CHAPTER 1

Introduction: Orientation, problems, and goals

These lecture notes contain the definitions, theorems, propositions etc. discussed in the lecture. Proofs are omitted. I will be grateful for your comments, questions, and in particular, your corrections.

CHAPTER 2

Elliptic boundary value problems

The material of this chapter follows in large parts the book by M. Taylor [**Tay96**], in particular Chapter 4 (Sections 1 and 4) and Chapter 5 (Sections 7, 11, and 12). For additional references on elliptic problems we refer to the book by Renardy and Rogers [**RR93**] and the book by Wloka, Rowley, and Lawruk [**WRL95**].

2.1. The Neumann problem for the Laplace operator

Throught this chapter $\Omega \subset \mathbb{R}^d$ is a connected, open, and bounded set with a C^∞ boundary. This is to say that for every point $x \in \partial\Omega$ there exists an open neighborhood $\mathscr{U} = \mathscr{U}(x)$ and a mapping $\varphi : \mathscr{U} \to \mathbb{R}^d$ of class C^∞ with C^∞ inverse such that

$$\varphi(\Omega \cap \mathscr{U}) \subset \mathbb{R}^d_+ = \{x \in \mathbb{R}^d \ : \ x_d > 0\} \quad \text{and} \quad \varphi(\partial\Omega \cap \mathscr{U}) \subset \{x \in \mathbb{R}^d \ : \ x_d = 0\} \equiv \mathbb{R}^{d-1} \ .$$

The domain of the inverse mapping is $\varphi(\mathscr{U})$. Using the compactness of the boundary, we know that we can select a finite number of such neighborhoods which will cover $\partial\Omega$. We have

$$\partial\Omega \subset \bigcup_{j=1}^{M} \mathscr{U}_j$$

and we denote the corresponding mappings of $\Omega \cap \mathscr{U}_j$ into the half space by φ_j. We will refer to Ω as a *smooth domain* and to the functions φ_j as *coordinate mappings*. Note that a smooth domain has a well-defined exterior unit normal field n along $\partial\Omega$ which is given by the formula

$$n(x) = -\frac{J^T_{\varphi_j}(x)e_d}{|J^T_{\varphi_j}(x)e_d|} \qquad \text{for } x \in \mathscr{U}_j$$

where J_{φ_j} is the Jacobian matrix of the function φ_j and e_d is the dth standard basis vector, that is $e_d = (0, ..., 0, 1)^T$. This formula follows from the fact that plane tangent to Ω at the point $x \in \partial\Omega \cap \mathscr{U}_j$ is spanned by the $d-1$ vectors of the form $J^{-1}_{\varphi_j}(x)e_q$ for $q = 1, ..., d-1$ and that $J^{-1}_{\varphi_j} = J_{\varphi_j^{-1}}$.

The *Neumann problem* for the Laplacian is the boundary value problem consists in finding a function u which satisfies

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega \ , \\ \frac{\partial u}{\partial n} &= 0 \quad \text{in } \partial\Omega \ , \end{aligned}$$

for a given function f. Here Δ denotes the Laplacian in $\mathbb{R}$, i.e.

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

and $\partial u/\partial n$ is the directional derivative of u in direction of the exterior unit normal n, that is

$$\frac{\partial u}{\partial n}(x) = n(x) \cdot \nabla u(x) \ .$$

We define a linear operator $\mathscr{L}_N \ : \ H^1(\Omega) \to H^1(\Omega)'$ by setting

$$(\mathscr{L}_N u, v) = \int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)} \, dx = (\nabla u, \nabla v)_{L_2(\Omega)} \ ,$$

for all $u, v \in H^1(\Omega)$. Here $H^1(\Omega)$ is the L_2-based Sobolev space of order one. This linear function space consists of all square integrable functions on Ω whose distributional gradient is also square integrable. A norm in this space is given by

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega |u(x)|^2 dx + \int_\Omega |\nabla u(x)|^2 dx \ .$$

The linear space $H^1(\Omega)'$ is the dual space of $H^1(\Omega)$ with respect to the L_2 inner product

$$(u, v) = \int_\Omega u(x)\overline{v(x)} \, dx \ .$$

We observe the algebraic and topological inclusions $H^1(\Omega) \subset L_2(\Omega) \subset H^1(\Omega)'$.

Proposition 2.1.1. *The map $\mathscr{L}_N + 1 \ : \ H^1(\Omega) \to H^1(\Omega)'$ is injective and surjective. Furthermore the operator $\mathscr{L}_N$ is self-adjoint and positive.*

Hence the operator $T_N = (\mathscr{L}_N + 1)^{-1} \ : \ H^1(\Omega)' \to H^1(\Omega)$ is well-defined. Restricting the domain of this operator to $L_2(\Omega)$ and enlarging the co-domain to $L_2(\Omega)$, the operator $T_N \ : \ L_2(\Omega) \to L_2(\Omega)$ is compact because of Rellich's Theorem.

Theorem 2.1.2. *For any $s \geq 0$ and $\sigma > 0$, the natural injection $j \ : \ H^{s+\sigma}(\Omega) \to H^s(\Omega)$ is compact. This is to say that bounded sets in $H^{s+\sigma}(\Omega)$ are precompact in $H^s(\Omega)$.*

Of course, to establish the compactness of T_N one needs to apply this theorem with $s = 0$ and $\sigma = 1$. Nevertheless, we chose to state a more general result, even though we have not introduced the Sobolev spaces $H^s(\Omega)$ for $s \in [0, \infty)$ except for $s = 0$ and $s = 1$. This will be done in the next section.

The spectral theorem for compact, self-adjoint operators on a Hilbert space tells us that $L_2(\Omega)$ has a orthonormal basis of eigenfunctions $\{u_k\}$ with corresponding eigenvalues $\mu_k > 0$. Furthermore, $\lim_{k\to\infty} \mu_k = 0$. The equation

$$T_N u_k = \mu_k u_k \ , \qquad \text{for } k = 1, 2, 3, ...$$

implies $u_k \in H^1(\Omega)$ for all $k \in \mathbb{N}$. Returning to the operator $\mathscr{L}_N$ gives then

$$\mathscr{L}_N u_k = \lambda_k u_k \ , \qquad \lambda_k = \frac{1}{\mu_k} - 1 \ .$$

Furthermore, one can show that for $u \in H^1(\Omega)$

$$\mathscr{L}_N u = -\Delta u$$

in the sense of distributions and also that

$$-\Delta u_k = \lambda_k u_k \qquad \text{in } \mathscr{D}'(\Omega) \ .$$

Here and henceforth, $\mathscr{D}'(\Omega)$ denotes the linear space of distributions on Ω. This is the dual space of $C_0^\infty(\Omega)$ which is a linear space with the topology induced by the seminorms

$$\sup_{x\in\Omega} |D^\alpha u(x)| \,, \qquad \alpha \in \mathbb{N}_0^d \,.$$

Eventually, we will show that the eigenfunctions on the Neumann Laplacian are smooth functions ($u_k \in C^\infty(\overline{\Omega})$) and that they satisfy the homogeneous Neumann condition $\partial u_k/\partial n = 0$ in $\partial\Omega$, $k = 1, 2, \ldots$.

One big step toward the proof of this statement is the following proposition.

Proposition 2.1.3. *For $f \in L_2(\Omega)$ the function $u = T_N f$ satisfies $u \in H^2(\Omega)$ and $\partial u/\partial n = 0$ in $\partial\Omega$. Furthermore, $-\Delta u + u = f$ and there exists a constant C depending only on Ω (and not on f)*

$$\|u\|_{H^2(\Omega)} \leq C \left\{ \|f\|^2_{L_2(\Omega)} + \|u\|^2_{H^1(\Omega)} \right\}$$

Of course, the Sobolev space $H^2(\Omega)$ is the space of all $L_2(\Omega)$ functions whose distributional derivatives up to order 2 are in $L_2(\Omega)$. This space is a Hilbert space with the inner product

$$(u, v)_{2,\Omega} = \sum_{|\alpha|\leq 2} (D^\alpha u, D^\alpha v) \,.$$

2.2. Sobolev spaces on bounded regions

Initially we will focus on Sobolev spaces on the half space $R^d_+ = \{x \in \mathbb{R}^d \;:\; x_d > 0\}$. For $k \in \mathbb{N}$ we set

$$H^k(\mathbb{R}^d_+) = \{u \in L_2(\mathbb{R}^d_+) \;:\; D^\alpha u \in L_2(\mathbb{R}^d_+) \text{ for } |\alpha| \leq k\} \tag{2.2.1}$$

This space is a Hilbert space with the inner product

$$(u, v)_{k,\mathbb{R}^d_+} = \sum_{|\alpha|\leq k} \int_{\mathbb{R}^d_+} D^\alpha u(x) D^\alpha v(x)\, dx$$

Sobolev functions defined on the half space can be extended to the full space without loss of regularity.

Lemma 2.2.1. *For all $N \in \mathbb{N}$ there exists a continuous extension map $E \;:\; H^k(\mathbb{R}^d_+) \to H^k(\mathbb{R}^d)$ for $k \leq N - 1$.*

Corollary 2.2.2. *The restriction operator $\rho \;:\; H^k(\mathbb{R}^d) \to H^k(\mathbb{R}^d_+)$ is surjective.*

Note that the subspace $\left\{ u \in H^s(\mathbb{R}^d) \;:\; u\Big|_{\mathbb{R}_+} = 0 \right\}$ is a closed subspace of $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$. Hence, an equivalent definition of $H^k(\mathbb{R}^d_+)$ is given by

$$H^k(\mathbb{R}^d_+) \approx H^k(\mathbb{R}^d) / \left\{ u \in H^k(\mathbb{R}^d) \;:\; u\Big|_{\mathbb{R}_+} = 0 \right\} \tag{2.2.2}$$

with the quotient norm

$$\|u\|_{H^s(\mathbb{R}^d_+)} = \inf_{U=u \text{ in } \mathbb{R}^d_+} \|U\|_{H^s(\mathbb{R}^d)} \,.$$

Note that $H^s(\mathbb{R}^d_+)$ is defined as a quotient space and that this definition is applicable for all $s \in \mathbb{R}$. The equality $u = 0$ on $\mathbb{R}^d_+$ has to be understood in the L_2 sense or in the sense of distributions.

PROPOSITION 2.2.3. *For $s = k \in \mathbb{N}$ the two definitions given by formulas (2.2.1) and (2.2.2) coincide.*

In the following we will use the abbreviation $x' = (x_1, ..., x_{d-1})$.

PROPOSITION 2.2.4. *For $s > 1/2$ the trace operator $\tau : C_0^\infty(\mathbb{R}^d) \to C_0^\infty(\mathbb{R}^{d-1})$ defined by $(\tau u)(x') = u(x', 0)$ can be extended to a continuous linear operator $\tau : H^s(\mathbb{R}^d) \to H^{s-1/2}(\mathbb{R}^{d-1})$.*

PROPOSITION 2.2.5. *The mapping τ is surjective for $s > 1/2$. In particular, for each $g \in H^{s-1/2}(\mathbb{R}^{d-1})$ there exists a $u \in H^s(\mathbb{R}^d)$ such that $\tau u = g$ and there exists a positive constant C depending only on s such that $\|u\|_{H^s(\mathbb{R}^d)} \le C\|g\|_{H^{s-1/2}(\mathbb{R}^{d-1})}$.*

The Sobolev spaces $H^k(\Omega)$ are introduced in a very similar fashion. We set

$$H^k(\Omega) = \{u \in L_2(\Omega) \ : \ D^\alpha u \in L_2(\Omega) \text{ for } |\alpha| \le k\}$$

This space is also a Hilbert space with the inner product

$$(u, v)_{k,\Omega} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha u(x) D^\alpha v(x)\, dx$$

As before, an equivalent definition of $H^s(\Omega)$ is given by

$$H^s(\Omega) \approx H^s(\mathbb{R}^d) \Big/ \left\{ u \in H^s(\mathbb{R}^d) \ : \ u\Big|_\Omega = 0 \right\}$$

with the equivalent norm

$$\|u\|_{H^s(\Omega)} = \inf_{U = u \text{ in } \Omega} \|U\|_{H^s(\mathbb{R}^d)} \ .$$

Again, the second definition has the advantage that it works for all $s \in \mathbb{R}$.

To define Sobolev spaces on the boundary we introduce a partition of unity $\{\chi_j\}_{j=1}^M$ subordinated to the covering $\{\mathscr{U}_j\}_{j=1}^M$ of the boundary which was introduced at the beginning of Section 2.1. We have $\chi_j \in C_0^\infty(\mathscr{U}_j)$ for $j = 1, 2, ..., M$ and $\sum_{j=1}^M \chi_j(x) = 1$ for all $x \in \partial\Omega$.

