WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #12 due 01/22/2016

Problem 1. Consider the following initial-boundary value problem for the heat equation

$$\begin{aligned} u_t - \Delta u &= f \in L_2(Q_T) \\ u &= 0 \text{ in } \Sigma = (0,T) \times \partial\Omega \\ u(0,\cdot) &= g \in L_2(\Omega) \ . \end{aligned}$$

a.) Construct a sequence of Faedo-Galerkin approximations, that is a sequence of functions $u_m : [0,T] \to \mathring{H}^1(\Omega)$ of the form $u_m(t) = \sum_{k=1}^m d_m^k(t) w_k$ where the coefficients d_m^k satisfy

$$d_m^k(0) = (g, w_k)_{L_2(\Omega)} \text{ and } (u_m', w_k)_{L_2(\Omega)} + (\nabla u_m, \nabla w_k)_{L_2(\Omega)} = (f, w_k)_{L_2(\Omega)}$$

for $k = 1, 2, ..., m$, where w_k are the orthonormal eigenfunctions of the Dirichlet Laplacian in Ω with respect to the L_2 inner product.

b.) Establish the apriori estimate

$$\max_{t\in[0,T]} \|u_m(t)\|_{L_2(\Omega)} + \|u_m\|_{L_2(0,T;\mathring{H}^1(\Omega))} + \|u_m'\|_{L_2(0,T;H^{-1}(\Omega))} \leq C \left(\|f\|_{L_2(Q_T)} + \|g\|_{L_2(\Omega)}\right) \ ,$$

where C is a positive constant which does not depend on m, g, and f.

Problem 2. Suppose that $u \in L_2(0,T;\mathring{H}^1(\Omega))$ satisfies $\partial u/\partial t \in L_2(0,T;H^{-1}(\Omega))$. Prove that $u \in C([0,T], L_2(\Omega))$.

Problem 3. Consider the semilinear elliptic boundary-value problem

$$\begin{aligned} -\Delta u + b(\nabla u) &= f \qquad \text{in } \Omega \ , \\ u &= 0 \qquad \text{in } \partial\Omega \ . \end{aligned}$$

Use Banach's fixed point theorem to show that there exists a unique solution $u \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$ provided $f \in L_2(\Omega)$ and $b : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous with a small enough Lipschitz constant.