WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #12 Key

Problem 1. Consider the following initial-boundary value problem for the heat equation

$$\begin{aligned} u_t - \Delta u &= f \in L_2(Q_T) \\ u &= 0 \text{ in } \Sigma = (0,T) \times \partial\Omega \\ u(0,\cdot) &= g \in L_2(\Omega) \ . \end{aligned}$$

a.) Construct a sequence of Faedo-Galerkin approximations, that is a sequence of functions $u_m : [0,T] \to \mathring{H}^1(\Omega)$ of the form $u_m(t) = \sum_{k=1}^m d_m^k(t) w_k$ where the coefficients d_m^k satisfy

$$d_m^k(0) = (g, w_k)_{L_2(\Omega)} \quad \text{and} \quad (u_m', w_k)_{L_2(\Omega)} + (\nabla u_m, \nabla w_k)_{L_2(\Omega)} = (f, w_k)_{L_2(\Omega)}$$

for $k = 1, 2, ..., m$, where w_k are the orthonormal eigenfunctions of the Dirichlet Laplacian in Ω with respect to the L_2 inner product.

Solution. Inserting $u_m(t) = \sum_{l=1}^m d_m^l(t) w_k$ into the identity

$$(u_m', w_k)_{L_2(\Omega)} + (\nabla u_m, \nabla w_k)_{L_2(\Omega)} = (f, w_k)_{L_2(\Omega)}$$

gives because of the othonormality of the Dirichlet eigenfunctions

$$\frac{d}{dt} d_m^k(t) + \lambda_k d_m^k(t) = (f(t,\cdot), w_k)_{L_2(\Omega)} =: F(t) \ .$$

Note that with the Cauchy-Schwarz inequality and Hölder's inequality

$$\begin{aligned} \int_0^T |F(t)| dt = \int_0^T \left| \int_\Omega f(t,x) w_k(x) \, dx \right| dt &\le \int_0^T \|f(t,\cdot)\|_{L_2(\Omega)} dt \, \|w_k\|_{L_2(\Omega)} \\ \le \int_0^T \|f(t,\cdot)\|^2 \, dt \, \|w_k\|_{L_2(\Omega)} &\le \|w_k\|_{L_2(\Omega)} \left(\int_0^T \|f(t,\cdot)\|_{L_2(\Omega)}^2 \right)^{1/2} \sqrt{T} \\ &\le \sqrt{T} \|f\|_{L_2(Q_T)} \|w_k\|_{L_2(\Omega)} \end{aligned}$$

which shows that the right hand side in the ODE above is in $L_1(0,T)$. Setting $(g, w_k)_{L_2(\Omega)} = g_k$, the unique solution to this ODE is given by

$$d_m^k(t) = e^{-\lambda_k t} g_k + \int_0^t e^{-\lambda_k (t-s)} F(s) \, ds$$

which is an absolutely continuous function. In conclusion, one obtains $u_m \in W_1^1(0,T; \mathring{H}^1(\Omega))$.

b.) Establish the apriori estimate

$$\max_{t \in [0,T]} \|u_m(t)\|_{L_2(\Omega)} + \|u_m\|_{L_2(0,T;\mathring{H}^1(\Omega))} + \|u_m'\|_{L_2(0,T;H^{-1}(\Omega))} \le C \left(\|f\|_{L_2(Q_T)} + \|g\|_{L_2(\Omega)} \right) \ ,$$

where C is a positive constant which does not depend on m, g, and f.

Solution. Multiplying each identity

$$(u_m', w_k)_{L_2(\Omega)} + (\nabla u_m, \nabla w_k)_{L_2(\Omega)} = (f, w_k)_{L_2(\Omega)}$$

by d_m^k and adding from $k = 1, 2, ...,$ one obtains, after integration over $(0, t)$ for some $0 \le t \le T$

$$\int_0^t (u_m', u_m)_{L_2(\Omega)} ds + \int_0^t (\nabla u_m, \nabla u_m)_{L_2(\Omega)} ds = \int_0^t (f, u_m)_{L_2(\Omega)} ds \ .$$

