WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #13 due 01/29/2016

Problem 1. a.) Suppose that $u : [a,b] \to [0,\infty)$ and $v : [a,b] \to \mathbb{R}$ are continuous functions and there exists a constant $C \in \mathbb{R}$ such that

$$v(t) \leq C + \int_a^t v(s)u(s)\,ds \qquad \text{for all } t \in [a,b]\ .$$

Prove that

$$v(t) \leq C\mathrm{exp}\left(\int_a^t u(s)\,ds\right) \qquad \text{for all } t \in [a,b]\ .$$

b.) Suppose that $u : [0,T] \to \mathbb{R}$ and $f : [0,T] \to \mathbb{R}$ are continuous functions, that f is non-negative, and that there exist two constant $C_0 \in \mathbb{R}$ and $C_1 > 0$ such that

$$u(t) \leq C_0 + C_1 \int_0^t [u(s) + f(s)]\,ds \qquad \text{for all } t \in [0,T]\ .$$

Prove that

$$u(t) \leq e^{C_1 t}\left(C_0 + C_1 \int_0^t f(s)\,ds\right) \qquad \text{for all } t \in [0,T]\ .$$

Both results are know as Gronwall's Lemma or Gronwall's inequality.

Problem 2. Suppose that $w_1 \in \mathring{H}^1(\Omega)$ is a first normalized eigenfunction of the Dirichlet Laplacian, that is $-\Delta w_1 = \lambda_1 w_1$ in Ω in the weak sense and that $\|w_1\|_{L_2(\Omega)} = 1$.

a.) Let $\lambda_1 > 0$ be the first (smallest) eigenvalue of the Dirichlet-Laplacian in Ω. Prove that

$$\lambda_1 = \min \int_\Omega |\nabla u|^2\,dx = \int_\Omega |\nabla w_1|^2\,dx\ ,$$

where the minimum is taken over all $u \in \mathring{H}^1(\Omega)$ such that $\|u\|_{L_2(\Omega)} = 1$. (Hint: Use the fact that there exists an orthonormal basis of Dirichlet eigenfunctions $w_1, w_2, \ldots$ in $L_2(\Omega)$.)

b.) Prove that we can choose $w_1 > 0$ in Ω.

c.) Show that λ_1 is a simple eigenvalue.