DEFINITION 2.2.6. *For $s > 0$ we define the Sobolev space*

$$H^s(\partial\Omega) = \{u \in L_2(\partial\Omega) \ : \ (\chi_j u) \circ \varphi_j^{-1} \in H^s(\mathbb{R}^{d-1}\} \ .$$

A norm is given by

$$\|u\|_{H^s(\partial\Omega)} = \sum_{j=1}^M \left\| (\chi_j u) \circ \varphi_j^{-1} \right\|_{H^s(\mathbb{R}^{d-1})} \ .$$

In order to prove the trace theorem for a region Ω, we will need the following transformation formula.

DEFINITION 2.2.7. *Let Ω and Ω' be open and bounded sets in $\mathbb{R}^d$. A bijection $\Phi : \Omega \to \Omega'$ is a diffeomorphism if $\Phi \in C^\infty(\overline{\Omega})$ and $\Phi^{-1} \in C^\infty(\overline{\Omega'})$. The pull back operator Φ^* is defined by*

$$\Phi^* u = u \circ \Phi$$

and the push forward operator $(\Phi^{-1})^*$ is defined by

$$(\Phi^{-1})^* v = v \circ \Phi^{-1}$$

for functions u and v defined on Ω' and Ω, respectively.

THEOREM 2.2.8. *Suppose that Ω and Ω' are bounded, open, and connected subsets of $\mathbb{R}^d$ with a smooth boundary. Let $\Phi : \Omega \to \Omega'$ be a diffeomorphism and let s be a non-negative real number. Then the pull back Φ^* is a continuous linear mapping from $H^s(\Omega')$ to $H^s(\Omega)$ and the push forward $(\Phi^{-1})^*$ is a continuous linear mapping from $H^s(\Omega)$ to $H^s(\Omega')$*

THEOREM 2.2.9. *For $s > 1/2$ the trace operator $T : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$ defined by*

$$Tu = u\Big|_{\partial\Omega}$$

can be extended to a continuous linear operator from $H^s(\Omega)$ onto $H^{s-1/2}(\partial\Omega)$.

COROLLARY 2.2.10. *For $s = 1$ we have*

$$\operatorname{Ker} T = \overset{\circ}{H}{}^1(\Omega)$$

where $\overset{\circ}{H}{}^s(\Omega)$ is the closure of the compactly supported smooth functions $C_0^\infty(\Omega)$ with respect to the $H^s(\mathbb{R}^d)$ topology for $s > 1/2$.

COROLLARY 2.2.11. *Let $m \in \mathbb{N}$ and $s \in \mathbb{R}$ such that $s - m > 1/2$. Then, there exists a linear continuous operator*

$$T_m : H^s(\Omega) \to H^{s-1/2}(\partial\Omega) \times H^{s-3/2}(\partial\Omega) \times \dots \times H^{s-m-1/2}(\partial\Omega)$$

with the property that

$$\tau_m u = \left(u\Big|_{\partial\Omega}, \frac{\partial u}{\partial n}\Big|_{\partial\Omega}, \frac{\partial^2 u}{\partial n^2}\Big|_{\partial\Omega}, \dots, \frac{\partial^m u}{\partial n^m}\Big|_{\partial\Omega} \right) .$$

2.3. Interior regularity

We introduce the $N \times N$ matrix differential operator P in $\Omega \subset \mathbb{R}^d$ of order m with smooth coefficients.

$$P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha \qquad a_\alpha(x) \in C^\infty(\overline{\Omega}, \mathbb{C}^{N\times N}), \quad |\alpha| \le m \tag{2.3.1}$$

The principal symbol of P is the matrix polynomial $P_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ since the coefficients a_α $N \times N$ are $N \times N$ matrices. Recall that $\alpha \in \mathbb{N}_0^d$ is a multiindex $\alpha = (\alpha_1, ..., \alpha_d)$ whose components are non-negative integers and that $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}...\xi_d^{\alpha_d}$.

DEFINITION 2.3.1. *The operator (2.3.1) is elliptic if $P_m(x,\xi)^{-1}$ exists for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^d \setminus \{0\}$.*

THEOREM 2.3.2. *Suppose $u \in L_2(\Omega)$, $Pu = f \in H^k(\Omega)$ for some $k \in \mathbb{N}_0$. Then $u \in H^{m+k}_{\mathrm{loc}}(\Omega)$ and for all $U \subset\subset V \subset\subset \Omega$ the estimate*

$$\|u\|_{H^{k+m}(U)} \leq C\left\{\|f\|_{H^k(V)} + \|u\|_{H^{m+k-1}(V)}\right\}$$

holds. Here U and V are open sets and C is a constant which depends on $U, V, \Omega,$, and k but not on u and f.

We write $U \subset\subset V$ if $\overline{U}$ is a compact subset of V.

2.4. Boundary regularity: Reduction to a boundary value problem on the half space

Let $B_j(x,D) = \sum_{|\alpha|\leq m_j} b_{\alpha,j}(x)D^\alpha$ be differential operators of order $m_j \leq m-1$ for $j = 1,2,...,l$ which are defined for x in a neighborhood $\mathscr{V}$ of $\partial\Omega$ in Ω. Here $b_{\alpha,j} \in C^\infty(\overline{\mathscr{V}}, \mathbb{C}^{N\times N})$. We consider the boundary value problem for the elliptic operator P, that is

$$P(x,D)u = f \quad \text{in } \Omega\,, \qquad B_j(x,D)u = g_j \quad \text{in } \partial\Omega\,, j = 1,2,...,l \tag{2.4.1}$$

where f and g_j are given function of specified regularity. For $u \in H^{m+k}(\Omega)$ we will try to establish estimates of the form

(2.4.2)
$$\|u\|_{H^{m+k}(\Omega)} \leq C\left\{\|P(x,D)u\|_{H^k(\Omega)} + \sum_{j=1}^{l}\|B_j(x,D)u\|_{H^{m+k-m_j-1/2}(\partial\Omega)} + \|u\|_{H^{m+k-1}(\Omega)}\right\}$$

Given $x \in \partial\Omega$ there is a neighborhood $\mathscr{U}$ and a diffeomorphism $\varphi : \overline{\mathscr{U}} \to \overline{\varphi(\mathscr{U})}$ such that $\varphi(\partial\Omega \cap \mathscr{U}) \subset \mathbb{R}^{d-1}$ and $\varphi(\Omega\cap\mathscr{U}) \subset \mathbb{R}^{d+1}_+$. For $y \in \overline{\varphi(\mathscr{U})}$ we define the operators $P(y,D)$ and $B_j(y,D)$ by

$$[P(x,D)u]\circ\varphi^{-1} = P(y,D)(u\circ\varphi^{-1}) \quad \text{and } [B_j(x,D)u]\circ\varphi^{-1} = B_j(y,D)(u\circ\varphi^{-1})$$

for $j = 1,2,...,l$. With $y = \varphi(x)$ we define the constant coefficient operators

$$P_x(D) = P(y,D) \quad \text{and} \quad B_{j,x}(D) = B_j(y,D)$$

for a given $x \in \partial\Omega \cap \mathscr{U}$.

PROPOSITION 2.4.1. *Suppose that for all $x \in \partial\Omega$ we have the estimate*

(2.4.3)
$$\|u\|_{H^{m+k}(\mathbb{R}^d_+)} \leq C\left\{\|P_x(D)u\|_{H^k(\mathbb{R}^d_+)} + \sum_{j=1}^{l}\|B_{j,x}(D)u\|_{H^{m+k-m_j-1/2}(\mathbb{R}^{d-1})} + \|u\|_{H^{m+k-1}(\mathbb{R}^d_+)}\right\}$$

for all $u \in C_0^\infty(\overline{\mathbb{R}^d_+})$, $k \in \mathbb{N}_0$ where the constant is independent of x. If $u \in H^m(\Omega)$, $P(x,D)u \in H^k(\Omega)$, $B_j(x,D) \in H^{m+k-m_j-1/2}(\partial\Omega)$ for $j = 1,2,...,l$, then $u \in H^{m+k}(\Omega)$ and the estimate (2.4.2) holds.

2.5. Elliptic boundary value problems on the half space with constant coefficients

In this section we will establish the estimate (2.4.3) provided the boundary operators $B_j(D)$ satisfy certain conditions. All the work in this section will be done in the half space. For simplicity we introduce new variables $y = y_d$ and $x = (y_1, ..., y_{d-1})$ and split the differentiations into normal and tangential ones

$$P(D) = \frac{\partial^m}{\partial y^m} + \sum_{j=0}^{m-1} A_j(D_x) \frac{\partial^j}{\partial y^j}$$

where A_j is a tangential operator of order $m - j$. The operator P will be transferred into a first-order operator as follows. The Fourier multiplier Λ defined by

$$\Lambda u(x) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} \langle \xi \rangle \hat{u}(\xi) e^{i\xi\cdot x} d\xi$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ is used to introduce for a smooth vector-valued function u with N components another vector-valued function v with Nm components by setting

$$v_1 = \Lambda^{m-1} u, \quad v_2 = \Lambda^{m-2} \frac{\partial u}{\partial y}, \quad ...., v_j = \Lambda^{m-j} \frac{\partial^{j-1} u}{\partial y^{j-1}}, \quad ..., v_m = \frac{\partial^{m-1} u}{\partial y^{m-1}}$$

Lemma 2.5.1. *Given $u \in C_0^\infty(\overline{\mathbb{R}^{d+1}})$, we have $P(D)u = f$ if and only if $\partial v/\partial y = K(D_x)v + F$ where*

$$F = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ f \end{bmatrix} \quad \text{and} \quad K(D_x) = \begin{bmatrix} 0 & \Lambda I_N & 0 & \dots & \dots & 0 \\ 0 & 0 & \Lambda I_N & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & & & \Lambda I_N \\ E_1 & E_2 & E_3 & \dots & \dots & E_m \end{bmatrix}$$

where $E_j(D_x) = -A_{j-1}(D_x)\Lambda^{j-m}$ for $j = 1, 2, ..., m$ are $N \times N$ matrices.

One can show that K is a linear continuous operator from $H^{s+1}(\mathbb{R}^{d-1})$ into $H^s(\mathbb{R}^d)$ and that $K(\xi) = K_1(\xi) + K_R(\xi)$ where $K_1(\xi)$ is homogeneous of degree one in ξ, that is $K_1(\lambda\xi) = \lambda K_1(\xi)$ for all $\lambda > 0$ and $K_R(\xi)$ is uniformly bounded, that is $|K_R(\xi)| \le C$ where C is a positive constant and $|\cdot|$ is a matrix norm. More specifically,

$$K_1(\xi) = \begin{bmatrix} 0 & |\xi| I_N & 0 & \dots & \dots & 0 \\ 0 & 0 & |\xi| I_N & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & & & |\xi| I_N \\ \tilde{E}_1 & \tilde{E}_2 & \tilde{E}_3 & \dots & \dots & \tilde{E}_m \end{bmatrix} \quad \text{with} \quad \tilde{E}_j = -\tilde{A}_{j-1}|\xi|^{j-m} \ ,$$

where $\tilde{A}_j(\xi)$ denotes the principal part of $A_j(\xi)$ which consists of all terms of order $m-j$, for $j = 0, ..., m-1$. We observe that $K_1 \in C^\infty(\mathbb{R}^{d-1} \setminus \{0\})$.

LEMMA 2.5.2. *The operator $P(D)$ is elliptic if and only if for all $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$ the matrix $K_1(\xi)$ has non purely imaginary eigenvalues. In particular we have that*

$$\det[i\eta I_{Nm} - K_1(\xi] = \det P_m(\xi, \eta) = \det \left[(i\eta)^m + \sum_{j=0}^{m-1} \tilde{A}_j(\xi)(i\eta)^j \right] .$$

Similarly the boundary conditions $B_j(x, D)u\Big|_{y=0} = g_j$ for $j = 1, ..., l$ can be reduced in a way so that they become essentially conditions of order zero. We expand the operators

$$B_j(x, D) = B_j\left(D_x, \frac{\partial}{\partial y}\right) = \sum_{k \le m_j} b_{jk}(D_x)\frac{\partial^k}{\partial y^k}$$

for $j = 1, 2, ..., l$, where the order of the tangential operator $b_{jk}(D_x)$ is $m_j - k$. Note that the matrices b_{jk} are of type $p_j \times N$ where p_j is a non-negative integer less than or equal to N. Then, as in Lemma 2.5.1 we have that

$$B_j\left(D_x, \frac{\partial}{\partial y}\right) u\Big|_{y=0} = g_j \quad \Longleftrightarrow \quad \sum_{k=0}^{m_j} b_{jk}(D_x)\Lambda^{k-m_j} v_{k+1}\Big|_{y=0} = \Lambda^{m-m_j-1} g_j =: h_j \ ,$$

for $j = 1, 2, ..., l$. Let $B = B(D_x)$ be now the (block) matrix with blocks $b_{jk}(D_x)\Lambda^{k+m_j}$ which can be considered as a matrix with $p = \sum_{j=1}^{l} p_j$ rows and Nm columns. The boundary condition is than rewritten as

$$B(D_x)v = h \quad \text{where} \quad h = [h_1\ h_2\ ....\ h_l]^T$$

and $B(\xi) = B_0(\xi) + B_R(\xi)$ where $B_0(\xi)$ is homogeneous of degree zero in ξ and $B_r(\xi)$ decays for large $|\xi|$ as $|\xi|^{-1}$.

