WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #10 due 01/08/2016

Problem 1. Let $a \in L_\infty(\mathbb{R}^d)$ and $u \in L_2(\mathbb{R}^d)$. Prove that $\|(au)^{(\varepsilon)} - au^{(\varepsilon)}\|_{L_2(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$. Here $u^{(\varepsilon)}$ denotes the regularization of u with respect to x.

Solution. Note that $au \in L_2(\mathbb{R}^d)$ (via Hölder's inequality). Hence, using Lemma 3.3.4 we have $\|(au)^{(\varepsilon)} - au\|_{L_2(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$. Thus, with the triangle inequality, Hölder inequality, and Lemma 3.3.4 one obtains

$$\begin{aligned}\|(au)^{(\varepsilon)} - au^{(\varepsilon)}\|_{L_2(\mathbb{R}^d)} &\leq \|(au)^{(\varepsilon)} - au\|_{L_2(\mathbb{R}^d)} + \|au - au^{(\varepsilon)}\|_{L_2(\mathbb{R}^d)} \\ &\leq \|(au)^{(\varepsilon)} - au\|_{L_2(\mathbb{R}^d)} + \|a\|_{L_\infty(\mathbb{R}^d)}\|u - u^{(\varepsilon)}\|_{L_2(\mathbb{R}^d)} \longrightarrow 0\end{aligned}$$

for $\varepsilon \to 0$ **Problem 2.** Show that the elastic wave equations

$$\rho\frac{\partial^2 u}{\partial t^2} - D(\partial)^T \mathscr{A} D(\partial) u = f \tag{1}$$

can be written has a symmetric hyperbolic system. Here $u : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the displacement,

$$D(\partial) = \begin{bmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{bmatrix},$$

the matrix $\mathscr{A}$ is a real symmetric positive definite 6×6 matrix which captures the stiffness of the elastic material, and $\rho > 0$ is the density. These coefficients are assumed to be smooth functions of time and space and ρ is uniformly positive and $\mathscr{A}$ is uniformly positive definite. Hint: Recall the reduction of the scalar wave equation to a symmetric hyperbolic system from Section 3.1.

Solution. Introduce a vector-valued function v with 9 components and 9×9 matrices A^j for $j = 0, 1, 2, 3$ via

$$v = \begin{bmatrix} \partial_t u \\ \mathscr{A} D(\partial) u \end{bmatrix}, \quad A^0 = \begin{bmatrix} \rho I_3 & 0 \\ 0 & \mathscr{A}^{-1} \end{bmatrix}, \quad \sum_{j=1}^{3} A^j \partial_j = \begin{bmatrix} 0 & D(\partial)^T \\ D(\partial) & 0 \end{bmatrix}.$$

Then, with the vector-valued function F with 9 components given by

$$F = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

one obtains from (1)

$$A^0(t,x)\partial_t v + \sum_{j=1}^{3} A^j \partial_j v - \begin{bmatrix} 0 & 0 \\ 0 & \mathscr{A}^{-1}\partial_t \mathscr{A} \end{bmatrix} v = F$$

and the matrices A^j, $j = 0, 1, 2, 3$ are symmetric by construction. Furthermore, the matrix A^0 is positive definite. This way the elastic wave equations are represented as a symmetric hyperbolic system.

Problem 3. Consider the Maxwell system

$$\partial_t(\varepsilon e) - \nabla \times h + \sigma e = f_1 \qquad \partial_t(\mu h) + \nabla \times e = f_2$$

with $f = (f_1, f_2)^T \in L_2(Q)^6$ and with initial data $e(0, \cdot) = e(x) \in L_2(\mathbb{R}^3)^3$ and $h(0, \cdot) = h(x) \in L_2(\mathbb{R}^3)^3$.

a.) Suppose that the coefficients ε, μ, σ are of class $W^1_\infty(Q)$ and that the matrices ε and μ are real symmetric and uniformly positive definite and the matrix σ is real symmetric and non-negative definite. What can you say about the solvability of the initial value problem ?

Solution. From Homework #8 we know that the Maxwell system is a symmetric hyperbolic system. Hence, using Theorem 3.3.3 this system has a unique solution $(e, h) \in C([0, T], L_2(\mathbb{R}^d)^6)$.
b.) Suppose that ε and μ are time-independent. Define

$$\mathscr{E}(t) = \int_{\mathbb{R}^3} [e^H \varepsilon e](t, x)\, dx + \int_{\mathbb{R}^3} [h^H \mu h](t, x)\, dx\ ,$$

which is known as the energy functional. Prove that the energy is non-increasing for a weak solution to the homogeneous Maxwell equations. Furthermore, show that the energy is time-independent if, in addition, $\sigma \equiv 0$. (Recall that $e^H \varepsilon e = \sum_{j=1}^3 \varepsilon_{jk} e_j \bar{e}_k$.)

Solution. From the proof of Theorem 3.3.3 we know that each weak solution (e, h) is the limit in $C([0, T], L_2(\mathbb{R}^d)^6)$ of its regularizations (in space only) $(e, h)^{(\varepsilon)} \in H^1(Q)$ for $\varepsilon \to 0$. Hence it will suffice to work with the regularizations since we can eventually take the limit $\varepsilon \to 0$. This has the advantage that we can differentiate in space and time. For brevity we will drop the superscript ε in the following formulas. Note that by construction that $\mathscr{E}$ is real valued. Using the symmetry and the time independence of ε and μ we have compute

$$\begin{aligned}
\mathscr{E}'(t) &= 2 \int_{\mathbb{R}^3} e^H \varepsilon \frac{\partial e}{\partial t}\, dx + 2 \int_{\mathbb{R}^3} h^H \mu \frac{\partial h}{\partial t}\, dx \\
&= \Re \int_{\mathbb{R}^3} [e^H \nabla \times h - h^H \nabla \times e]\, dx - \int_{\mathbb{R}^3} e^H \sigma e\, dx \\
&= \Re(e, \nabla \times h)_{L_2(\mathbb{R}^3)} - \Re(\nabla \times e, h)_{L_2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} e^H \sigma e\, dx = - \int_{\mathbb{R}^3} e^H \sigma e\, dx \le 0
\end{aligned}$$

where we relied also on the Maxwell equations, the integration by parts formula for the curl from Homework #7 and the fact that σ is assumed to be non-negative definite. Moreover, if $\sigma \equiv 0$ the last integral will vanish and we obtain $\mathscr{E}'(t) = 0$ for all $t \in [0, T]$.