WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #11 Key

The first two problems are dedicated to a more elementary proof of the Brouwer Fixed Point Theorem (Theorem 4.1.2). The goal is to replace the argument given in the lecture using differential forms by more elementary means.

Problem 1. Given a $d \times d$ matrix P, denote its *cofactor matrix* by $\mathrm{cof}\, P$. From linear algebra recall the identity $(\det P) I_d = P^T \mathrm{cof}\, P$. Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field with C^2 components and introduce a vector field $G : \mathbb{R}^d \to \mathbb{R}^d$ by $G_k = (\mathrm{cof}\, Du)_{jk}$ for some $j \in \{1, 2, ..., d\}$. This vector field is a row of the cofactor matrix of the Jacobian matrix of u. Show that G is divergence free, that is $\nabla \cdot G = 0$ for all $x \in \mathbb{R}^d$.

Proof. Recall from Linear Algebra that

$$(\mathrm{cof}\, Du)_{jk} = \det \begin{bmatrix} \frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u}{\partial x_{k-1}} & e_j & \frac{\partial u}{\partial x_{k+1}} & \cdots & \frac{\partial u}{\partial x_d} \end{bmatrix} .$$

Here e_j denotes the jth standard basis vector. Then for some j with $1 \leq j \leq d$ one obtains

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k} (\mathrm{cof}\, Du)_{jk} = \sum_{k=1}^{d} \sum_{\substack{l=1 \\ l \neq k}}^{d} \det \begin{bmatrix} \frac{\partial u}{\partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_k \partial x_l} & \cdots & e_j & \cdots & \frac{\partial u}{\partial x_d} \end{bmatrix}$$

where we used the definition of the determinant and the product rule of differential calculus. Exchanging the lth with the kth column, using $u_{x_l x_k} = u_{x_k x_l}$, and exchanging the order of the summation gives

$$\begin{aligned} \sum_{k=1}^{d} \frac{\partial}{\partial x_k} (\mathrm{cof}\, Du)_{jk} &= - \sum_{l=1}^{d} \sum_{\substack{k=1 \\ l \neq k}}^{d} \det \begin{bmatrix} \frac{\partial u}{\partial x_1} & \cdots & e_j & \cdots & \frac{\partial^2 u}{\partial x_l \partial x_k} & \cdots & \frac{\partial u}{\partial x_d} \end{bmatrix} \\ &= \sum_{k=1}^{d} \frac{\partial}{\partial x_l} (\mathrm{cof}\, Du)_{jl} , \end{aligned}$$

which proves that $\nabla \cdot G = -\nabla \cdot G$. □

alternative solution.

Proof. Denote the components of P by p_{jk}. Then

$$\det P \; \delta_{jk} = \sum_{l=1}^{d} p_{lj} (\mathrm{cof} P)_{lk} \quad \text{for } j, k = 1, ..., d, \tag{1}$$

in particular,

$$\det P = \sum_{l=1}^{d} p_{lj} (\mathrm{cof} P)_{lj} \, .$$

Differentiating this last equation with respect to p_{ij} gives

$$\frac{\partial \det P}{\partial p_{ij}} = (\operatorname{cof} P)_{ij} \ .$$

Here it is important that the (l,j)th entry of the cofactor matrix does not depend on p_{lj}. Setting $P = Du$ in (1) and using the chain rule gives

$$\sum_{k=1}^{d} \frac{\partial \det Du}{\partial x_k} \delta_{jk} = \sum_{k,l,m=1}^{d} \frac{\partial \det P}{\partial p_{lm}} \frac{\partial p_{lm}}{\partial x_k} \delta_{jk} = \sum_{k,l,m=1}^{d} (\operatorname{cof} Du)_{lm} \frac{\partial^2 u_l}{\partial x_k \partial x_m} \delta_{jk} \ , \tag{2}$$

for $j = 1, 2, ..., d$. On the other hand, from equation (1) we know that

$$\det Du \ \delta_{jk} = \sum_{l=1}^{d} \frac{\partial u_l}{\partial x_j} (\operatorname{cof} Du)_{lk}$$

and hence,

$$\sum_{k=1}^{d} \frac{\partial \det Du}{\partial x_k} \delta_{jk} = \sum_{k,l=1}^{d} \frac{\partial^2 u_l}{\partial x_j x_k} (\operatorname{cof} Du)_{lk} + \sum_{l,k=1}^{d} \frac{\partial u_l}{\partial x_j} \frac{\partial (\operatorname{cof} Du)_{lk}}{\partial x_k}$$