The desired estimate (2.4.3) can then be written as

$$\|v\|_{H^{k+1}(\mathbb{R}^d_+)} \le C \left\{ \|Lv\|_{H^k(\mathbb{R}^d_+)} + \|Bv(0)\|_{H^{k+1/2}(\mathbb{R}^{d-1})} + \|v\|_{H^k(\mathbb{R}^d_+)} \right\} \tag{2.5.1}$$

where $L = \partial/\partial y - K(D_x)$.

The spectral projection onto the eigenspace corresponding to all eigenvalues with positive real part is given by

$$E_+(\xi) = \frac{1}{2\pi i}\int_{\gamma(\xi)} (\zeta I_{Nm} - K_1(\xi))^{-1} d\zeta \ , \tag{2.5.2}$$

for $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$, where $\gamma(\xi)$ is a smooth, simple, closed curve in the right half plane which includes the eigenvalues of $K_1(\xi)$. The matrix function E_+ is smooth for all $\xi \neq 0$ and homogeneous of degree zero.

LEMMA 2.5.3. *The matrix function $\mathscr{A}(\xi) = (2E_+(\xi) - I_{Nm})K_1(\xi)$ is homogeneous of degree one in ξ and all its eigenvalues has positive real part.*

PROPOSITION 2.5.4. *The matrix function*

$$P(\xi) = |\xi| \int_{-\infty}^{0} e^{t\mathscr{A}(\xi)^H} e^{t\mathscr{A}(\xi)}\, dt$$

is a Hermitian positive definite matrix with entries in $C^\infty(\mathbb{R}^{d-1}\setminus\{0\})$ *and is homogeneous of degree zero in* ξ*. Furthermore,*

$$2\Re[P(\xi)\mathscr{A}(\xi)] := P(\xi)\mathscr{A}(\xi) + \mathscr{A}(\xi)^H P(\xi) = \frac{1}{2}|\xi| I_{Nm}\ .$$

The Sobolev space $H_{(k,s)}(\mathbb{R}^d_+)$. For a non-negative integer k and $s\in\mathbb{R}$ we set

$$H_{(k,s)}(\mathbb{R}^d_+) = \left\{ u\in\mathcal{E}'(\mathbb{R}^d)\ :\ \int_0^\infty\int_{\mathbb{R}^{d-1}} |D_y^j\hat{u}(\xi,y)|^2\langle\xi\rangle^{2(k-j+s)}d\xi dy < \infty \text{ for } j=0,...,k \right\}\ ,$$

where $\hat{u}(\xi,y)$ denotes the Fourier transform of $u(x,y)$ with respect to the x variables and $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. This space is a Hilbert space and the norm is given by

$$\|u\|^2_{(k,s)} = \sum_{j=0}^{k}\int_0^\infty\int_{\mathbb{R}^{d-1}} |D_y^j\hat{u}(\xi,y)|^2\langle\xi\rangle^{2(k-j+s)}d\xi dy \tag{2.5.3}$$

PROPOSITION 2.5.5. *If for* $g\in C_0^\infty(\mathbb{R}^{d-1})$ *and* $s\in\mathbb{R}$ *the inequality*

$$|g|^2_{H^{s+1/2}} \le C\left[|B_0 g|^2_{H^{s+1/2}} + |E_+ g|^2_{H^{s+1/2}}\right] \tag{2.5.4}$$

holds, then for all non-negative integers k *and real numbers* $\sigma < s$ *we have*

$$\|u\|_{(k,s)} \le C\left[\|Lu\|^2_{(k-1,s)} + |Bu(0)|^2_{H^{k+s-1/2}} + \|u\|^2_{(0,\sigma)}\right]$$

for all $u\in C_0^\infty(\overline{\mathbb{R}^d_+})$.

PROPOSITION 2.5.6. *The inequality (2.5.4) is equivalent to the following conditions.*
(i) For all $\xi\in\mathbb{R}^{d-1}\setminus\{0\}$ $\ker B_0(\xi)\cap\ker E_+(\xi) = \{0\}$.
(ii) For all $\xi\in\mathbb{R}^{d-1}\setminus\{0\}$ *there is no non-trivial bounded solution on the interval* $[0,\infty)$ *to the system of ordinary differential equations of order one with parameter* ξ

$$\frac{d\varphi}{dy} = K_1(\xi)\varphi$$

subject to the boundary condition $B_0(\xi)\varphi(0) = 0$.
(iii) For all $\xi\in\mathbb{R}^{d-1}\setminus\{0\}$ *there is no non-trivial bounded solution on the interval* $[0,\infty)$ *to the ordinary differential equation(s) of order* m *with parameter* ξ

$$\frac{d^m\Phi}{dy^m} + \sum_{j=0}^{m-1}\tilde{A}_j(\xi)\frac{d^j\Phi}{dy^j} = 0$$

subject to the boundary conditions $\tilde{B}_j(\xi)\varphi(0) = 0$ *for* $j=1,2,...,l$.

COROLLARY 2.5.7. *For all* $\xi\in\mathbb{R}^{d-1}\setminus\{0\}$ *the following three conditions are equivalent.*
(i) For all $\eta\in\mathbb{C}^p$ *there exists a unique solution* $\varphi\in L_2(\mathbb{R}_+)$ *to the system of ordinary differential equations of order one with parameter* ξ

$$\frac{d\varphi}{dy} = K_1(\xi)\varphi$$

subject to the boundary condition $B_0(\xi)\varphi(0) = \eta$.
(ii) For all $\eta_j\in\mathbb{C}^{p_j}$ *for* $j=1,2,...,l$ *there exists a unique solution* $\Phi\in L_2(\mathbb{R}_+)$ *to the*

ordinary differential equation(s) of order m with parameter ξ

$$\frac{d^m\Phi}{dy^m} + \sum_{j=0}^{m-1} \tilde{A}_j(\xi)\frac{d^j\Phi}{dy^j} = 0$$

subject to the boundary conditions $\tilde{B}_j(\xi)\varphi(0) = \eta_j$ for $j = 1, 2, ..., l$.
(iii) The mapping $B_0(\xi) : \ker E_+(\xi) \to \mathbb{C}^p$ is bijective.

Note that (iii) implies that $p = \dim\ker E_+(\xi)$.

DEFINITION 2.5.8. *The boundary value problem $P(D), B_1(D), ..., B_l(D)$ with constant coefficients on the half space is regular, if one of the conditions of Corollary 2.5.7 is satisfied.*

2.6. The Lopatinskii condition and the Fredholm property

DEFINITION 2.6.1. *The boundary value problem $(P(x, D), B_1(x, D), ..., B_l(x, D))$ satisfies the Lopatinskii condition, if for all $\underline{x} \in \partial\Omega$ the constant coefficient problem $(P(\underline{x}, D), B_1(\underline{x}, D), ..., B_l(\underline{x}, D))$, after a transformation to the half space, is a regular boundary value problem. In this case we say that the boundary value problem is a regular elliptic problem.*

REMARK 2.6.2. *The constant coefficient problem $(P(\underline{x}, D), B_1(\underline{x}, D), ..., B_l(\underline{x}, D))$ can be analyzed, even without a coordinate transformation to the half space. The Lopatinskii condition can be formulated as follows: For all $\underline{x} \in \Omega$ and $\eta_j \in \mathbb{C}^{p_j}$ there exists a unique solution to the frozen problem*

$$\begin{aligned} P(\underline{x}, D)u &= 0 \qquad \text{in } \{x \in \mathbb{R}^d \ : \ (x - \underline{x}) \cdot n(\underline{x}) < 0\} \ , \\ B_j(\underline{x}, D)u &= \eta_j \qquad \text{in } \{x \in \mathbb{R}^d \ : \ (x - \underline{x}) \cdot n(\underline{x}) = 0\}, \quad \text{for } j = 1, 2, ..., l \ , \end{aligned}$$

of the form $u(x) = e^{i\xi\cdot(x-\underline{x})}v(y)$ with $\xi \in \mathbb{R}^d \setminus \{0\}$, $\xi \cdot n(\underline{x}) = 0$, $y = (x - \underline{x}) \cdot n(\underline{x})$, and $v(y) \to 0$ as $y \to -\infty$.

For $k \geq 0$, a non-negative integer define the operator

$$T \ : \ H^{m+k}(\Omega) \to \mathscr{H}^k := H^k(\Omega) \times H^{m+k-m_1-1/2}(\partial\Omega) \times \cdots \times H^{m+k-m_l-1/2}(\partial\Omega)$$

by

$$Tu = (P(x, D)u, B_j(x, D)u, ..., B_l(x, D)u) \ . \tag{2.6.1}$$

Note that T is a continuous a linear operator from $H^{m+k}(\Omega)$ into $\mathcal{H}^k$. The following theorem is the main result of this chapter.

THEOREM 2.6.3. *The following two statements are equivalent.*
(i) The boundary value problem $(P(x, D), B_1(x, D), ..., B_l(x, D))$ satisfies the Lopatinskii condition.
(ii) The operator T is a Fredholm operator, that is $\dim\ker T < \infty$ *and the range of T is closed with finite co-dimension.*
Furthermore, each of them implies that for all $u \in H^{m+k}(\Omega)$ there exists a constant

depending only on the non-negative integer k, Ω, P, and the boundary operators B_j for $j = 1, 2, .., l$ such that

(2.6.2)

$$\|u\|_{H^{m+k}(\Omega)} \leq C \left\{ \|P(x,D)u\|_{H^k(\Omega)} + \sum_{j=1}^{l} \|B_j(x,D)u\|_{H^{m+k-m_j-1/2}(\partial\Omega)} + \|u\|_{H^{m+k-1}(\Omega)} \right\} .$$

The proof of this Theorem is broken down into several propositions. At first we note that the last statement of the theorem follows from Proposition 2.4.1 and 2.5.5. An important part of statement (i) is equivalent to statement (ii) follows from

Proposition 2.6.4. *Let X, Y, Z be Hilbert spaces such that the embedding $X \subset Y$ is compact and that the operator $T : X \to Z$ is linear and continuous. Then* $\dim \ker T < \infty$ *and $T(X)$ is closed if and only if there exists a constant $c > 0$ such that*

$$\|x\|_X \leq c[\|Tx\|_Z + \|x\|_Y] \qquad \textit{for all } x \in X \ .$$

Applying this proposition one sees that (2.6.2) is equivalent to $\dim \ker T < \infty$ and range of T closed. Hence, the only thing left is to show that the Lopatinskii condition implies that the image of T has finite co-dimension and vice versa. In this context the following result from Functional Analysis is of interest.

Proposition 2.6.5. *The linear continuous operator $T : X \to Z$ is Fredholm if and only if there exist linear and continuous operators S_1 and S_2 from Z to X such that*

$$S_1 T = I + K_1 \quad \textit{and} \quad T S_2 = I + K_2$$

where $K_1 : X \to X$ and $K_2 : Z \to Z$ are compact operators.

The operators S_1 and S_2 are a left Fredholm inverse and a right Fredholm inverse, respectively. As a matter of fact, one can show that the conditions of Proposition 2.6.4 are equivalent to the existence of a left Fredholm inverse. Hence, the following result is of interest.

Proposition 2.6.6. *If the boundary value problem $(P(x,D), B_1(x,D), ..., B_l(x,D))$ satisfies the Lopatinskii condition, then there exists a right Fredholm inverse for the operator T.*

This proposition implies the proof of the important implication (i) ⇒ (ii) of Theorem 2.6.3.

Corollary 2.6.7 (Weyl's Lemma). *If $u \in H^m(\Omega)$, $P(x,D)u \in C^\infty(\overline{\Omega})$, and $B_j(x,D)u \in C^\infty(\partial\Omega)$ for $j = 1, 2, ..., l$, then $u \in C^\infty(\overline{\Omega})$.*

Corollary 2.6.8. *If the statements of Theorem 2.6.3 are true for $P(x,D)$ and $B_j(x,D)$ for $j = 1, 2, ..., l$, then they are also valid for all operators $\tilde{P}$ with boundary conditions $\tilde{B}_j$ as long as those have the same principal symbols as P and B_j for $j = 1, ..., l$.*

Corollary 2.6.9. *The index of the operator T defined in (2.6.1), i.e.*

$$\mathrm{ind}\, \mathrm{T} = \dim \ker \mathrm{T} - \dim \operatorname{coker} \mathrm{T}$$

is independent of k. Furthermore the index is stable with respect to small perturbations of the coefficients of the operator and the boundary conditions. Also $\ker T$ *is a finite-dimensional subspace of $C^\infty(\overline{\Omega})^N$ and the image of T is the orthogonal complement in $L_2(\Omega)^N \times L_2(\partial\Omega)^p$ of a finite-dimensional subspace of $C^\infty(\overline{\Omega})^N \times C^\infty(\partial\Omega)^p$.*

For clarity we have added an upper index for the vector-valued functions which occur in this setting. The function u maps from Ω into $\mathbb{C}^N$ and the vector $P(x,D)u$ has N components and the vector $B_1(x,D)u, ..., B_l(x,D)u$ has p components.

