentiable on $(0, \infty)$, and either ϕ is affine strictly positive and $1/\phi''$ is concave.

$\phi(x) = x^2$ corresponds to Efron-Stein.

$x \log x$ is subadditivity of entropy.

We may consider $\phi(x) = x^p$ for $p \in (1, 2]$.

generalized efron-stein

Define

$$Z'_i = f(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n) \ ,$$

$$V^+ = \sum_{i=1}^{n} (Z - Z'_i)_+^2 \ .$$

generalized efron-stein

Define

$$Z_i' = f(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n) \ ,$$

$$V^+ = \sum_{i=1}^{n} (Z - Z_i')_+^2 \ .$$

For $q \geq 2$ and $q/2 \leq \alpha \leq q - 1$,

$$\mathbb{E}\left[(Z - \mathbb{E}Z)_+^q\right]$$
$$\leq \mathbb{E}\left[(Z - \mathbb{E}Z)_+^\alpha\right]^{q/\alpha} + \alpha\,(q - \alpha)\,\mathbb{E}\left[V^+\,(Z - \mathbb{E}Z)_+^{q-2}\right] \ ,$$

and similarly for $\mathbb{E}\left[(Z - \mathbb{E}Z)_-^q\right]$.

moment inequalities

We may solve the recursions, for $\mathbf{q} \geq \mathbf{2}$.

moment inequalities

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If $\mathbf{V}^+ \leq \mathbf{c}$ for some constant $\mathbf{c} \geq \mathbf{0}$, then for all integers $\mathbf{q} \geq \mathbf{2}$,

$$\left(\mathbb{E}\left[(\mathbf{Z} - \mathbb{E}\mathbf{Z})_+^{\mathbf{q}}\right]\right)^{\mathbf{1/q}} \leq \sqrt{\mathbf{Kqc}}\ ,$$

where $\mathbf{K} = \mathbf{1}/\left(\mathbf{e} - \sqrt{\mathbf{e}}\right) < \mathbf{0.935}$.

moment inequalities

We may solve the recursions, for $\mathbf{q} \geq \mathbf{2}$.

If $\mathbf{V}^{+} \leq \mathbf{c}$ for some constant $\mathbf{c} \geq \mathbf{0}$, then for all integers $\mathbf{q} \geq \mathbf{2}$,

$$\left(\mathbb{E}\left[(\mathbf{Z}-\mathbb{E}\mathbf{Z})_{+}^{\mathbf{q}}\right]\right)^{1/\mathbf{q}} \leq \sqrt{\mathbf{Kqc}} ,$$

where $\mathbf{K} = \mathbf{1}/\left(\mathbf{e}-\sqrt{\mathbf{e}}\right) < \mathbf{0.935}$.

More generally,

$$\left(\mathbb{E}\left[(\mathbf{Z}-\mathbb{E}\mathbf{Z})_{+}^{\mathbf{q}}\right]\right)^{1/\mathbf{q}} \leq \mathbf{1.6}\sqrt{\mathbf{q}}\left(\mathbb{E}\left[\mathbf{V}^{+\mathbf{q}/2}\right]\right)^{1/\mathbf{q}} .$$

sums: khinchine's inequality

Let $\mathbf{X_1, \ldots, X_n}$ be independent Rademacher variables and $\mathbf{Z = \sum_{i=1}^n a_i X_i}$. For any integer $\mathbf{q \geq 2}$,

$$\left(\mathbb{E}\left[\mathbf{Z_+^q}\right]\right)^{\mathbf{1/q}} \leq \sqrt{\mathbf{2Kq}}\sqrt{\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{a_i^2}}$$

sums: khinchine's inequality

Let $\mathbf{X_1, \ldots, X_n}$ be independent Rademacher variables and $\mathbf{Z = \sum_{i=1}^n a_i X_i}$. For any integer $\mathbf{q} \geq \mathbf{2}$,

$$\left(\mathbb{E}\left[\mathbf{Z}_+^{\mathbf{q}}\right]\right)^{\mathbf{1/q}} \leq \sqrt{\mathbf{2Kq}}\sqrt{\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{a_i^2}}$$

Proof:

$$\mathbf{V}^+ = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbb{E}\left[(\mathbf{a_i}(\mathbf{X_i} - \mathbf{X_i'}))_+^2 \mid \mathbf{X_i}\right] = \mathbf{2}\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{a_i^2} \mathbb{1}_{\mathbf{a_i X_i > 0}} \leq \mathbf{2}\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{a_i^2} \ ,$$

Aleksandr Khinchin
(1894–1959)

sums: rosenthal's inequality

Let $\mathbf{X_1, \dots, X_n}$ be independent real-valued random variables with $\mathbb{E}\mathbf{X_i = 0}$. Define

$$\mathbf{Z} = \sum_{i=1}^{n} \mathbf{X_i}\,, \quad \sigma^2 = \sum_{i=1}^{n} \mathbb{E}\mathbf{X_i^2}\,, \quad \mathbf{Y} = \max_{i=1,\dots,n} |\mathbf{X_i}|\,.$$

