WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #10 due 01/08/2016

Problem 1. Let $a \in L_\infty(\mathbb{R}^d)$ and $u \in L_2(\mathbb{R}^d)$. Prove that $\|(au)^{(\varepsilon)} - au^{(\varepsilon)}\|_{L_2(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$. Here $u^{(\varepsilon)}$ denotes the regularization of u with respect to x.

Problem 2. Show that the elastic wave equations

$$\rho \frac{\partial^2 u}{\partial t^2} - D(\partial)^T \mathscr{A} D(\partial) u = f$$

can be written has a symmetric hyperbolic system. Here $u : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the displacement,

$$D(\partial) = \begin{bmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{bmatrix} ,$$

the matrix $\mathscr{A}$ is a real symmetric positive definite 6×6 matrix which captures the stiffness of the elastic material, and ρ is the density. These coefficients are assumed to be smooth functions of time and space and ρ is uniformly positive and $\mathscr{A}$ is uniformly positive definite. Hint: Recall the reduction of the scalar wave equation to a symmetric hyperbolic system from Section 3.1.

Problem 3. Consider the Maxwell system

$$\partial_t(\varepsilon e) - \nabla \times h + \sigma e = f_1 \qquad \partial_t(\mu h) + \nabla \times e = f_2$$

with $f = (f_1, f_2)^T \in L_2(Q)^6$ and with initial data $e(0, \cdot) = e(x) \in L_2(\mathbb{R}^3)^3$ and $h(0, \cdot) = h(x) \in L_2(\mathbb{R}^3)^3$.

a.) Suppose that the coefficients ε, μ, σ are of class $W^1_\infty(Q)$ and that the matrices ε and μ are real symmetric and uniformly positive definite and the matrix σ is real symmetric and non-negative definite. What can you say about the solvability of the initial value problem ?

b.) Suppose that ε and μ are time-independent. Define

$$\mathscr{E}(t) = \int_{\mathbb{R}^3} [e^H \varepsilon e](t, x)\, dx + \int_{\mathbb{R}^3} [h^H \mu h](t, x)\, dx ,$$

which is known as the energy functional. Prove that the energy is non-increasing for a weak solution to the homogeneous Maxwell equations. Furthermore, show that the energy is time-independent if, in addition, $\sigma \equiv 0$. (Recall that $e^H \varepsilon e = \sum_{j=1}^3 \varepsilon_{jk} e_j \bar{e}_k$.)