2.7. Strongly elliptic operators

Definition 2.7.1. *An $N \times N$ elliptic operator (in the sense of Definition 2.3.1) of even order $m = 2\nu$ is strongly elliptic if and only if there exists a positive constant such that*

$$\Re P_m(x,\xi) \geq c|\xi|^m \qquad \text{for all } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^d \, .$$

Recall that for all $N \times N$ matrix A, $\Re A$ denotes its Hermitian part and that $A \geq c$ means $v^H A v \geq c|v|^2$ for all $v \in \mathbb{C}^N$.

Definition 2.7.1. *The Dirichlet boundary conditions for a strongly elliptic P are*

$$B_1 u = u \ , \quad B_2 u = \frac{\partial u}{\partial n} \ , \dots , B_\nu u = \frac{\partial^{\nu-1}}{\partial n^{\nu-1}} \ .$$

Here n is a vector field in $\mathbb{R}^d$ which coincides with the exterior unit normal vector field to Ω on $\partial\Omega$.

Proposition 2.7.2. *For strongly elliptic operator P the Dirchlet boundary conditions satisfy the Lopatinskii condition.*

In order to find simpler ways of verifying the Lopatinskii condition, we will consider a special coordinate transform. The goal is to essentially avoid the transformation into the half space. For that purpose we introduce the oriented distance function for our smooth domain Ω

$$b_\Omega(x) = d_\Omega(x) - d_{\mathbb{R}^d \setminus \Omega}(x) \ ,$$

$d_A(x) = \inf_{y \in A} |x - y|$ is the distance function. The following result is taken from the book of M. Delfour and J.-P. Zolésio [**DZ11**, Chapter 7, Theorem 8.2] where much more material concerning the oriented distance function can be found.

Theorem 2.7.3. *The function b_Ω is smooth in a neighborhood $\mathscr{V}$ of $\partial\Omega$ in $\mathbb{R}^d$ and $\nabla b_\Omega(x) = n(x)$ for all $x \in \partial\Omega$.*

With the help of the oriented distance function we will construct special coordinate mappings. Fix $\underline{x} \in \partial\Omega$. Due to our assumptions on the set Ω there exists a coordinate mapping $\varphi \ : \ \mathscr{U}(\underline{x}) \to \mathbb{R}^d$ such that

$$\varphi(\Omega \cap \mathscr{U}(\underline{x}) \subset \mathbb{R}^d_+ \quad \text{ and } \quad \varphi(\partial\Omega \cap \mathscr{U}) \subset \{y \in \mathbb{R}^d \ : \ y_d = 0\}$$

Without loss of generality we have $\mathscr{U}(\underline{x}) \subset \mathscr{V}$. Define now for $x \in \mathscr{U}(\underline{x})$

$$\psi(x) = (\varphi_1(z), ..., \varphi_{d-1}(z), b_\Omega(x)) = (\varphi'(z), b_\Omega(x)) \ , \quad z = x - b_\Omega(x)\nabla b_\Omega(x) \ .$$

Here φ_j denotes the jth component of the vector-valued function φ and $\varphi' = (\varphi_1, ..., \varphi_{d-1})$. Note that $z = z(x)$ is a projection of x on $\partial\Omega$. One can show that ψ is a coordinate mapping, that is ψ is of class C^∞ on $\overline{\mathscr{U}}(\underline{x})$ and it has a C^∞ inverse.

The Jacobian matrix of ψ is

$$J_\psi(x) = \begin{bmatrix} J_{\varphi'}(z)[I - \nabla b_\Omega(x)\nabla b_\Omega^T(x) - b_\Omega(x)\nabla^2 b_\Omega(x)] \\ \nabla b_\Omega(x) \end{bmatrix}$$

Given a diffeomorphism $\Phi : \Omega \to \Omega'$ (see Definition 2.2.7) we have

$$[P(x, D)u] \circ \Phi^{-1} = P(\Phi^{-1}(y), J_\Phi(\Phi^{-1})D)(u \circ \Phi^{-1})$$

where J_ϕ is the Jacobi matrix (derivative) of Φ that is

$$J_\Phi = \left(\frac{\partial \phi_j}{\partial x_k}\right)_{1\le j,k\le d}$$

Note that a tangent vector $w \in T_x\Omega$ transfers into the tangent vector $J_\Phi(x)w \in T_{\Phi(x)}\Omega$. A cotangent vector $\xi \in T_x^*\Omega'$ becomes $J_\Phi^{-T}(x)\xi \in T_{\Phi(x)}^*\Omega'$. Here A^{-T} is the transpose of the inverse of the square matrix A.

Proposition 2.7.4. *The principal symbol of an operator P transforms correspondingly to the transformation rule for the cotangent bundle. With $y = \Phi(x)$, the principal symbol $P_m(x, \xi)$ transforms into $\mathbb{P}(y, \eta) = P_m\left(\Phi^{-1}(y), J_\Phi^T(\Phi^{-1}(y))\eta\right)$.*

Observe that $x = \Phi^{-1}(y)$ and $\xi = J_\Phi^T(\Phi^{-1}(y))\eta$, that is $\eta = J_\Phi^{-T}(x)\xi$.

Corollary 2.7.5. *Under the coordinate transformation ψ defined above, the principal symbol of our differential operator $P(x, D)$ can be written in the form*

$$P_m(x, \xi + \tau n(x)) \qquad \textit{for all } x \in \mathscr{U}(\underline{x})$$

Hence, in order to verify the Lopatinskii condition it suffices to work with the ordinary differential operators

$$P_m\left(x, \xi - n(x)\frac{1}{i}\frac{d}{dy}\right) \qquad \textit{and} \qquad \tilde{B}_j\left(x, \xi - n(x)\frac{1}{i}\frac{d}{dy}\right)$$

for $j = 1, 2, ..., l$. No explicit transformation of the operator into the half space is necessary.

2.8. Stationary linear elasticity: an elliptic system of order 2

The region $\Omega \subset \mathbb{R}^3$ is now considered as an elastic body and the function $u : \Omega \to \mathbb{R}^3$ measures the (elastic) displacement due to external forces. The Cauchy stress tensor is defined by

$$\sigma(u)_{jk} = \sum_{l,m=1}^{3} a^{jklm}(x)e_{lm} \,,$$

where $e_{lm} = [\partial_l u_m + \partial_m u_l]/2$ is the strain tensor. Note that the strain tensor is the symmetric Jacobian of u. Suppose that $f : \Omega \to \mathbb{R}^3$ is a force acting on the elastic body.

The equations of stationary linear elasticity are

$$-\sum_{k=1}^{3} \frac{\partial \sigma(u)_{jk}}{\partial x_k} = f_j \ , \qquad j = 1,2,3 \ . \tag{2.8.1}$$

The tensor of material stiffness a^{jklm} is a fourth order tensor with real components and satisfies the following symmetry relations.

$$a^{jklm} = a^{jkml} = a^{kjlm} = a^{lmjk} \ ,$$

which means there are up to 21 different coefficients which capture the elastic properties of the medium. Furthermore, this tensor is assumed to be positive definite, that is, there exists a positive constant c such that

$$\sum_{j,k,l,m=1}^{3} \overline{w}_j k a^{jklm} w_{lm} \geq c \sum_{k,j=1}^{3} |w_{jk}|^2 \ , \qquad \text{for all } x \in \overline{\Omega} \ ,$$

for all real symmetric 3×3 matrices (w_{jk}). This property can be used to establish the fact that the operator $E(x,D) = -\sum_{k=1}^{3} \frac{\partial \sigma(u)_{jk}}{\partial x_k}$ is strongly elliptic. Because of the symmetry relations, one can rewrite the operator with a 6×6 positive definite coefficient matrix $\mathscr{A}$ instead of the fourth order tensor. For that purpose one groups the first two indices and the last two indeces of the of the tensor a^{jklm} into pairs and re-labels them as follows

$$11 \to 1, \quad 22 \to 2, \quad 33 \to 3, \quad 23 = 32 \to 4, \quad 31 = 13 \to 5, \quad 12 = 21 \to 6 \ .$$

With the operator

$$D(\partial) = \begin{bmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{bmatrix}$$

the equation (2.8.1) turns into

$$-D(\partial)^T \mathscr{A}(x) D(\partial) u = f \qquad \text{in } \Omega \ . \tag{2.8.2}$$

We discuss now the Dirichlet problem for the stationary equations of elasticity, that is the equation above is complemented by the boundary condition

$$u = g \qquad \text{in } \partial\Omega \ .$$

Since the operator E is strongly elliptic, the Dirichlet boundary condition satisfies the Lopatinskii condition, see Proposition 2.7.2. From Theorem 2.6.3 we know that, given a non-negative integer k, there exists a constant C such that

$$\|u\|_{H^{k+m}(\Omega)} \leq C \left\{ \|E(x,D)u\|_{H^k(\Omega)} + \|u\|_{H^{k+3/2}(\partial\Omega)} + \|u\|_{H^{k+1}(\Omega)} \right\}$$

for all $u \in H^{2+k}(\Omega)^3$. Furthermore, the operator $T : H^{2+k}(\Omega)^3 \to H^k(\Omega)^3 \times H^{3/2+k}(\partial\Omega)^3$ defined by $Tu = (E(x,D)u, u|_{\partial\Omega})$ is a Fredholm operator.

We claim that $\ker T = \{0\}$. To see this one can use the fact that $u \in \ker T$ implies that $u \in C^\infty(\Omega)^3$ by Corollary 2.6.9. Furthermore, using integration by parts and the positive definiteness of $\mathscr{A}$ implies for all $u \in \ker T$ that

$$
\begin{aligned}
0 &= \int_\Omega E(x,D)u \cdot \overline{u(x)}\,dx = \int_\Omega \mathscr{A}(x)D(\partial)u(x)\overline{D(\partial)u(x)}\,dx \\
&\quad - \int_{\partial\Omega} D(n(x))^T \mathscr{A}(x)D(\partial)u(x)\overline{u(x)}\,ds \\
&= \int_\Omega \mathscr{A}(x)D(\partial)u(x)\overline{D(\partial)u(x)}\,dx \geq C\int_\Omega |[\nabla u]^s(x)|^2\,dx\,.
\end{aligned}
\tag{2.8.3}
$$

This implies that the symmetric Jacobian $([\nabla u]^s_{jk}) = (e_{jk})$ vanishes in Ω. The following result know as *Korn's first inequality* shows that this implies that $u \equiv 0$. For the formulation we will use the Sobolev space $\mathring{H}^1(\Omega)$, see Corollary 2.2.10.

Proposition 2.8.1. *Suppose that $u \in \mathring{H}^1(\Omega)$. Then*

$$
\frac{1}{3}\int_\Omega |\nabla u(x)|^2\,dx = \frac{1}{3}\sum_{j,k=1}^{3}\int_\Omega |\partial_j u_k(x)|^2\,dx \leq \int_\Omega |D(\partial)u(x)|^2\,dx\,.
$$

In order to solve the Dirichlet problem we need also to determine the Co-kernel of T, that is the orthogonal complement of $T(H^{2+k}(\Omega))$ in $L_2(\Omega)^3 \times L_2(\partial\Omega)^3$. For that purpose we define a linear, unbounded operator $\mathscr{P}$ on $L_2(\Omega)$ by setting $\mathscr{D}(\mathscr{P}) = H^2(\Omega)^3 \cap \mathring{H}^1(\Omega)^3$ and $\mathscr{P}u = E(x,D)u$ for all $u \in \mathscr{D}(\mathscr{P})$.

We will show that $\mathscr{P}$ is self-adjoint, that is $\mathscr{P}^* = \mathscr{P}$ where $\mathscr{P}^*$ is the Hilbert space adjoint of $\mathscr{P}$ in the sense of unbounded operators. Recall that the definition of the domain of the domain of $\mathscr{P}^*$ is

$$
\mathscr{D}(\mathscr{P}^*) = \{v \in L_2(\Omega)^3 \ :\ |(\mathscr{P}u, v)| \leq C(v)\|u\|_{L_2(\Omega)}\}
\tag{2.8.4}
$$

For the proof of self-adjointness of $\mathscr{P}$ via the Green formula, the following result is needed. For that we recall the Sobolev space $H_{m,k}(\mathbb{R}^d_+)$ and its norm introduced in formula (2.5.3). With the introduction of local normal coordinates in the previous section we can introduce this Sobolev space also in a neighborhood of the boundary $\partial\Omega$ in Ω. Let $\mathscr{V}$ be the neighborhood of the boundary $\partial\Omega$ introduced in Theorem 2.7.3. With the local coordinate mapping ψ, this neighborhood $\mathscr{V}$ is diffeomorphic to the set $(-\delta,\delta)\times\partial\Omega$ for some $\delta > 0$. Set $\mathscr{C} = \mathscr{V} \cap \partial\Omega$ and for $k \in \mathbb{N}_0$ and $s \in \mathbb{R}$

$$
H_{k,s}(\mathscr{C}) = \left\{ u \in \mathcal{E}(\mathbb{R}^d) \ :\ \|u\|^2_{(k,s)} = \sum_{j=0}^{k}\int_0^\delta \|D^j_y u(\cdot,y)\|^2_{H^{k-j+s}(\partial\Omega)}dy < \infty \right\}\,.
$$

The following result is an extension of Proposition 2.4.1.