Noting that $2(u_m', u_m)_{L_2(\Omega)} = \frac{d}{dt}\|u_m\|^2_{L_2(\Omega)}$ and using the Cauchy-Schwarz inequality on the right-hand side gives

$$\frac{1}{2}\|u_m(t)\|^2_{L_2(\Omega)} - \frac{1}{2}\|u_m(0)\|^2_{L_2(\Omega)} + \|u_m\|^2_{L_2(0,t;\mathring{H}^1(\Omega))} \le \|f\|_{L_2(Q_T)^2}\|u_m\|_{L_2(Q_T)}$$

for all $0 \le t \le T$. Hence, for all $\varepsilon > 0$ one gets

$$\begin{aligned} \sup_{0<t<T} \frac{1}{2}\|u_m(t)\|^2_{L_2(\Omega)} + \|u_m\|^2_{L_2(0,T;\mathring{H}^1(\Omega))} &\le \frac{1}{4\varepsilon}\|f\|^2_{L_2(Q_T)^2} + \varepsilon\|u_m\|^2_{L_2(Q_T)} + \frac{1}{2}\|u_m(0)\|^2_{L_2(\Omega)} \\ &\le \frac{1}{4\varepsilon}|f\|^2_{L_2(Q_T)^2} + \varepsilon C\|u_m\|^2_{L_2(0,T;\mathring{H}^1(\Omega))} + \frac{1}{2}\|g\|^2_{L_2(\Omega)} \end{aligned}$$

where one uses also the Poincaré inequality and

$$u_m(0) = \sum_{k=1}^m g_k w_k \ .$$

Choosing $\varepsilon = 1/(2C)$ allows us to move the second term on the right-hand side into the left-hand side. Then

$$\sup_{0<t<T} \|u_m(t)\|^2_{L_2(\Omega)} + \|u_m\|^2_{L_2(0,T;\mathring{H}^1(\Omega))} \le C|f\|^2_{L_2(Q_T)^2} + \|g\|^2_{L_2(\Omega)} \ . \tag{1}$$

Finally, note that

$$\begin{aligned} \|u_m'\|_{L_2(0,T;H^{-1}(\Omega))} &= \sup_{\|v\|_{L_2(0,T;\mathring{H}^1(\Omega))}=1} |(u_m', v)_{L_2(Q_T)}| \\ &= \sup \left| -\left(\nabla u_m, \sum_{k=1}^m \beta_k(t)\nabla w_k\right)_{L_2(Q_T)} + \left(f, \sum_{k=1}^m \beta(t) w_k\right)_{L_2(Q_T)} \right| \end{aligned}$$

where the sup is taken over all continuous functions β_k such that

$$\int_0^T \sum_{k=1}^m |\beta_k(t)|^2 dt = 1 \ .$$

The sum terminates at m since u_m is a linear combination of the first m basis functions. Using the Cauchy-Schwarz inequality gives

$$\|u_m'\|_{L_2(0,T;H^{-1}(\Omega))} \le \|u_m\|_{L_2(0,T;\mathring{H}^1(\Omega))} + \|f\|_{L_2(Q_T)} \ ,$$

and the proof is finished by inserting the last inequality into (1).

Problem 2. Suppose that $u \in L_2(0, T; \mathring{H}^1(\Omega))$ satisfies $\partial u/\partial t \in L_2(0, T; H^{-1}(\Omega))$. Prove that $u \in C([0, T], L_2(\Omega))$.

Proof. For $\varepsilon, \delta > 0$ consider the regularization $u^{(\varepsilon)}$ and $u^{(\delta)}$ of u in time and space. In order to use the regularization as introduced in Chapter 3 the functions u is extended by zero outside of Ω and outside of the interval $[0, T]$. Then, using the duality between the Sobolev spaces $\mathring{H}^1(\Omega)$ and $H^{-1}(\Omega)$, we have

$$\frac{d}{dt}\|u^{(\varepsilon)}(\tau) - u^{(\delta)}(\tau)\|_{L_2(\Omega)} = 2\left(u^{(\varepsilon)}(\tau) - u^{(\delta)}(\tau), \frac{d}{dt}u^{(\varepsilon)}(\tau) - \frac{d}{dt}u^{(\delta)}(\tau)\right)$$
$$\leq \|u^{(\varepsilon)}(\tau) - u^{(\delta)}(\tau)\|^2_{\mathring{H}^1(\Omega)} + \left\|\frac{d}{dt}u^{(\varepsilon)}(\tau) - \frac{d}{dt}u^{(\delta)}(\tau)\right\|^2_{H^{-1}(\Omega)} .$$