Then for any integer $\mathbf{q} \geq \mathbf{2}$,

$$\left(\mathbb{E}\left[\mathbf{Z_+^q}\right]\right)^{1/q} \leq \sigma\sqrt{10q} + 3q\left(\mathbb{E}\left[\mathbf{Y_+^q}\right]\right)^{1/q} \;.$$

influences

If $\mathbf{A} \subset \{-1,1\}^n$ and $\mathbf{X} = (X_1, \ldots, X_n)$ is uniform, the influence of the $\mathbf{i}$-th variable is

$$I_i(A) = \mathbb{P}\left\{\mathbb{1}_{X \in A} \neq \mathbb{1}_{X^{(i)} \in A}\right\}$$

where $X^{(i)} = (X_1, \ldots, X_{i-1}, 1 - X_i, X_{i+1}, \ldots, X_n)$.

The total influence is

$$I(A) = \sum_{i=1}^{n} I_i(A) .$$

influences

If $A \subset \{-1,1\}^n$ and $X = (X_1, \ldots, X_n)$ is uniform, the influence of the i-th variable is

$$I_i(A) = \mathbb{P}\left\{\mathbb{1}_{X \in A} \neq \mathbb{1}_{X^{(i)} \in A}\right\}$$

where $X^{(i)} = (X_1, \ldots, X_{i-1}, 1 - X_i, X_{i+1}, \ldots, X_n)$.

The total influence is

$$I(A) = \sum_{i=1}^{n} I_i(A) \ .$$

Note that

$$I(A) = 2^{-(n-1)} |\partial_E(A)| \ .$$

influences: examples

dictatorship: $\mathbf{A = \{x : x_1 = 1\}}$. $\mathbf{I(A) = 1}$.

parity: $\mathbf{A = \{x : \sum_i \mathbb{1}_{x_i=1} \text{ is even}\}}$. $\mathbf{I(A) = n}$.

majority: $\mathbf{A = \{x : \sum_i x_i > 0\}}$. $\mathbf{I(A) \approx \sqrt{2n/\pi}}$.

$$\text{by Efron-Stein,} \quad \mathbf{P(A)(1 - P(A)) \leq \frac{I(A)}{4}}$$

so dictatorship has smallest total influence (if $\mathbf{P(A) = 1/2}$).

improved efron-stein on the hypercube

Recall that for any $\mathbf{f} : \{-1, 1\}^n \to \mathbb{R}$ under the uniform distribution,

$$\mathrm{Ent}(f^2) \leq 2\mathcal{E}(f)$$

where $\mathrm{Ent}(f^2) = \mathbf{E}\left[f^2 \log(f^2)\right] - \mathbf{E}\left[f^2\right] \log \mathbf{E}\left[f^2\right]$ and

$$\mathcal{E}(f) = \frac{1}{4}\mathbb{E}\left[\sum_{i=1}^{n}\left(f(X) - f(\overline{X}^{(i)})\right)^2\right]$$

This implies, for any non-negative $f : \{-1, 1\}^n \to [0, \infty)$,

$$\mathbf{E}\left[f^2\right] \log \frac{\mathbf{E}\left[f^2\right]}{\mathbf{E}\left[f\right]^2} \leq 2\mathcal{E}(f) \ .$$

improved efron-stein on the hypercube

Recall the Doob-martingale representation $\mathbf{f(X) - Ef = \sum_{i=1}^n \Delta_i}$.
One easily sees that

$$\mathcal{E}(\mathbf{f}) = \sum_{i=1}^{n} \mathcal{E}(\Delta_i) .$$

But then, by the previous lemma,

$$\begin{aligned}
\mathcal{E}(\mathbf{f}) &\geq \sum_{j=1}^{n} \mathcal{E}(|\Delta_j|) \geq \frac{1}{2} \sum_{j=1}^{n} \mathbf{E}\left[\Delta_j^2\right] \log \frac{\mathbf{E}\left[\Delta_j^2\right]}{(\mathbf{E}|\Delta_j|)^2} \\
&= -\frac{1}{2} \mathrm{Var}(\mathbf{f}) \sum_{j=1}^{n} \frac{\mathbf{E}\left[\Delta_j^2\right]}{\mathrm{Var}(\mathbf{f})} \log \frac{(\mathbf{E}|\Delta_j|)^2}{\mathbf{E}\left[\Delta_j^2\right]} \\
&\geq -\frac{1}{2} \mathrm{Var}(\mathbf{f}) \log \frac{\sum_{j=1}^{n} (\mathbf{E}|\Delta_j|)^2}{\mathrm{Var}(\mathbf{f})}
\end{aligned}$$

improved efron-stein on the hypercube

We obtained that for any $\mathbf{f : \{-1,1\}^n \to \mathbb{R}}$,

$$\mathbf{Var(f)} \log \frac{\mathbf{Var(f)}}{\sum_{j=1}^{n} (\mathbf{E}|\Delta_j|)^2} \le 2\mathcal{E}(\mathbf{f}) \ .$$