Proposition 2.8.2. *Suppose that $P(x,D)$ is an elliptic operator of order m and $\{P(x,D), B_j(x,D), j = 1,...,l\}$ is a regular elliptic boundary value problem. If $u \in H_{(m,\sigma)}(\mathscr{C})$ for some $\sigma \in \mathbb{R}$, $P(x,D) \in H_{(k,s)}(\mathscr{C})$, $B_j(x,D)u \in H^{m+k-m_j-1/2+s}(\partial\Omega)$ for $j = 1,2,...,l$, then $u \in H_{(m,\sigma)}(\mathscr{C})$ together with the corresponding estimate.*

The proof of this proposition follows with the techniques developed in Section 2.5.

COROLLARY 2.8.3. *If* $u \in L_2(\Omega)$, $P(x,D)u \in L_2(\Omega)$, *and* $B_j(x,D)u \in H^{m-m_j-1/2}(\partial\Omega)$, *then* $u \in H^m(\Omega)$.

Now we return to the operator $\mathscr{P}$ defined above. In order to determine $\mathscr{D}(\mathscr{P}^*)$ we will need, that for $u, v \in H^2(\Omega)^3$ we have

$$(E(x,D)u, v)_\Omega - (u, E(x,D)v)_\Omega = (u, Nv)_{\partial\Omega} - (Nu, v)_{\partial\Omega}\ . \tag{2.8.5}$$

Here $(\cdot,\cdot)_A$ denotes the scalar product in $L_2(A)$ and

$$(Nu)(x) = D(n(x))^T \mathscr{A} D(\partial)u(x) = \sigma(u)n(x)\ , \qquad x \in \partial\Omega$$

denotes the Neumann operator in the context of stationary elasticity which occurred already in (2.8.3). Because of the similarity of (2.8.5) with the second Green formula, this formula is also referred to as Green formula. It is a feature of all regular elliptic boundary value problems that such a Green formula can be found. As we will see with our example for stationary elasticity, this formula is a useful tool to determine the cokernel of the Fredholm operator T.

Note that formula (2.8.5) is also valid for $u \in \mathscr{D}(\mathscr{P})$ and $v \in L_2(\Omega)^3$ and $E(x,D)v \in L_2(\Omega)^3$. In this case one can show that $v|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)^3$. In view of formula (2.8.5) one needs for the estimate in the definition (2.8.4) to hold that $v \in L_2(\Omega)^3$, $E(x,D)v \in L_2(\Omega)^3$ and that $v|_{\partial\Omega} = 0$. Corollary 2.8.3 implies that $v \in H^2(\Omega)^3$ and thus $\mathscr{D}(\mathscr{P}^*) = \mathscr{D}(\mathscr{P})$. From formula (2.8.5) we infer that $(\mathscr{P}u, v)_\Omega = (u, \mathscr{P}v)_\Omega$ which shows that $\mathscr{P}$ is self-adjoint.

Since we know that $\ker T = \ker \mathscr{P} = \{0\}$, using the formula

$$\operatorname{range} \mathscr{P}^* = \ker \mathscr{P}^\perp$$

and the self-adjointness proves that $\mathscr{P}$ is surjective. Hence, the operator $T\ :\ H^2(\Omega) \to L_2(\Omega)^3 \times H^{3/2}(\partial\Omega)^3$ is surjective. Hence $\operatorname{ind} T = 0$. Since the index of the operator does not depend on k and the $\ker T$ can only become smaller with increasing k, we have shown that the operator $T\ :\ H^{2+k}(\Omega) \to H^k(\Omega)^3 \times H^{3/2+k}(\partial\Omega)^3$ is surjective for all $k \in \mathbb{N}$.

A similar analysis can be made for the Neumann problem of stationary elasticity, that is for the problem

$$E(x,D)u = f \text{ in } \Omega\ , \qquad Nu = g \text{ in } \partial\Omega\ .$$

Throughout our lectures we have acquired all the necessary tools, except for the following version of *Korn's second inequality*.

PROPOSITION 2.8.4. *If* $u \in H^1(\Omega)^3$, $E(x,D)u \in L_2(\Omega)^3$ *and* $Nu|_{\partial\Omega} = 0$, *then there exists a constant* C *such that*

$$\|u\|_{H^1(\Omega)} \le C \int_\Omega |D(\partial)u(x)|^2\, dx\ .$$

CHAPTER 3

Hyperbolic systems of partial differential equations

3.1. First order systems: Definitions

Throughout this chapter the space variable will be denoted by $x \in \mathbb{R}^d$ and the time will be $t \in [0, \infty) = \overline{\mathbb{R}_+}$. Given matrix functions $A^j, D : [0, \infty) \times \mathbb{R}^d \to \mathbb{C}^{N \times N}$ for $j = 0, 1, ..., d$ a differential operator of first order is defined by

$$P(t, x; D) = A^0(t, x)\frac{\partial}{\partial t} + \sum_{j=1}^{d} A^j(t, x)\frac{\partial}{\partial x_j} + D(t, x) \ .$$

The principal part of the operator is

$$P_1(t, x; D) = A^0(t, x)\frac{\partial}{\partial t} + \sum_{j=1}^{d} A^j(t, x)\frac{\partial}{\partial x_j}$$

and the principal symbol is

$$P_1(t, x; \tau, \xi) = iA^0(t, x)\tau + i\sum_{j=1}^{d} A^j(t, x)\xi_j + D(t, x) \ .$$

Definition 3.1.1. *The operator P is symmetric hyperbolic if the matrices A^j, $j = 0, 1, ..., d$ are Hermitian and the matrix A^0 is uniformly positive definite, that is there exists a positive constant such that $w^H A^0(t, x)w \geq c|w|^2$ for all $(t, x) \in [0, \infty) \times \mathbb{R}^d$ and $w \in \mathbb{C}^N$.*

Definition 3.1.2. *The operator P is strictly hyperbolic if for all $\xi \in S^{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ and $(t, x) \in [0, \infty) \times \mathbb{R}^d$ the matrix*

$$(A^0(t, x))^{-1}\sum_{j=1}^{d} A^j(t, x)\xi_j$$

has only simple real eigenvalues.

The operator P is constantly hyperbolic if the eigenvalues are real, semi-simple and their algebraic multiplicity does not change with (t, x, ξ).

Recall that eigenvalues are semi-simple if their algebraic and geometric multiplicities coincide.

3.2. The apriori estimate for symmetric hyperbolic systems

Given $X \subset \mathbb{R}^d$, recall the Sobolev space

$$W^1_\infty(X) = \left\{u \in L_\infty(X) \ : \ \frac{\partial u}{\partial x_j} \in L_\infty(X) \text{ for } j = 1, 2, ..., d\right\}$$

In what follows we abbreviate $Q = (0,T) \times \mathbb{R}^d$ where T is a positive number or ∞.

LEMMA 3.2.1. *Suppose that $u \in L_\infty(Q)$. Then $u \in W^1_\infty(Q)$ if and only if u is Lipschitz, that is there exists a positive constant L such that $|u(s,x) - u(t,y)| \leq L[|t-s| + |x-y|]$*

PROPOSITION 3.2.2. *Suppose that P is symmetric hyperbolic, that the coefficients $A^j \in W^1_\infty(Q)$ for $j = 0, ..., d$, and that $D \in L_\infty(Q)$. Then there exists constants $C > 0$ and $\gamma_0 > 0$ such that*

$$\|e^{-\gamma T} u(T)\|^2_{L_2(\mathbb{R}^d)} + \gamma \|e^{-\gamma t} u\|^2_{L_2(Q)} \leq \frac{1}{\gamma} \|e^{-\gamma t} P(t,x;D) u\|^2_{L_2(Q)} + \|u(0)\|^2_{L_2(\mathbb{R}^d)}$$

for all $u \in H^1(Q)$ and $\gamma \geq \gamma_0$.

By the trace theorem (Theorem 2.2.4), we know that $u(t) \in H^{1/2}(\mathbb{R}^d)$ for $0 \leq t \leq T$ so all expressions in the estimate above are well defined.

Note that the function space $H^1(Q)$ can be expressed as the intersection of two functions spaces of Hilbert space valued functions of one variable. This approach is useful to show that for $u \in H^1(Q)$ the expression $u(t)$ is defined as an L_2 function.

$$H^1(Q) = H^1(0,T; L_2(\mathbb{R}^d)) \cap L_2(0,T; H^1(\mathbb{R}^d))$$

where

$$L_2(0,T; H^1(\mathbb{R}^d)) = \left\{ u(t) \in H^1(\mathbb{R}^d) \text{ for } 0 < t < T \ : \ \int_0^T \|u(t)\|^2_{H^1(\mathbb{R}^d)} dt < \infty \right\}$$

and

$$H^1(0,T); L_2(\mathbb{R}^d)) = \left\{ u(t), u'(t) \in L_2(\mathbb{R}^d) \text{ for } 0 < t < T \ : \ \begin{array}{l} \int_0^T \|u(t)\|^2_{L_2(\mathbb{R}^d)} dt < \infty \\ \int_0^T \|u'(t)\|^2_{L_2(\mathbb{R}^d)} dt < \infty \end{array} \right\} .$$

Here $u'(t)$ denotes the distributional derivative of the $L_2(\mathbb{R}^d)$ valued function $u(t)$ with respect to t. By definition this means that for all function $\psi \in C_0^\infty(0,T; L_2(\mathbb{R}^d))$ we have

$$\int_0^T (u'(t), \psi(t))_{L_2(\mathbb{R}^d)} dt = -\int_0^T (u(t), \psi'(t))_{L_2(\mathbb{R}^d)} dt$$

where $\psi'(t)$ satisfies

$$\|\psi(t+h) - \psi(t) - \psi'(t) h\|_{L_2(\mathbb{R}^d)} = o(h) \quad \text{as } h \to 0$$

for $0 < t < T$. Since every function in $H^1(0,T)$ is continuous, one infers that every function in $H^1(Q)$ is also in $C([0,T], L_2(\Omega))$.

3.3. Existence and uniqueness of weak solutions to symmetric hyperbolic systems

Throughout this section we will assume that P is symmetric hyperbolic with coefficients $A^j \in W^1_\infty(Q, \mathbb{C}^{N \times N})$ and $D \in L_\infty(Q, \mathbb{C}^{N \times N})$. Let $u, v \in H^1(Q)$. Then

$$(Pu, v)_{L_2(Q)} = (u, P^* v)_{L_2(Q)} + (A^0 u, v)_{L_2(\mathbb{R}^d)} \Big|_{t=0}^{t=T}$$

where

$$P^*v = -\frac{\partial}{\partial t}[A^0 v](t,x) - \sum_{j=1}^{d} \frac{\partial}{\partial x_j}[A^j v](t,x) + [D^H v](t,x)$$

LEMMA 3.3.1. *The operator $-P^*$ is symmetric hyperbolic and there exists positive constants C and γ_0 such that*

$$\|v(0)\|^2_{L_2(\mathbb{R}^d)} + \gamma\|e^{-\gamma t}v\|^2_{L_2(Q)} \leq C\left\{\frac{1}{\gamma}\|e^{-\gamma t}P^*v\|^2_{L_2(Q)} + \|e^{-\gamma T}v(T)\|^2_{L_2(\mathbb{R}^d)}\right\}$$

for $\gamma \geq \gamma_0$ and all $v \in H^1(Q)$.

For the time being we restrict ourselves to the case $T < \infty$.

PROPOSITION 3.3.2. *The initial value problem $Pu = f \in L_2(Q)$, $u(0,\cdot) = g \in L_2(\mathbb{R}^2)$ admits a weak solution $u \in L_2(Q) \cap C([0,T], H^{-1}(\mathbb{R}^d))$.*

THEOREM 3.3.3. *The weak solution u satisfies $u \in C([0,T], L_2(\mathbb{R}^d))$ and it is unique and satisfies the estimate*

$$\|u(T)\|_{L_2(\mathbb{R}^d)} + \|u\|_{L_2(Q)} \leq C_T\left\{\|f\|_{L_2(Q)} + \|g\|_{L_2(\mathbb{R}^d)}\right\}$$

The proof of this Theorem will be performed by means of regularization. This is a technique developed by K. O. Friedrichs and applied to symmetric hyperbolic systems in his classical work [**Fri54**]. Since we are dealing with an initial value problem, we will perform the regularization only with respect to x.

