WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #1 Solutions

Problem 1. Consider the homogeneous Maxwell equations for the electromagnetic field $(e, h) = (e_1, e_2, e_3, h_1, h_2, h_3)$ in $\mathbb{R}_t \times \mathbb{R}^3_x$,

$$\begin{aligned} \partial_t(\varepsilon e) - \nabla \times h + \sigma e &= 0 \\ \partial_t(\mu h) + \nabla \times e &= 0 \\ \nabla \cdot (\varepsilon e) &= 0 \\ \nabla \cdot (\mu h) &= 0 \ . \end{aligned}$$

where ε is the electric permittivity, μ the magnetic permeability, and σ is the conductivity. These parameters depend on the medium and they are 3×3 symmetric, positive definite matrices which may depend on space and time.

a) Show that if ε and μ are constant multiples of the identity matrix and if $\sigma = 0$, that then each component of (e, h) satisfies the wave equation $u_{tt} - c^2 \Delta u = 0$ with some positive constant c. Hint: Apply the time derivative to the first (vector) equation and the curl to the second (vector) equation.

Solution. Differentiating the first equation with respect to t and taking the curl of the second equation divided by μ gives

$$\begin{aligned} \varepsilon \partial_t^2 e - \nabla \times \partial_t h &= 0 \\ \nabla \times \partial_t h + \frac{1}{\mu} \nabla \times \nabla \times e &= 0 \ . \end{aligned}$$

Using now the identity $\nabla \times \nabla u = \nabla \nabla \cdot u - \Delta u$ and the third equation allows us to write the second equation as

$$\nabla \times \partial_t h - \frac{1}{\mu} \Delta e = 0 \ .$$

Adding this equation to the equation $\varepsilon \partial_t^2 e - \nabla \times \partial_t h = 0$ results in

$$\varepsilon \mu \partial_t^2 e - \Delta e = 0 \ .$$

With the same procedure, the same equation is shown to be valid for h, provided one applies the curl to the first equation and the time derivative to the second equation. This time one has to make use also of the fourth equation.

b) Show that if ε and μ are multiples of the identity matrix which depend on t and x, that then the Maxwell equations above can be rewritten as a second order system whose principal part is diagonal.

Solution. One uses the same procedure as in the first part. However, this time one has to account for the derivatives of the coefficients which are assumed to be two times differentiable.

Applying the time derivative to the first equation and the curl to the second equation divided by μ gives

$$\varepsilon\partial_t^2 e + 2\partial_t\varepsilon\partial_t e + (\partial_t^2\varepsilon)e - \nabla\times\partial_t h + \partial_t(\sigma e) = 0$$
$$\nabla\times\partial_t h + \frac{1}{\mu}\nabla\times\nabla\times e + \nabla\left(\frac{1}{\mu}\right)\times(\nabla\times e) = 0\ .$$

Note that $\nabla\cdot(\varepsilon e) = \nabla\varepsilon\cdot e + \varepsilon\nabla\cdot e$ which gives

$$\nabla\cdot\varepsilon = -\frac{\nabla\varepsilon\cdot e}{\varepsilon}\ ,$$

Using the identity $\nabla\times\nabla\times e = \nabla\nabla\cdot e - \Delta e$ gives now

$$\nabla\times\nabla\times e = -\Delta e - \nabla\frac{\nabla\varepsilon\cdot e}{\varepsilon}\ .$$

Combining our efforts gives

$$\varepsilon\mu\partial_t^2 e - \Delta e + 2\mu\partial_t\varepsilon\partial_t e + \mu(\partial_t^2\varepsilon)e - \nabla\frac{\nabla\varepsilon\cdot e}{\varepsilon} + \mu\partial_t(\sigma e) + \mu\nabla\left(\frac{1}{\mu}\right)\times(\nabla\times e) = 0\ .$$

