WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #1 due 10/23/2015

Problem 1. Consider the homogeneous Maxwell equations for the electromagnetic field $(e, h) = (e_1, e_2, e_3, h_1, h_2, h_3)$ in $\mathbb{R}_t \times \mathbb{R}^3_x$,

$$\begin{aligned} \partial_t(\varepsilon e) - \nabla \times h + \sigma e &= 0 \\ \partial_t(\mu h) + \nabla \times e &= 0 \\ \nabla \cdot (\varepsilon e) &= 0 \\ \nabla \cdot (\mu h) &= 0 \; . \end{aligned}$$

where ε is the electric permittivity, μ the magnetic permeability, and σ is the conductivity. These parameters depend on the medium and they are 3×3 symmetric, positive definite matrices which may depend on space and time.

a) Show that if ε and μ are constant multiples of the identity matrix and if $\sigma = 0$, that then each component of (e, h) satisfies the wave equation $u_{tt} - c^2 \Delta u = 0$ with some positive constant c. Hint: Apply the time derivative to the first (vector) equation and the curl to the second (vector) equation.

b) Show that if ε and μ are multiples of the identity matrix which depend on t and x, that then the Maxwell equations above can be rewritten as a second order system whose principal part is diagonal.

c) Show that if $\sigma = 0$ and μ is the identity matrix and ε is a diagonal matrix which is not a scalar multiple of the identity matrix, that then Maxwell's equations cannot be diagonalized by means of differentiation.

Problem 2. Consider the Dirichlet problem in $\Omega = \{x \in \mathbb{R}^d \; : \; |x| < R\}$ for the equation of constant mean curvature equal to H

$$-\sum_{j=1}^{d} \frac{\partial}{\partial x_j} \frac{u_{x_j}}{\sqrt{1 + |\nabla u|^2}} = dH \quad \text{ in } \Omega \; ,$$

satisfying $u = 0$ in $\partial\Omega$.

In the case $0 \le H < 1/R$ use spheres to find an explicit solution for this equation. Hint: The mean curvature $\kappa = \kappa(x)$ of a surface $z = u(x)$ in $\mathbb{R}^{d+1}$ in the point $(x, u(x)) \in \mathbb{R}^{d+1}$ is given by

$$\kappa(x) = -\frac{1}{d} \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \frac{u_{x_j}}{\sqrt{1 + |\nabla u|^2}} \; .$$