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\int \varphi(x)\,dx = 1$, $\varphi(x) = \varphi(-x)$, $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^d$ and $\varphi(x) = 0$ for $|x| \geq 1$. Then introduce the set of functions

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi\left(\frac{x}{\varepsilon}\right)$$

and for a function $u \in L_{1,\mathrm{loc}}(R^d, \mathbb{C}^N)$, set

$$u^{(\varepsilon)}(x) = [\varphi_\varepsilon * u](x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)u(y)\,dy\ .$$

Since the convolution with smooth, compactly supported functions is defined even for distributions one can define the regularization $u^{(\varepsilon)}$ for $u \in \mathcal{D}(\mathbb{R}^d)$ by

$$u^{(\varepsilon)}(x) = \langle \varphi(x-\cdot), u\rangle\ .$$

LEMMA 3.3.4. *Let $s \in \mathbb{R}$. For $u \in H^s(\mathbb{R}^d)$, we have $\|u^{(\varepsilon)} - u\|_{H^s(\mathbb{R}^d)} \to 0$ for $\varepsilon \to 0$.*

The same result holds for $u \in L_p(\mathbb{R}^d)$ for $1 \leq p < \infty$, that is $\|u^{(\varepsilon)} - u\|_{L_p(\mathbb{R}^d)} \to 0$ and if $u \in C(\mathbb{R}^d)$, then $u^{(\varepsilon)}$ converges to u uniformly on compact subsets.

The proof of Theorem 3.3.3 is rather straightforward in the case of constant coefficients since in this case the operator $P = P(D)$ and the regularization operator commute, that is $(P(D)u)^{(\varepsilon)} = P(D)u^{(\varepsilon)}$. Since we are studying an initial value problem, the regularization is applied only to the space variables in an effort not to destroy the initial condition. Given $u \in L_2(Q)$ we have $u^{(\varepsilon)} \in L_2(0,T,C^\infty(\mathbb{R}^d))$.

In the case of variable coefficients the issue is much more subtle and relies on the following result, see also see Lemma 11.27 in [**RR93**]

PROPOSITION 3.3.5 (Friedrich's Lemma). *Suppose that $a \in W^1_\infty(\mathbb{R}^d)$ and $u \in L_2(\mathbb{R}^d)$. Then*

$$\left\| \left(a \frac{\partial u}{\partial x_j} \right)^{(\varepsilon)} - a \frac{\partial u^{(\varepsilon)}}{\partial x_j} \right\|_{L_2(\mathbb{R}^d)} \longrightarrow 0 \quad \text{as } \varepsilon \to 0 \,, \qquad \text{for all } j = 1, 2, ..., d \,.$$

COROLLARY 3.3.6. *Suppose that $a \in L_\infty(\mathbb{R}^d)$ and $u \in L_2(\mathbb{R}^d)$. Then $\|(au)^{(\varepsilon)} - au^{(\varepsilon)}\|_{L_2(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$.*

3.4. Differentiable solutions of symmetric hyperbolic systems

THEOREM 3.4.1. *Suppose that P is symmetric hyperbolic with all coefficients in $W^1_\infty(Q)$. Then the initial value problem $Pu = f \in H^1(Q)$, $u(0, \cdot) = g \in H^1(\mathbb{R}^d)$ has a unique solution $u \in C([0,T], H^1(\mathbb{R}^d)) \cap C^1([0,T], L_2(\mathbb{R}^d))$. Furthermore, the solution satisfies the estimate*

$$\|u(T)\|_{H^1(\mathbb{R}^d)} + \|u\|_{H^1(Q)} \leq C_T \left\{ \|f\|_{H^1(Q)} + \|g\|_{H^1(\mathbb{R}^d)} \right\}$$

The proof of this theorem builds on the techniques developed earlier in this chapter. At first the *a priori* estimate

$$\|e^{-\gamma T} u(T, \cdot)\|^2_{1,\gamma,\mathbb{R}^d} + \gamma \|e^{-\gamma t} u\|^2_{1,\gamma,Q} \leq C \left\{ \frac{1}{\gamma} \|e^{-\gamma t} Pu\|^2_{1,\gamma,Q} + \|u(0, \cdot)\|^2_{1,\gamma,\mathbb{R}^d} \right\}$$

is established for all $u \in H^2(Q)$. For convenience, one uses the weighted norms

$$\|w\|^2_{1,\gamma,\mathbb{R}^d} = \gamma^2 \|w\|^2_{L_2(\mathbb{R}^d)} + \sum_{j=1}^d \|\partial_j w\|^2_{L_2(\mathbb{R}^d)} \qquad \text{and}$$

$$\|w\|^2_{1,\gamma,Q} = \gamma^2 \|w\|^2_{L_2(Q)} + \sum_{j=1}^d \|\partial_j w\|^2_{L_2(Q)} + \|\partial_t w\|^2_{L_2(Q)}$$

After that procedure one invokes Theorem 3.3.3 and regularizes the weak solution $u \in C([0,T], L_2(\mathbb{R}^d))$ with respect to the space variable x. The resulting net $u^{(\varepsilon)} \in C([0,T], C^\infty(\mathbb{R}^d))$ satisfies the equation

$$A^0 \frac{\partial u^{(\varepsilon)}}{\partial t} = - \sum_{j=1}^d A^j \frac{\partial u^{(\varepsilon)}}{\partial x_j} - D u^{(\varepsilon)} + f_\varepsilon$$

where $f_\varepsilon \in H^1(0,T; C^\infty(\mathbb{R}^d))$ and $\|f_\varepsilon(t, \cdot) - f(t, \cdot)\|_{L_2(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$ (but f_ε is not the regularization of f in x). This gives $u^{(\varepsilon)} \in H^1(0,T; C^\infty(\mathbb{R}^d))$. Differentiating the equation with respect to time can be used to show that $u^{(\varepsilon)} \in H^2(0,T; C^\infty(\mathbb{R}^d)) \subset H^2(Q)$. Then the *a priori* estimate can be used and establishes the desired regularity of the solution u.

3.5. Finite speed of propagation

Consider the homogeneous wave equation with constant coefficients in $\mathbb{R} \times \mathbb{R}^d$, that is $u_{tt} - c^2 \Delta u = 0$. A co-vector (τ, ξ) is characteristic if $P(\tau, \xi) = -\tau^2 + c^2|\xi|^2 = 0$, that is $\tau = \pm c|\xi|$. A solution to the initial value problem of the homogeneous wave equation can be investigated in the following way. Given a given point $(\underline{t}, \underline{x}) \in Q$, which values of the initial data determine the value $u(\underline{t}, \underline{x})$? This question can be answered by analyzing the explicit solution formula for the initial value problem or by relying on Holmgren's uniqueness theorem. The *domain of determinacy* of $(\underline{t}, \underline{x})$ is the set

$$\mathcal{M} = \{x \in \mathbb{R}^d \ : \ |x - \underline{x}| \leq c\underline{t}\} \ .$$

Indeed, one can show that the solution to the wave equation with initial data equals zero on $\mathcal{M}$ has to be zero at the point $(\underline{t}, \underline{x})$. Another notion is the *domain of influence* of a subset (usually a ball) of $\mathbb{R}^d$. Here one investigates (using the same tools) which points in space time Q the initial data in the given subset influences. The *domain of influence* of the ball $B(\underline{x}, R) = \{x \in \mathbb{R}^d \ : \ |x - \underline{x}| < R\}$ is the set

$$\mathcal{N} = \{(t, x) \in Q \ : \ |x - \underline{x}| \leq R + tc\} \ .$$

These concepts carry over to hyperbolic systems. The following discussion is based on a symmetric hyperbolic system with coefficients in $W^1_\infty(Q)$. A each point $(t, x) \in Q$ one defines the characteristic cone

$$\operatorname{char}(t, x) = \left\{(\tau, \xi) \in \mathbb{R}^{d+1} \ : \ \det \left[\tau A^0(t, x) + \sum_{j=1}^{d} A^j(t, x)\xi_j\right] = 0\right\}$$

and the forward cone

$$\Gamma(t, x) = \{(\tau, \xi) \in \mathbb{R}^{d+1} \ : \ w^H P_1(t, x; \tau, \xi)w/i > 0 \text{ for all } w \in \mathbb{C}^N\} \ .$$

In other words, the forward cone characterizes all $(\tau, \xi) \in \mathbb{R}^{d+1}$ such that the Hermitian matrix $\tau A^0(t, x) + \sum_{j=1}^d A^j(t, x)\xi_j$ is positive definite.

Definition 3.5.1. *Let $\mathscr{H}$ be a smooth hyper-surface in $\overline{Q}$ and let $n(t, x)$ be the unit normal with non-negative first component.*
(i) The hyper-surface $\mathscr{H}$ is characteristic at $(t, x) \in \mathscr{H}$ if and only if $n(t, x) \in \mathrm{char}\,(\mathrm{t}, \mathrm{x})$.
(ii) The hyper-surface is spacelike at $(t, x) \in \mathscr{H}$ if and only if $n(t, x) \in \Gamma\,(t, x)$.

Theorem 3.5.2. *Suppose that $u \in C^1(\overline{Q})$ solve the homogeneous equation $Pu = 0$ in Q. Furthermore, suppose that $\mathscr{L}$ is a lens-shaped region which is bounded by two space-like hyper-surfaces $\mathscr{H}$ and $\mathscr{K}$ where the vector $n(t, x)$ on $\mathscr{H}$ points into the interior $\mathscr{L}$ and the vector $n(t, x)$ on $\mathscr{K}$ points into the exterior of $\mathscr{L}$.*
If $u \equiv 0$ on $\mathscr{H}$, then $u \equiv 0$ on $\mathscr{K}$.

For the proof we need a multi-dimensional version of Gronwall's lemma.

Lemma 3.5.3. *Suppose that $\mathscr{L} \subset \mathbb{R}^d$ is a region foliated by smooth hypersurfaces $\mathscr{H}_\varepsilon$ that is $\mathscr{L} = \cup_{0 \leq \varepsilon \leq 1} \mathscr{H}_\varepsilon$ and set*

$$\mathscr{L}_\theta = \bigcup_{0 \leq \varepsilon \leq \theta} \mathscr{H}_\theta \subset \mathscr{L} \qquad \text{for } \theta \in [0, 1] \ .$$

Suppose that $u \in C^1$ in a neighborhood of $\mathscr{L}$ and that for some $C > 0$ we have

$$\int_{\mathscr{H}_\theta} |u| \, dS \leq C \left\{ \int_{\mathscr{H}_0} |u| \, dS + \int_{\mathscr{L}_\theta} |u| \, dx \right\}$$

for all $\theta \in [0, 1]$. Then there exists a constant $c > 0$ such that

$$\int_{\mathscr{H}_1} |u| \, dS \leq c \int_{\mathscr{H}_0} |u| \, dS \ .$$

3.6. Concluding Remarks

There are many interesting questions which we have not addressed in our discussion on hyperbolic problems. The initial value problem for constantly hyperbolic systems of first order can be solved as well. However, in order to obtain the apriori estimates one needs to construct a symmetrizer which makes the system symmetric hyperbolic. In case of constant coefficients this can be achieved by means of a Fourier multiplier. In the case of variable coefficients the symmetrizer is more involved and turns out to be a pseudo-differential operator. For more information about the topic we refer to the book by S. Benzoni-Gavage and D. Serre [**BGS07**] where also initial-boundary value problem for hyperbolic systems are discussed in details.

CHAPTER 4

Nonlinear equations

This chapter showcases a variety of methods for nonlinear problems and follows Chapters 9, 11 & 3 the book by L.C. Evans [**Eva98**]. For the last section on conservation laws we refer also to the book by D. Serre [**Ser99**]

4.1. Monotonicity methods

Throughout this section we consider the homogeneous Dirichlet problem following quasilinear PDE in divergence form

$$\begin{aligned} -\nabla \cdot a(\nabla u) &= f \quad \text{in } \Omega \, , \\ u &= 0 \quad \text{in } \partial\Omega \end{aligned} \tag{4.1.1}$$

Here $f \in L_2(\Omega)$ and $a : \mathbb{R}^d \to \mathbb{R}^d$ is a C^∞ vector field.

Definition 4.1.4. *A vector field $a : \mathbb{R}^d \to \mathbb{R}^d$ is monotone if $(a(p)-a(q))\cdot(p-q) \geq 0$ for all $p, q \in \mathbb{R}^d$. The vector field a is strict monotone if there exists a constant $\theta > 0$ such that $(a(p) - a(q)) \cdot (p - q) \geq \theta|p - q|^2$ for all $p.q \in \mathbb{R}^d$.*

Throughout this section we will assume that $a \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ is monotone and that there exist positive constanct C and α and a non-negative constant $\beta \geq 0$ such that

$$|a(p)| \leq C(1 + |p|) \quad \text{and} \quad a(p) \cdot p \geq \alpha|p|^2 - \beta$$

for all $p \in \mathbb{R}^d$. Our goal is to establish the existence of a weak solution to the boundary value problem above. This will be done by means of *Galerkin approximations*. Suppose that $\{w_k\}_{k=1}^\infty$ is an orthonormal basis in $\mathring{H}^1(\Omega)$ with respect to the inner product $(u, v)_1 = (\nabla u, \nabla v)_{L_2(\Omega)}$. We will look for a sequence of functions

$$u_m = \sum_{k=1}^m d_m^k w_k, \qquad m = 1, 2, ... \tag{4.1.2}$$

where the coefficients d_m^k are chosen such that

$$\int_\Omega a(\nabla u_m(x)) \cdot \nabla w_k(x)\, dx = \int_\Omega f(x) w_k(x)\, dx \tag{4.1.3}$$

for $k = 1, 2, ..., m$. At first we will establish the existence of such a sequence. Then we will try to take the limit as $m \to \infty$, establish convergence in some sense and show that the limit function is the solution to the boundary value problem. In the following we will abbreviate the closed unit ball in $\mathbb{R}^d$ centered at the origin by B, that is $B = \overline{B(0, 1)}$.