Comparing this last formula with formula (2) gives

$$0 = \sum_{l,k=1}^{d} \frac{\partial u_l}{\partial x_j} \frac{\partial (\operatorname{cof} Du)_{lk}}{\partial x_k} = \sum_{l=1}^{d} \frac{\partial u_l}{\partial x_j} \sum_{k=1}^{d} \frac{\partial (\operatorname{cof} Du)_{lk}}{\partial x_k}$$

for $j = 1, 2, ..., d$. All these formulas are valid for all $x \in \mathbb{R}^d$. Now fix $\underline{x} \in \mathbb{R}^d$. If $\det Du(\underline{x}) \neq 0$, then the columns of the Jacobian are linearly independent and hence

$$\sum_{k=1}^{d} \frac{\partial (\operatorname{cof} Du(\underline{x}))_{lk}}{\partial x_k} = 0$$

for $l = 1, 2, ..., d$. If $\det Du(\underline{x}) = 0$, then for all sufficiently small $\varepsilon > 0$ one has $\det(\varepsilon I_d + Du(\underline{x}) \neq 0$ and the previous steps of the proof give

$$\sum_{k=1}^{d} \frac{\partial (\operatorname{cof}[\varepsilon I_d + Du(\underline{x})])_{lk}}{\partial x_k} = 0$$

for $l = 1, 2, ..., d$ and $\varepsilon > 0$. Letting $\varepsilon \to 0$ gives the conclusion also in this case. ☐

Problem 2. Suppose that $w \in C^2(\overline{B(0,1)})$ is a retraction of the closed unit ball to its boundary, that is $w : \overline{B(0,1)} \to S^{d-1}$ and $w(x) = x$ for all $x \in S^{d-1}$. Recall from the proof of Theorem 4.1.2 that $\det Dw = 0$ for all $x \in \overline{B(0,1)}$. Introduce a vector field F by setting $F_j = w_1 (\operatorname{cof} Dw)_{1j}$ for $j = 1, 2, ..., d$.

(i) Use Problem 1 to show that F is divergence free.

Proof. Compute

$$\nabla \cdot F = \sum_{j=1}^{d} \frac{\partial w_1}{\partial x_j} (\operatorname{cof} Dw)_{1j} + w_1 \sum_{j=1}^{d} \frac{\partial (\operatorname{cof} Dw)_{1j}}{\partial x_j} = \det Dw = 0$$

where we used the statement of Problem 1 as well as formula (1). ☐

Show that in the case $d = 3$ we have $F = w_1(\nabla w_2 \times \nabla w_3) = w_1 \nabla \times (w_2 \nabla w_3)$.

Solution. One computes that

$$(\operatorname{cof} Dw)_{1k} = \begin{bmatrix} \partial_2 w_2 \partial_3 w_3 - \partial_2 w_3 \partial_3 w_2 \\ \partial_3 w_2 \partial_1 w_3 - \partial_3 w_3 \partial_2 w_2 \\ \partial_1 w_2 \partial_2 w_3 - \partial_2 w_2 \partial_1 w_3 \end{bmatrix} = \nabla w_2 \times \nabla w_3 = \nabla \times (w_2 \nabla w_3) = \nabla \times (w_3 \nabla w_2) \; .$$

Note that this field is divergence free because every irrotational field is.

Problem 3. Use the divergence theorem (Gauss's Theorem) to prove that a retraction of the closed ball to its boundary of class C^2 does not exist.

