WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #11 due 01/15/2016

The first two problems are dedicated to a more elementary proof of the Brouwer Fix Point Theorem (Theorem 4.1.2). The goal is to replace the argument given in the lecture using differential forms by more elementary means.

Problem 1. Given a $d \times d$ matrix P, denote its *cofactor matrix* by $\operatorname{cof} P$. From linear algebra recall the identity $\det P I_d = P^T \operatorname{cof} P$. Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field with C^2 components and introduce a vector field $G : \mathbb{R}^d \to \mathbb{R}^d$ by $G_k = (\operatorname{cof} Du)_{jk}$ for some $j \in \{1, 2, ..., d\}$. This vector field is a row of the cofactor matrix of the Jacobian matrix of u. Show that G is divergence free, that is $\nabla \cdot G = 0$ for all $x \in \mathbb{R}^d$.
(Hint: Depending on the strategy of the proof it may be helpful to assume at first that $\det Du(\underline{x}) \neq 0$ at the point of consideration $\underline{x} \in \mathbb{R}^d$. If $\det Du(\underline{x}) = 0$ replace u by $u + \varepsilon x$ for $\varepsilon > 0$ and consider the limit as $\varepsilon \to 0$.)

Problem 2. Suppose that $w \in C^2(\overline{B(0,1)})$ is a retraction of the closed unit ball to its boundary, that is $w : \overline{B(0,1)} \to S^{d-1}$ and $w(x) = x$ for all $x \in S^{d-1}$. Recall from the proof of Theorem 4.1.2 that $\det Dw = 0$ for all $x \in \overline{B(0,1)}$. Introduce a vector field F by setting $F_j = w_1 (\operatorname{cof} Dw)_{1j}$ for $j = 1, 2, ..., d$.

(i) Use Problem 1 to show that F is divergence free.

(ii) Show that in the case $d = 3$ we have $F = w_1(\nabla w_2 \times \nabla w_3)$.

Problem 3. Use the divergence theorem (Gauss's Theorem) to prove that a retraction of the closed unit ball to its boundary of class C^2 in $d = 3$ does not exist.