Theorem 4.1.5 (Brouwer's Fixed Point Theorem). *Suppose that $w : B \to B$ is continuous. Then w has a fix point.*

LEMMA 4.1.6. *Let v be a continuous vector field and there exists a $r > 0$ such that $v(x) \cdot x \geq 0$ for all $|x| = r$. There there exists an $x \in \overline{B(0,r)}$ such that $v(x) = 0$.*

PROPOSITION 4.1.7. *For $m = 1, 2, ...$ there exists a function u_m defined in formula (4.1.2) which satisfies the relation (4.1.3).*

In order to discuss the limit for $m \to \infty$ we need

PROPOSITION 4.1.8. *There exists a positive constant C dependent only on Ω and a such that*

$$\|u_m\|_{\mathring{H}^1(\Omega)} \leq C\left(1 + \|f\|_{L_2(\Omega)}\right)$$

for $m = 1, 2, ...$

The existence of a weak solution is formulated in in the following theorem

THEOREM 4.1.9. *Under the standing assumptions made above, the boundary value problem (4.1.1) has a weak solution $u \in \mathring{H}^1(\Omega)$.*

COROLLARY 4.1.10. *If a is in additions strict monotone, then the weak solution is unique.*

4.2. Fixed point methods

THEOREM 4.2.1 (Banach). *Let $A : X \to X$ be an operator on the Banach space X. Suppose there exists a constant $0 < \gamma < 1$ such that*

$$\|Au - Av\| < \gamma\|u - v\| \qquad \text{for all } u, v \in X\ .$$

Then A has a unique fixed point.

Consider now the following semilinear initial boundary value problem for the heat equation.

$$\begin{aligned} u_t - \Delta u &= f(u) && \text{in } Q_T = (0,T) \times \Omega\,, \\ u &= 0 && \text{in } \Sigma_T = (0,T) \times \partial\Omega \\ u(0,\cdot) &= g && \text{in } \Omega \end{aligned} \tag{4.2.1}$$

Here $g \in \mathring{H}^1(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz, that is, there exists a constant $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$.

A weak solution to the boundary value problem (4.2.1) is defined as a function $u \in L_2(0,T;\mathring{H}^1(\Omega))$ with its time derivative $\partial u/\partial t \in L_2(0,T;H^{-1}(\Omega))$ such that $u(0,\cdot) = g$ and

$$\left(\frac{\partial u}{\partial t}, v\right)_{[H^{-1}(\Omega),\mathring{H}^1(\Omega)]} + (\nabla u, \nabla v)_{L_2(\Omega)} = (f(u), v)_{L_2(\Omega)}$$

for all $v \in \mathring{H}^1(\Omega$ and almost everywhere in t for $0 \leq t \leq T$. Here $(\cdot,\cdot)_{[H^{-1}(\Omega),\mathring{H}^1(\Omega)]}$ denotes the dual pairing between the dual spaces $H^{-1}(\Omega)$ and $\mathring{H}^1(\Omega)$ where $L_2(\Omega)$ is identified with its own dual space.

REMARK 4.2.2. *If $u \in L_2(0,T;\mathring{H}^1(\Omega))$ and $\partial u/\partial t \in L_2(0,T;H^{-1}(\Omega))$, then $u \in C([0,T], L_2(\Omega))$.*

With the help of Theorem 4.2.1 one can now prove the following result.

THEOREM 4.2.3. *There exists a unique weak solution to the initial-boundary value problem (4.2.1).*

THEOREM 4.2.4 (Schauder). *Suppose that $K \subset X$ is compact and convex and suppose that $A : X \to X$ is continuous. Then the operator A has a fixed point.*

DEFINITION 4.2.5. *An operator $A : X \to X$ is compact, if for every bounded sequence $\{x_k\}_{k=1}^{\infty} \subset X$ the sequence $\{Au_k\}_{k=1}^{\infty}$ has a convergent subsequence.*

COROLLARY 4.2.6 (Schaefer). *Suppose that $A : X \to X$ is continuous and compact. In addition, the set $\{x \in X \ : \ u = \lambda Au$ for some $0 \le \lambda \le 1\}$ is bounded. Then A has a fixed point.*

Consider now the semilinear elliptic boundary value problem

$$\begin{aligned} -\Delta u + b(\nabla u) + \mu u &= 0 \qquad \text{in } \Omega \ , \\ u &= 0 \qquad \text{in } \partial\Omega \ , \end{aligned}$$

where $b : \mathbb{R}^d \to \mathbb{R}$ is globally Lipschitz.

THEOREM 4.2.7. *If $\mu > 0$ is sufficiently large, there exists a solution $u \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$.*

4.3. Non-existence

This section serves as an example, that non-linear problem may not be solvable in the same manner linear problems are. As an example consider the semilinear problem for the heat equation

$$\begin{aligned} u_t - \Delta u &= u^2 \qquad \text{in } Q_T \ , \\ u &= 0 \qquad \text{in } \Sigma_T \ , \\ u(0, \cdot) &= g \qquad \text{in } \Omega \ . \end{aligned}$$

THEOREM 4.3.1 (minimum principle). *Suppose that $u \in C^2(Q_T) \cap C(\overline{Q_T})$ satisfies $u_t - \Delta u \ge 0$. Then*

$$\min_{\overline{Q_T}} u = \min_{\partial' Q_T} u$$

where $\partial' Q_T = \overline{\Sigma_T} \cup (\Omega \times \{0\})$. Furthermore, if there exists a point $(\underline{t}, \underline{x}) \in Q_T$ such that

$$u(\underline{t}, \underline{x}) = \min_{\overline{Q_T}} u \ ,$$

then u is constant in $Q_{\underline{t}}$.

For references on the maximum principle we refer to [**Joh91**, Chapter 7] and [**Eva98**, Section 7.1, Theorem 11].

LEMMA 4.3.2. *For $T > 0$ and $g \ge 0$ sufficiently large, there does not exist a smooth solution $u \in C^2(Q_T) \cap C(\overline{Q_T})$.*

The proof of this lemma relies on Theorem 4.3.1 and on the following fact:
The first eigenfunction w_1 of the Dirichlet Laplacian can be chosen to be positive in Ω [**Eva98**, Section 6.5, Theorem 2]. (The eigenvalues of the Dirichlet Laplacian are all positive and have no finite point of accumulation. The function $w_1 \in \mathring{H}^1(\Omega)$ satisfies $-\Delta w_1 = \lambda_1 w_1$ where $\lambda_1 > 0$ is the smallest eigenvalue.)

4.4. Conservation Laws

Throughout this section we discuss the non-linear partial differential system in two independent variables (t, x)

$$u_t + F(u)_x = 0 \qquad (t, x) \in \mathbb{R}^2 . \tag{4.4.1}$$

$u = u(t, x)$ is a vector-valued function with N components and $F : \mathbb{R}^N \to \mathbb{R}^N$ is a C^1 vector field. We will be mainly concerned about the initial value problem

$$\begin{aligned} u_t + F(u)_x &= 0 \qquad t > 0, x \in \mathbb{R} , \\ u(0, x) &= g(x) \qquad x \in \mathbb{R} \end{aligned}$$

4.4.1. Integral solutions and the Rankine Hugoniot condition.

Definition 4.4.1. *A function $u \in L_\infty(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}^N)$ is an integral solution of the initial value problem above provided*

$$\int_0^\infty \int_{\mathbb{R}} u \cdot v_t + F(u) \cdot v_x \, dxdt + \int_{\mathbb{R}} g \cdot v(0) \, dx = 0 ,$$

for all $v \in C_0^\infty(\mathbb{R}^2)$.

Suppose that u is an integral solution of the initial value problem in $(0, \infty) \times \mathbb{R}$ and that C is a C^1-curve in time and space which divides the set $(0, \infty) \times \mathbb{R}$ into V_l and V_r. If u is continuously differentiable in V_l and V_r then the Rankine-Hugoniot condition

$$[u]\sigma = [F(u)] , \qquad \text{for all } (t, x) \in C$$

holds. Here $[u]$ is the jump of u, $[F(u)]$ is the jump of $F(u)$ across the curve C, and σ is the speed of the curve C. More specifically, if $C = \{(t, x) \ : \ x = x(t)\}$ with $x(t)$ is differentiable and

$$u_l(t, x) = \lim_{V_l \ni (t_n, x_n) \to (t,x)} u(t_n, x_n) , \quad u_r(t, x) = \lim_{V_r \ni (t_n, x_n) \to (t,x)} u_n(t_n, x_n) , \qquad (t, x) \in C$$

then $[u] = u_l - u_r$, $[F(u)] = F(u_l) - F(u_r)$, and $\sigma(t, x) = x'(t)$ for $(t, x) \in C$.

4.4.2. traveling waves, hyperbolic conservation laws. Since F is assumed to be of class C^1, a C^1 solution of the conservation law (4.4.1) will satisfy

$$u_t = DF(u)u_x$$

where $DF =: B$ is the Jacobian matrix of F. A traveling wave is a function of the form $u(t, x) = v(x - \sigma t)$ where v is the profile and σ is the speed. In order to be a solution to the system above one needs

$$-\sigma v'(x - \sigma t) + B(v(x - \sigma t))v'(x - \sigma t) = 0 ,$$

in other words, v' needs to be an eigenvector for the matrix B with eigenvalue σ.

Definition 4.4.2. *The system (4.4.1) is strictly hyperbolic, if the matrix B has only real, simple eigenvalues for all $z \in \mathbb{R}^N$.*

THEOREM 4.4.3. *Suppose that $u \in C^1(\mathbb{R}^2)$ is a solution to the system $u_t + B(u)u_x$ and that $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 diffeomorphism with inverse Ψ. Then the function $\tilde{u} = \Phi(u) \in C^1(\mathbb{R}^2)$ is a solution to the system*

$$\tilde{u}_t + B(\tilde{u})\tilde{u}_x$$

where $B(\tilde{u}) = D\Phi(\Psi(\tilde{z})B(\Psi(\tilde{z})D\Psi(\tilde{z})$ for all $\tilde{z} \in \mathbb{R}^N$.

COROLLARY 4.4.4. *If the matrix B is strictly hyperbolic, so is the matrix $\tilde{B}$.*

In the strictly hyperbolic case we denote the eigenvalues by $\lambda_1(z) < \lambda_2(z) < ... < \lambda_N(z)$ and denote a right eigenvector corresponding to the eigenvalue $\lambda_k(z)$ by $r_k(z)$. Similarly, a left eigenvector is denoted by $l_k(z)$. We have

$$B(z)r_k(z) = \lambda_k(z), \quad B^T l_k(z) = \lambda_k(z), \quad l_k(z)\cdot r_j(z) = 0 \quad \text{for all } k = 1, 2, ..., N \text{ and } j \neq k\ .$$

The next theorem states that the eigenvalues and the eigenvectors are C^1 functions as long as the entries of B are of class C^1.

THEOREM 4.4.5. *Suppose that $B \in C^1(\mathbb{R}^N)$ and that B is strictly hyperbolic. Then $\lambda_k \in C^1(\mathbb{R}^N)$ and there exist left and right eigenvectors $l_k \in C^1(\mathbb{R}^N)$ and $r_k \in C^1(\mathbb{R}^N)$ which satisfy also*

$$|r_k(z)| = |l_k(z)| = 1 \qquad \textit{for all } k = 1, 2, ..., N \textit{ and } z \in \mathbb{R}^N\ .$$

4.4.3. The Riemann Problem. The following initial value problem

$$u_t + F(u)_x = 0 \text{ in } \mathbb{R}_+ \times \mathbb{R} \qquad u(0,x) = \begin{cases} u_l & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases}, \tag{4.4.2}$$

is known as Riemann problem. Here u_l and u_r are different constant vectors. We consider at first the case $N = 1$. In this case the initial value problem can be solved using characteristics. The curve $y(s) = (t(s), x(s))$ is a characteristic if a solution to the conservation law u is constant along this curve, i.e. $u((t(s), x(s)) = const.$. Using the chain rule, one obtains

$$u_t(y(s))t'(s) + u_x(y(s))x'(s) = 0$$

which in view of (4.4.2) suggests that $t'(s) = 1$ and $x'(s) = F'(u(s))$. With the initial value problem in mind we obtain the family of curves $y(t) = (t, F'(g(x_0))t + x_0)$ where g denotes the given initial values and $x_0 \in \mathbb{R}$. Each of these curves originates at the point $(0, x_0)$.