Integrating this identity over the interval $(s, t) \subset [0, T]$ one obtains

$$\|u^{(\varepsilon)}(t) - u^{(\delta)}(t)\|_{L_2(\Omega)} \leq \|u^{(\varepsilon)}(s) - u^{(\delta)}(s)\|_{L_2(\Omega)}$$
$$+ \|u^{(\varepsilon)} - u^{(\delta)}\|^2_{L_2(0,T;\mathring{H}^1(\Omega))} + \left\|\frac{d}{dt}u^{(\varepsilon)} - \frac{d}{dt}u^{(\delta)}\right\|^2_{L_2(0,T;H^{-1}(\Omega))}$$

Choose now $s \in (0, T)$ such that $u^{(\varepsilon)}(s) \to u(s)$ in $L_2(\Omega)$ as $\varepsilon \to 0$. This can be done since convergence in $L_2(0, T, \mathring{H}^1(\Omega))$ implies convergence almost everywhere with respect to t. Then the identity above shows that $u(\varepsilon)$ is a Cauchy sequence in the function space $C([0, T], L_2(\Omega))$. Hence, $u^{(\varepsilon)} \to v \in C([0, T], L_2(\Omega))$ and $u = v$ almost everywhere in t in $L_2(\Omega)$ for $t \in [0, T]$. □

Problem 3. Consider the semilinear elliptic boundary-value problem

$$-\Delta u + b(\nabla u) = f \qquad \text{in } \Omega ,$$
$$u = 0 \qquad \text{in } \partial\Omega .$$

Use Banach's fixed point theorem to show that there exists a unique solution $u \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$ provided $f \in L_2(\Omega)$ and $b : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous with a small enough Lipschitz constant.

Proof. Suppose that $u \in \mathring{H}^1(\Omega)$ and consider the linear elliptic boundary value problem

$$-\Delta w = -b(\nabla u) + f \qquad \text{in } \Omega ,$$
$$u = 0 \qquad \text{in } \partial\Omega .$$

Since b is Lipschitz, we know that $|b(p)| \leq C(1 + |p|)$ for all $p \in \mathbb{R}^d$. Hence

$$\int_\Omega |b(\nabla u)|^2\, dx \leq C^2 \int_\Omega (1 + |\nabla u|)^2 \leq 2C^2\left(1 + \|u\|^2_{\mathring{H}^1(\Omega)}\right) .$$

which shows that $b(\nabla u) \in L_2(\Omega)$. Using the theory from Chapter 2, one sees that the linear problem above has a unique solution in $w \in \mathring{H}^1(\Omega) \cap H^2(\Omega)$. Introduce a non-linear operator $A : \mathring{H}^1(\Omega) \to \mathring{H}^1(\Omega)$ by setting $Au = w$. We will show that this operator is a contraction provided the Lipschitz constant L of b is sufficiently small. Let $A\tilde{u} = \tilde{w}$ and observe that

$$(\nabla w - \nabla\tilde{w}, \nabla v)_{L_2(\Omega)} = -(b(\nabla u) - b(\nabla\tilde{u}), v)_{L_2(\Omega)} \qquad \text{for all } v \in \mathring{H}^1(\Omega) ,$$

Compute, with $v = w - \tilde{w}$ in the identity above, using the Cauchy-Schwarz inequality

$$\|w - \tilde{w}\|^2_{\mathring{H}^1(\Omega)} \leq \|b(\nabla u) - b(\nabla\tilde{u})\|_{L_2(\Omega)}\|w - \tilde{w}\|_{L_2(\Omega)} \leq CL\|u - \tilde{u}\|_{\mathring{H}^1(\Omega)}\|w - \tilde{w}\|_{\mathring{H}^1(\Omega)} ,$$

where C is the constant in Poincaré's inequality. Hence,

$$\|w - \tilde{w}\|_{\mathring{H}^1(\Omega)} \leq CL\|u - \tilde{u}\|_{\mathring{H}^1(\Omega)}$$

which proves that A is a contraction as long as $CL < 1$. By Banach's Fixed Point Theorem, the operator A has a unique fixed point $u \in \mathring{H}^1(\Omega)$ which is then the only possible solution to the semilinear problem above. Note that elliptic regularity implies that $u \in H^2(\Omega)$. □