Proof. Suppose a retraction exists. Then, using the divergence theorem on the vector field F introduced in Problem 2 over the unit ball gives

$$\int_{S^{d-1}} x \cdot F(x)\, dS = \int_B \nabla \cdot F \, dx = 0$$

since on the unit sphere S^{d-1} the exterior unit normal at the point $x \in S^{d-1}$ is equal to vector x. (This identity is valid in all dimensions.) On the other hand

$$\int_{S^2} x \cdot F(x)\, dS = \int_{S^2} w_1 x \cdot (\nabla w_2 \times \nabla w_3)\, dS \; .$$

Use now the vector identity $a \cdot (b \times c) = c \cdot (a \times b)$. Then

$$\int_{S^2} x \cdot F(x)\, dS = \int_{S^2} w_1 \nabla w_3 \cdot (x \times \nabla w_2)\, dS \; .$$

Since x is the exterior normal vector and $w_2 = x_2$ on S^2, we know that

$$x \times \nabla w_2 = x \times \nabla x_2 \; .$$

By a similar argument ∇w_3 can be replaced by ∇x_3. Hence, with e_j denoting the jth standard basis vector we have

$$\begin{aligned} \int_{S^2} x \cdot F(x)\, dS &= \int_{S^2} x_1 x \cdot (\nabla x_2 \times \nabla x_3)\, dS \\ &= \int_{S^2} x_1 x \cdot (e_2 \times e_3)\, dS = \int_{S^2} x_1 x \cdot e_1 \, dS = \int_{S^2} x_1^2 \, dS = \frac{4}{3}\pi \neq 0 \; . \end{aligned}$$

Hence we have a contradiction and the retraction cannot exist. □

It is not too difficult to obtain an argument which works for all dimensions. Note that

$$(\operatorname{cof} Dw)_{1j} = \det \begin{bmatrix} e_j^T \\ [\nabla w_2]^T \\ \vdots \\ [\nabla w_d]^T \end{bmatrix}$$

Hence

$$x \cdot F(x) = x_1 \det \begin{bmatrix} x^T \\ [\nabla w_2]^T \\ \vdots \\ [\nabla w_d]^T \end{bmatrix} \; .$$

For $x \in S^{d-1}$ we know that $w_j = x_j$ and for $j = 2, ..., d$ and that x is the unit exterior normal vector. Decomposing the gradient into tangential and normal components gives

$$\nabla w_j = \nabla_{\tan} w_j + (x \cdot \nabla w_j) x = \nabla_{\tan} x_j + (x \cdot \nabla w_j) x \ .$$

This formula needs to be justified. The tangential gradient $\nabla_{\tan} f$ of a function f at $x \in S^{d-1}$ is the projection of the gradient into the tangent plane at x. This is best understood in a fairly general setting. Suppose that $x \ : \ U \subset \mathbb{R}^{d-1} \to S$ is a smooth parametrization of the hypersurface S in $\mathbb{R}^d$. Then we know that the Jacobian $J = Dx$ is a matrix with d rows and $d-1$ columns. The columns of this matrix $J(\underline{u})$ span the tangent space of S at $\underline{x} = x(\underline{u})$. Hence, working with the projection into the tangent space we know by the the chain rule that

$$\nabla_{\tan} f(\underline{x}) = J(J^T J)^{-1} J^T \nabla f = J(J^T J)^{-1} \nabla_u f(x(\underline{u})) \ .$$

This shows that the tangential gradient at $\underline{x}$ depends only on the values of f on the surface S. Hence, the tangential gradient of w_j and x_j is the same a every point $x \in S^{d-1}$.

In the rows $j = 2, ..., d$ we subtract now $(x \cdot \nabla w_j) x^T$ and add $(x \cdot \nabla x_j) x^T$ which are both multiples of the first row. This operation does not change the determinant and hence

$$x \cdot F(x) = x_1 \det \begin{bmatrix} x^T \\ [\nabla x_2]^T \\ \vdots \\ [\nabla x_d]^T \end{bmatrix} = x_1 \det \begin{bmatrix} x^T \\ e_2^T \\ \vdots \\ e_d^T \end{bmatrix} = x_1^2$$

and thus

$$\int_{S^{d-1}} x \cdot F(x) \ dS = \int_{S^{d-1}} x_1^2 \, dS = \mathrm{Vol}(B) \ .$$