With a similar computation one obtains

$$\varepsilon\mu\partial_t^2 h - \Delta h + 2\varepsilon\partial_t\mu\partial_t h + \varepsilon(\partial_t^2\mu)h - \nabla\frac{\nabla\mu\cdot h}{\mu} + \nabla\times\frac{\sigma e}{\varepsilon} + \varepsilon\nabla\left(\frac{1}{\varepsilon}\right)\times(\nabla\times h) = 0\ .$$

Here we note that in both equations, only the first two expression contain derivatives of second order in e and h. All the other terms contain at most derivatives of first order in e and h. The last two equations can be written in the form

$$P_2(x,\partial)\begin{bmatrix} e \\ h \end{bmatrix} + P_1(x,\partial)\begin{bmatrix} e \\ h \end{bmatrix} + P_0(x,\partial)\begin{bmatrix} e \\ h \end{bmatrix} = 0$$

where P_j are differential operators of order j and moreover,

$$P_2 = \begin{bmatrix} \varepsilon\mu\partial_t^2 - \Delta & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon\mu\partial_t^2 - \Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon\mu\partial_t^2 - \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon\mu\partial_t^2 - \Delta & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon\mu\partial_t^2 - \Delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon\mu\partial_t^2 - \Delta \end{bmatrix}.$$

The principal part of this system of PDE is diagonal.

c) Show that if $\sigma = 0$ and μ is the identity matrix and ε is a diagonal matrix which is not a scalar multiple of the identity matrix, that then Maxwell's equations cannot be diagonalized by means of differentiation.

Solution. Let now $\varepsilon = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$ and $\varepsilon_1 \neq \varepsilon_2$. If one tries to proceed as before one hits a road block pretty soon. Applying a time derivative to the first (vector) equation and the curl to the second vector equation gives

$$\varepsilon\partial_t^2 e - \nabla\times\partial_t h = 0 \quad \text{and } \nabla\times\partial_t h + \nabla\times\nabla\times e = 0$$

We obtain

$$\varepsilon\partial_t^2 e + \nabla\times\nabla\times e = 0$$

which is a 3×3 system of second order which cannot be simplified any further since the third Maxwell equation $\nabla \cdot (\varepsilon e) = 0$ does not provide us with an the expression for $\nabla \cdot e$ as it does when ε is a diagonal matrix.

Problem 2. Consider the Dirichlet problem in $\Omega = \{x \in \mathbb{R}^d \ : \ |x| < R\}$ for the equation of constant mean curvature equal to H

$$-\sum_{j=1}^{d} \frac{\partial}{\partial x_j} \frac{u_{x_j}}{\sqrt{1 + |\nabla u|^2}} = dH \qquad \text{in } \Omega \, ,$$

satisfying $u = 0$ in $\partial\Omega$.
In the case $0 \leq H < 1/R$ use spheres to find an explicit solution for this equation. Hint: The mean curvature $\kappa = \kappa(x)$ of a surface $z = u(x)$ in $\mathbb{R}^{d+1}$ in the point $(x, u(x)) \in \mathbb{R}^{d+1}$ is given by

$$\kappa(x) = -\frac{1}{d} \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \frac{u_{x_j}}{\sqrt{1 + |\nabla u|^2}} \, .$$

Solution. The upper hemisphere in $\mathbb{R}^{d+1}$ is given by the graph of the function

$$u(x) = \sqrt{R^2 - |x|^2}$$

with the domain Ω. One computes $\kappa(x) = 1/R$. Hence, this function is a solution to the equation of constant mean curvature equal to $H = 1/R$. To find a solution with for $H \in (0, 1/R)$ one uses spheres with radius equal to $1/H$. Then

$$u(x) = \sqrt{\frac{1}{H^2} - |x|^2} - \sqrt{\frac{1}{H^2} - \frac{1}{R^2}}$$

has the curvature $\kappa(x) = H$ for all $x \in \Omega$. The second term in the formula for u is needed to satisfy the homogeneous Dirichlet conditions. Finally, a solution for the constant mean curvature equation with $H = 0$ is given by the function $u \equiv 0$.