If every point in the upper half plane $\mathbb{R}_t \times \mathbb{R}$ lies exactly on one of these characteristic curves, then a (global) unique solution to the initial value problem may be found. Note that due to the non-linearity, the characteristics depend on the initial data. However, as long as the initial data are smooth, the initial value problem for a scalar conservation law can be solved for small times.

More precisely, one can show that for each $g \in C^1(\mathbb{R})$ there exists a $T^* > 0$ such that , the initial value problem

$$u_t + F(u)_x = 0 \text{ in } \mathbb{R}_+ \times \mathbb{R} \qquad u(0,x) = g(x) \tag{4.4.3}$$

has a unique classical solution $u \in C^1([0, T^*) \times \mathbb{R})$. At T^* two characteristics starting from different points $(0, x)$ and $(0, y)$ may intersect which will produce a discontinuity.

Riemann problems are more difficult since the initial data are not continuous. It may occur that the family of characteristics does not cover the upper half plane. On the other hand there may be intersections of two different characteristics for any $t > 0$. Crossing characteristics lead to the formation of *shocks*, whereas the first phenomenon does not allow a unique solution. In this case one is forced to add an additional condition (entropy condition) which leads to a solution in form of *rarefaction waves*.

A continuous, convex function e is called an entropy and the function

$$f(s) = F'(s)e(s) - F'(0)e(0) - \int_0^s F''(y)e(y)\,dy$$

is called the corresponding entropy flux. If e is differentiable we have $f' = F'e'$.

Definition 4.4.6. *An integral solution u is an entropy solution, if for each pair of entropy and entropy flux (e, f) and for all $\varphi \geq 0$, $\varphi \in C_0^\infty(\mathbb{R}^2)$ the inequality*

$$\int_0^\infty \int_{\mathbb{R}} [e(u)\varphi_t + f(u)\varphi_x]dxdt + \int_{\mathbb{R}} e(g(x))\varphi(0,x)\,dx \geq 0$$

holds.

The motivation for this definition is connected with the so-called viscosity solutions. Instead of (4.4.2) one considers for small, positive ε the equation

$$u_t + f(u)_x = \varepsilon u_{xx} \qquad t > 0, x \in \mathbb{R}\,. \tag{4.4.4}$$

This is a parabolic equation and one can show that the initial value problem for this equation with initial data

$$u(0,x) = g(x) \qquad x \in \mathbb{R}$$

has a unique classical solution which satisfies the maximum principle.

Lemma 4.4.7. *Suppose that $g \in C(\mathbb{R})$ is a bounded function and denote the classical solution of the initial value problem to equation (4.4.4) by u^ε. Suppose that $u^\varepsilon(t,x) \to u(t,x) \in L_\infty(\mathbb{R}_+ \times \mathbb{R})$ almost everywhere. Then u is an integral solution to the initial value problem for the conservation law $u_t + F(u)_x = 0$.*

The most important entropy - entropy flux pair is given by the functions

$$e(u) = |u-k| \quad \text{and} \quad f(u) = [F(u) - F(k)]\mathrm{sgn}(u-k) \qquad k \in \mathbb{R}$$

where

$$\mathrm{sgn}(z) = \begin{cases} 1 & \text{if} \quad z > 0 \\ 0 & \text{if} \quad z = 0 \\ -1 & \text{if} \quad z < 0 \end{cases}.$$

Hence the every entropy solution satisfies the inequality

$$\int_Q \{\varphi_t|u-k| + \varphi_x[F(u) - F(k)]\mathrm{sgn}(u-k)\}\,dxdt + \int_{\mathbb{R}} |g(x) - k|\varphi(0,x)\,dx \tag{4.4.5}$$

for all $\varphi \geq 0$, $\varphi \in C_0^\infty(\mathbb{R}^2)$ and $k \in \mathbb{R}$. Somewhat surprisingly, the converse is true as well.

Proposition 4.4.8. *The function $u \in L_\infty(\mathbb{R}_+ \times \mathbb{R})$ is an entropy solution to the initial value problem for the conservation law (4.4.4) with $g \in L_\infty(\mathbb{R})$ if and only if inequality (4.4.5) holds for all $k \in \mathbb{R}$ and $\varphi \geq 0$, $\varphi \in C_0^\infty(\mathbb{R}^2)$*

In case of a uniformly convex function F, the solution to the Riemann problem in the case $N = 1$ can be given explicitly.

THEOREM 4.4.9. *Suppose that F is uniformly convex, that is $F''(z) \geq \theta > 0$ for some constant theta > 0.*
1.) If $u_l > u_r$, then the unique solution to the Riemann problem (4.4.2) is

$$u(t,x) = \begin{cases} u_l & \text{if} \quad x/t < \sigma \\ u_r & \text{if} \quad x/t > \sigma \end{cases} ,$$

where $\sigma = \dfrac{F(u_l) - F(u_r)}{u_l - u_r}$.
2.) If $u_l < u_r$, then the unique entropy solution to the Riemann problem (4.4.2) is given by

$$u(t,x) = \begin{cases} u_l & \text{if} \quad x/t < F'(u_l) \\ G(x/t) & \text{if} \quad F'(u_l) < x/t < F'(u_r) \\ u_r & \text{if} \quad x/t > F'(u_r) \end{cases} ,$$

where G is the inverse function of F'.

The first solution has a discontinuity along the straight line $x = \sigma t$ which is a shock front moving with velocity σ. The second solution is continuous and is referred to as a *rarefaction wave.* If a entropy solution is discontinuous along a differentiable curve $(t, x(t))$, then the inequality

$$F'(u_l) \geq x'(t) \geq F'(u_r) \tag{4.4.6}$$

holds. This is the *Lax shock condition* which expresses the fact, that shock occur only when characteristics starting at $(0, x)$ and $(0, y)$ intersect for some $t > 0$. Here $x \neq y$. The Lax shock condition prohibits that two characteristics passing through the points (t, x) and (t, y), respectively, for some $t > 0$ intersect for some smaller positive $\tilde{t}$.

Now we discuss the Riemann problem for $N > 1$. In this case we look for a simple wave solution, that is a solution of the form $u(t,x) = v(w(t,x))$ where $v : \mathbb{R} \to \mathbb{R}^N$ and $w : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. Going with this ansatz into the equation (4.4.2) gives

$$v'(w)w_t + DF(v(w))v'(w)w_x = 0 .$$

Hence, for some $k \in \{1, ..., N\}$ the functions v and w need to solve the equations

$$\begin{aligned} v'(s) &= r_k(v(s)) \\ w_t + \lambda_k(v(w))w_x &= 0 \end{aligned} \tag{4.4.7}$$

In this case we call u a k-simple wave. Note that the second equation in (4.4.7) represents a scalar conservation law.

In what follows we will investigate the conditions which are needed to produce a continuous solution to (4.4.2). For that purpose we define for all $z \in \mathbb{R}^N$ the kth rarefaction curve $R_k(z)$ as the solution to the first equation in (4.4.7) which passes through z.

With $F_k(s) = \int_0^s \lambda_k(v(t))\,dt$ we consider the Riemann problem for the scalar conservation law $w_t + F_k(w)_x = 0$. This problem can be solve using Theorem 4.4.9 as long as F_k is uniformly convex or uniformly concave.

Compute

$$F_k''(s) = \nabla\lambda_k(v(s)) \cdot v'(s) = \nabla\lambda_k(v(s)) \cdot r_k(v(s)) .$$

DEFINITION 4.4.10. *The pair (λ_k, r_k) is genuinely non-linear if $\nabla\lambda_k(z) \cdot r_k(z) \neq 0$ for all $z \in \mathbb{R}^N$. The pair is degenerate linear if $\nabla\lambda_k(z) \cdot r_k(z) = 0$ for all $z \in \mathbb{R}^N$.*

THEOREM 4.4.11. *Suppose that the pair (λ_k, r_k) is genuinely non-linear for some $k \in \{1, ..., N\}$ and that*

$$u_r \in R_k^+(u_l) = \{z \in R_k(u_l) \ : \ \lambda_k(z) > \lambda_k(u_l)\}$$

Then there exists a continuous integral solution to (4.4.2) which is a k-simple wave which is constant along the rays emanating from the origin.

DEFINITION 4.4.12. *For a given $\underline{z} \in \mathbb{R}^N$, the set*

$$S(\underline{z}) = \{z \in \mathbb{R}^N \ : \ F(z) - F(\underline{z}) = \sigma(z - \underline{z})\}$$

is the shock set. Here $\sigma = \sigma(z, \underline{z})$ is a real number.

PROPOSITION 4.4.13. *For all $\underline{z} \in \mathbb{R}^N$ there exists a neighborhood $\mathscr{U}(\underline{z})$ such that*

$$S(\underline{z}) = \bigcup_{k=1}^{N} S_k(\underline{z}) \ ,$$

where each $S_k(\underline{z})$ is a smooth curve and
(i) the vector $r_k(\underline{z})$ is tangent to the curve $S_k(\underline{z})$ at the point $\underline{z}$.
(ii) $\lim_{z \to \underline{z}} \sigma(z, \underline{z}) = \lambda_k(\underline{z})$
(iii) For $z \in S_k(\underline{z})$ and $z \to \underline{z}$ we have

$$\sigma(z, \underline{z}) = \frac{\lambda_k(z) + \lambda_k(\underline{z})}{2} + O(|z - \underline{z}|^2) \ .$$

PROPOSITION 4.4.14. *Suppose that for $k \in \{1, ..., N\}$ the pair (λ_k, r_k) is linear degenerate. Then, for all $\underline{z} \in \mathbb{R}^N$ we have $R_k(\underline{z}) = S_k(\underline{z})$ and $\sigma(z, \underline{z}) = \lambda_k(z) = \lambda_k(\underline{z})$ for all $z \in S_k(\underline{z})$.*

If the pair $\lambda_k, r_k)$ is linear degenerate and $u_r \in S_k(u_l)$, then the function

$$u(t, x) = \begin{cases} u_l & \text{for} \quad x < \sigma t \\ u_r & \text{for} \quad x > \sigma t \end{cases} \tag{4.4.8}$$

with $\sigma = \sigma(u_r, u_l) = \lambda(u_l) = \lambda(u_r)$ is a solution to the Riemann problem. This solution is referred to as contact discontinuity.

If the pair (λ_k, r_k) is genuinely nonlinear and $u_r \in S_k(u_l)$, then the function (4.4.8) with $\sigma = \sigma(u_r, u_l)$ is an integral solution. However, two cases have to be distinguished, $\lambda_k(u_l) < \lambda_k(u_r)$ and $\lambda_k(u_l) > \lambda_k(u_r)$. Because of Proposition 4.4.13 we have in the first case

$$\lambda_k(u_l) < \sigma(u_r, u_l) < \lambda_k(u_r)$$

and in the second case

$$\lambda_k(u_l) > \sigma(u_r, u_l) > \lambda_k(u_r) \ .$$

Note that the first inequality contradicts the *Lax shock condition* (4.4.6). The first case is a non-physical shock and will be discarded.

DEFINITION 4.4.15. *Suppose that the pair* (λ_k, r_k) *is genuinely nonlinear. The pair* (u_l, u_r) *is admissible if and only if* $u_r \in S_k(u_l)$ *and* $\lambda_k(u_l) > \sigma(u_r, u_l) > \lambda_k(u_r)$. *Then the solution (4.4.8) is a k-shock wave.*

DEFINITION 4.4.16. *Suppose that the pair* (λ_k, r_k) *is genuinely nonlinear. Then*

$$\begin{aligned} S_k^+(\underline{z}) &= \{z \in S_k(\underline{z}) \ : \ \lambda_k(\underline{z}) < \sigma(z, \underline{z}) < \lambda_k(z)\} \\ S_k^-(\underline{z}) &= \{z \in S_k(\underline{z}) \ : \ \lambda_k(z) < \sigma(z, \underline{z}) < \lambda_k(\underline{z})\} \\ T_k(\underline{z}) &= R_k^+(\underline{z}) \cup \{\underline{z}\} \cup S_k^-(\underline{z}) \end{aligned}$$

With this definition one observes that the pair (u_l, u_r) is admissible if and only if $u_r \in S_k^+(u_l)$. If the pair (λ_k, r_k) is linear degenerate, then $T_k(\underline{z}) = R_k(\underline{z}) = S_k(\underline{z})$.

THEOREM 4.4.17. *Suppose that for each* $k \in \{1, ..., N\}$ *the pairs* (λ_k, r_k) *are either genuinely nonlinear or degenerate linear and that* u_l *is given. Then, for* u_r *sufficiently close to* u_l *there exists an integral solution to the Riemann problem (4.4.2) which is constant on lines through the origin.*

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