(Falik and Samorodnitsky, 2007; Rossignol, 2006).

improved efron-stein on the hypercube

We obtained that for any $\mathbf{f : \{-1, 1\}^n \to \mathbb{R}}$,

$$\mathbf{Var(f) \log \frac{Var(f)}{\sum_{j=1}^{n} (E|\Delta_j|)^2} \leq 2\mathcal{E}(f)} \ .$$

(Falik and Samorodnitsky, 2007; Rossignol, 2006).
"Slightly" better than Efron-Stein.

improved efron-stein on the hypercube

We obtained that for any $\mathbf{f : \{-1, 1\}^n \to \mathbb{R}}$,

$$\mathrm{Var}(f) \log \frac{\mathrm{Var}(f)}{\sum_{j=1}^{n} (\mathbf{E}|\Delta_j|)^2} \le 2\mathcal{E}(f) \ .$$

(Falik and Samorodnitsky, 2007; Rossignol, 2006).
"Slightly" better than Efron-Stein.

Use this for $f(x) = \mathbb{1}_{x \in A}$ for $A \subset \{-1, 1\}^n$:

$$P(A)(1 - P(A)) \log \frac{4P(A)(1 - P(A))}{\sum_i I_i(A)^2} \le \frac{I(A)}{4}$$

kahn, kalai, linial

Corollary: (Kahn, Kalai, Linial, 1988).

$$\max_i \mathbf{I_i(A)} \geq \frac{\mathbf{P(A)(1 - P(A)) \log n}}{\mathbf{n}}$$

If the influences are equal,

$$\mathbf{I(A) \geq P(A)(1 - P(A)) \log n}$$

Another corollary: (Friedgut, 1998).
If $\mathbf{I(A) \leq c}$, $\mathbf{A}$ (basically) depends on a bounded number of variables. $\mathbf{A}$ is a "junta."

threshold phenomena

Let $\mathbf{A} \subset \{-1, 1\}^{\mathbf{n}}$ be a monotone set and let $\mathbf{X} = (\mathbf{X_1}, \ldots, \mathbf{X_n})$ be such that

$$\mathbb{P}\{\mathbf{X_i} = 1\} = \mathbf{p} \qquad \mathbb{P}\{\mathbf{X_i} = -1\} = 1 - \mathbf{p}$$

$$\mathbf{P_p(A)} = \sum_{\mathbf{x} \in \mathbf{A}} \mathbf{p}^{\|\mathbf{x}\|}(1 - \mathbf{p})^{\mathbf{n} - \|\mathbf{x}\|}$$

is an increasing function of $\mathbf{p} \in [0, 1]$.

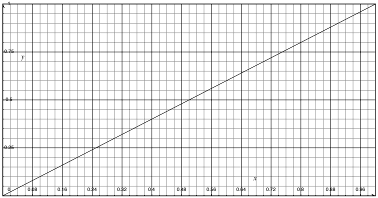
Let $\mathbf{p_a}$ be such that $\mathbf{P_{p_a}(A) = a}$.

Critical value $= \mathbf{p_{1/2}}$

Threshold width: $\mathbf{p_{1-\varepsilon} - p_\varepsilon}$

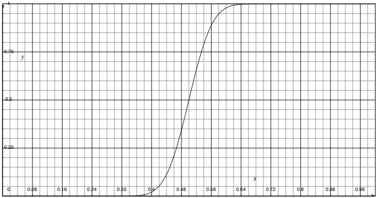
two (extreme) examples

dictatorship



threshold width $= \mathbf{1 - 2}\varepsilon$

majority (with $\mathbf{n = 101}$)



$\leq \sqrt{\mathbf{log(1/}\varepsilon\mathbf{)/(2n)}}$

In what cases do we have a quick transition?

russo's lemma

If $\mathbf{A}$ is monotone,

$$\frac{dP_p(A)}{dp} = I^{(p)}(A)$$

The Kahn, Kalai, Linial result, generalized for $p \neq 1/2$, implies that
if $\mathbf{A}$ is such that $I_1^{(p)} = I_2^{(p)} = \cdots = I_n^{(p)}$, then

$$p_{1-\varepsilon} - p_\varepsilon = O\left(\frac{\log\frac{1}{\varepsilon}}{\log n}\right)$$

On the other hand, if $p_{3/4} - p_{1/4} \geq c$ then $\mathbf{A}$ is (basically) a junta.

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