WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #13 Key

Problem 1. a.) Suppose that $u : [a, b] \to [0, \infty)$ and $v : [a, b] \to \mathbb{R}$ are continuous functions and there exists a constant $C \in \mathbb{R}$ such that

$$v(t) \le C + \int_a^t v(s)u(s)\,ds \qquad \text{for all } t \in [a, b]\ .$$

Prove that

$$v(t) \le C\text{exp}\left(\int_a^t u(s)\,ds\right) \qquad \text{for all } t \in [a, b]\ .$$

Proof. Start with

$$v(t) - \int_a^t v(s)u(s)\,ds \le C$$

and multiply this inequality with

$$u(t)\text{exp}\left\{-\int_a^t u(s)\,ds\right\}\ .$$

Because of our assumptions this expression is non-negative. Thus,

$$\left[v(t)u(t) - u(t)\int_a^t v(s)u(s)\,ds\right]\text{exp}\left\{-\int_a^t u(s)\,ds\right\} \le Cu(t)\text{exp}\left\{-\int_a^t u(s)\,ds\right\}\ .$$

Using the product rule this inequality can be written in the form

$$\frac{d}{dt}\left[\int_a^t v(s)u(s)\,ds\ \text{exp}\left\{-\int_a^t u(s)\,ds\right\}\right] \le -C\frac{d}{dt}\text{exp}\left\{-\int_a^t u(s)\,ds\right\}\ .$$

Integrating over the interval $[a, t]$ with $t \in [a, b]$ gives

$$\int_a^t v(s)u(s)\,ds\ \text{exp}\left\{-\int_a^t u(s)\,ds\right\} \le C\left[1 - \text{exp}\left\{-\int_a^t u(s)\,ds\right\}\right]$$

which results in

$$\int_a^t v(s)u(s)\,ds \le C\left[\text{exp}\left\{\int_a^t u(s)\,ds\right\} - 1\right] \qquad \text{for all } t \in [a, b]$$

The proof is finished by using the assumption one more time:

$$v(t) \le C + \int_a^t v(s)u(s)\,ds \le C\text{exp}\left\{\int_a^t u(s)\,ds\right\} \qquad \text{for all } t \in [a, b]\ .$$

□

b.) Suppose that $u : [0,T] \to \mathbb{R}$ and $f : [0,T] \to \mathbb{R}$ are continuous functions, that f is non-negative, and that there exist two constant $C_0 \in \mathbb{R}$ and $C_1 > 0$ such that

$$u(t) \leq C_0 + C_1 \int_0^t [u(s) + f(s)]\, ds \qquad \text{for all } t \in [0,T] \,.$$

Prove that

$$u(t) \leq e^{C_1 t} \left(C_0 + C_1 \int_0^t f(s)\, ds \right) \qquad \text{for all } t \in [0,T] \,.$$

Proof. Let

$$v(t) = C_0 + C_1 \int_0^t [u(s) + f(s)] ds \,, \qquad t \in [0,T] \,.$$

Note that v is continuously differentiable and that

$$v'(t) = C_1[u(t) + f(t)] \leq C_1[v(t) + f(t)] \,.$$

This inequality can be rewritten as

$$\frac{d}{dt}[e^{-C_1 t} v] \leq e^{-C_1 t} f(t) \,.$$

Integrating over $[0,t]$ with $t \in [0,T]$ gives

$$e^{-C_1 t} v(t) \leq v(0) + \int_0^t e^{-C_1 s} f(s)\, ds$$

and thus, since f is non-negative,

$$v(t) \leq e^{C_1 t} v(0) + \int_0^t e^{C_1 (t-s)} f(s)\, ds \leq e^{C_1 t} \left(v(0) + \int_0^t f(s)\, ds \right) \,.$$

Finally, not that $v(0) = C_0$ and since $u(t) \leq v(t)$ the claim has been proved. □

Both results are know as Gronwall's Lemma or Gronwall's inequality.

Problem 2. Suppose that $w_1 \in \mathring{H}^1(\Omega)$ is a first normalized eigenfunction of the Dirichlet Laplacian, that is $-\Delta w_1 = \lambda_1 w_1$ in Ω in the weak sense and that $\|w_1\|_{L_2(\Omega)} = 1$.

a.) Let $\lambda_1 > 0$ be the first (smallest) eigenvalue of the Dirichlet-Laplacian in Ω. Prove that

$$\lambda_1 = \min \int_\Omega |\nabla u|^2\, dx = \int_\Omega |\nabla w_1|^2\, dx \,,$$

where the minimum is taken over all $u \in \mathring{H}^1(\Omega)$ such that $\|u\|_{L_2(\Omega)} = 1$. (Hint: Use the fact that there exists an orthonormal basis of Dirichlet eigenfunctions $w_1, w_2, ...$ in $L_2(\Omega)$.)

Proof. Suppose that $u \in \mathring{H}^1(\Omega)$ and that $\|u\|_{L_2(\Omega)} = 1$. Then

$$u = \sum_{n=1}^\infty u_n w_n \quad \text{with} \quad u_n = (u, w_n)_{L_2(\Omega)} \quad \text{and} \quad \sum_{n=1}^\infty u_n^2 = 1 \quad \sum_{n=1}^\infty \lambda_n u_n^2 < \infty$$

Then

$$\int_\Omega |\nabla u|^2\, dx = \sum_{n=1}^\infty \lambda_n u_n^2 \geq \lambda_1 \,.$$

□

Observe that the proof shows that for all $u \in \mathring{H}^1(\Omega)$ the inequality

$$\int_\Omega |\nabla u|^2\,dx \ge \lambda_1 \int_\Omega |u|^2\,dx$$

holds. One sees this when one replace u in the proof above with $u/\|u\|_{L_2(\Omega)}$. In other words, the constant $C = \lambda_1^{-1}$ is the best (i.e. smallest) constant in Poincaré's inequality.

b.) Prove that we can choose $w_1 > 0$ in Ω.

Proof. Let $w^+ = max\{0, w_1\}$ and $w^- = \min\{0, w_1\}$ be the positive and negative part of w_1, respectively. Then

$$\nabla w^+ = \begin{cases} \nabla w_1 & \text{a.e. on} \quad w_1 > 0 \\ 0 & \text{a.e. on} \quad w_1 < 0 \end{cases} \quad \text{and} \quad \nabla w^- = \begin{cases} \nabla w_1 & \text{a.e. on} \quad w_1 < 0 \\ 0 & \text{a.e. on} \quad w_1 > 0 \end{cases} .$$

(This statement is not obvious. It may deserve a proof.) Then with

$$a = \int_\Omega |w^+|^2\,dx \quad \text{and} \quad b = \int_\Omega |w^-|^2\,dx$$

one has, using a.), in particular the remark following the proof,

$$\lambda_1 = \int_\Omega |\nabla w_1|^2\,dx = \int_\Omega |\nabla w^+|^2\,dx + \int_\Omega |\nabla w^-|^2\,dx \ge \lambda_1(a+b) = \lambda\ .$$

Hence, both w^+ and w^- are eigenfunctions for the Dirichlet Laplacian with eigenvalue λ_1. At least one of these two functions, say w^+, cannot be identically zero. Then because of the strong maximum principle for second order elliptic equations one obtains $w_1 > 0$ in Ω and $w^- > 0$. □

c.) Show that λ_1 is a simple eigenvalue.

Solution. If there are two linearly independent eigenfunctions, according to part b.), they must be both positive. However, then the cannot be orthogonal to each other.