WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #14 due 02/05/2016

Problem 1. Verify that the shallow water equations

$$\phi_t + (v\phi)_x = 0$$
$$v_t + \left(\frac{v^2}{2} + \phi\right)_x = 0$$

form a strictly hyperbolic system as long as $\phi > 0$.

Solution. Note that

$$\begin{bmatrix} \phi \\ v \end{bmatrix}_t = \begin{bmatrix} v & \phi \\ 1 & v \end{bmatrix} \begin{bmatrix} \phi \\ v \end{bmatrix}_x .$$

The Matrix $B = \begin{bmatrix} v & \phi \\ 1 & v \end{bmatrix}$ has the eigenvalues $\lambda = v \pm \sqrt{\phi}$ which are real and distinct if and only if $\phi > 0$.

Problem 2. Consider the matrix function

$$B(z) = \begin{cases} e^{-1/z^2} \begin{bmatrix} \cos(2/z) & \sin(2/z) \\ \sin(2/z) & -\cos(2/z) \end{bmatrix} & \text{for } z \neq 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{for } z = 0 \end{cases} .$$

a.) Show that $B \in C^\infty(\mathbb{R}; \mathbb{R}^{2\times 2})$.

Proof. Recall that the function

$$f(z) = \begin{cases} e^{-1/z^2} & \text{for} \quad z \neq 0 \\ 0 & \text{for} \quad z = 0 \end{cases}$$

is in C^∞ and that all derivatives at $z = 0$ vanish. Furthermore, each derivative decays to zero faster than z^m as $z \to 0$ for all $m \in \mathbb{N}$. Hence, each entry of the function $B(z)$ is also in C^∞ since all derivatives can be continuously extended to $z = 0$ by zero. □

b.) Prove that there do not exist eigenvectors $r_1(z)$, $r_2(z)$ depending continuously on z near 0. What happens to the eigenspaces as $z \to 0$?

Proof. Compute, for $z \neq 0$

$$\det \begin{bmatrix} \lambda - e^{-1/z^2}\cos(2/z) & -e^{-1/z^2}\sin(2/z) \\ -e^{-1/z^2}\sin(2/z) & \lambda + e^{-1/z^2}\cos(2/z) \end{bmatrix}$$
$$= \lambda^2 - e^{-1/z^4}\cos^2\frac{2}{z} - e^{-1/z^4}\sin^2\frac{2}{z} = \lambda^2 - e^{-1/z^4}$$

which gives the two eigenvalues $\lambda_1 = -e^{-1/z^2}$ and $\lambda_2 = e^{-1/z^2}$. The corresponding normalized eigenvectors are

$$r_1(z) = \begin{cases} \dfrac{1}{\sqrt{2-2\cos(2/z)}} \begin{bmatrix} 1-\cos(2/z) \\ -\sin(2/z) \end{bmatrix} & \text{for} \quad \cos(2/z) \neq 1 \ , \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{for} \quad \cos(2/z) = 1 \ , \end{cases}$$

and

$$r_2(z) = \begin{cases} \dfrac{1}{\sqrt{2+2\cos(2/z)}} \begin{bmatrix} 1+\cos(2/z) \\ \sin(2/z) \end{bmatrix} & \text{for} \quad \cos(2/z) = -1 \ , \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{for} \quad \cos(2/z) = -1 \ . \end{cases}$$

With a little bit of trigonometry, these eigenvectors can be written as

$$r_1(z) = \begin{bmatrix} \sin(1/z) \\ -\cos(1/z) \end{bmatrix} \quad \text{and} \quad r_2(z) = \begin{bmatrix} \cos(1/z) \\ \sin(1/z) \end{bmatrix} \ ,$$

which has the advantage that it shows that the normalized eigenvectors are C^∞ for all $z \neq 0$. However, there is no limit for $z \to 0$. Indeed, for all $\varepsilon > 0$ there exist $|z_j| < \varepsilon$ for $j = 1, 2$ such that $r_1(z_1) = (1, 0)^T$ and $r_1(z_2) = (0, 1)^T$. □

Problem 3. a.) Consider the initial value problem (Riemann Problem) for Burgers's equation

$$u_t + uu_x = 0 \qquad \text{for } t > 0, x \in \mathbb{R} \ ,$$
$$u(0, x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases} \ .$$

Prove that

$$u(t, x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \frac{x}{t} & \text{for} \quad 0 < x < t \\ 1 & \text{for} \quad x > t \end{cases} \quad \text{and} \quad \tilde{u}(t, x) = \begin{cases} 0 & \text{for} \quad x < t/2 \\ 1 & \text{for} \quad x > t/2 \end{cases}$$

are both integral solutions to Burgers's equation.

Proof. Recall that an integral solution u of a conservation law is an essentially bounded function which satisfies the integral identity

$$\int_0^\infty \int_{\mathbb{R}} [uv_t + f(u)v_x] \, dxdt = -\int_{\mathbb{R}} g(x)v(0, x) \, dx$$

for all $v \in C_0^\infty(\mathbb{R}^2)$ where g are the initial data.

In order to verify that u is an integral solution, one computes for $v \in C_0^\infty(\mathbb{R}^2)$ the integral

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}} [uv_t + u^2 v_x/2] dx dt &= \int_0^\infty \int_0^t \left[\frac{x}{t} v_t + \frac{x^2}{2t^2} v_x\right] dx dt + \int_0^\infty \int_t^\infty \left[v_t + \frac{1}{2} v_x\right] dx dt \\
&= \int_0^\infty \int_x^\infty \frac{x}{t} v_t \, dt dx + \int_0^\infty \int_0^t \frac{x^2}{2t^2} v_x \, dx dt \\
&+ \int_0^\infty \int_0^x v_t \, dt dx + \int_0^\infty \int_t^\infty \frac{1}{2} v_x \, dx dt \\
&= \int_0^\infty \int_x^\infty \frac{x}{t^2} v \, dt dx - \int_0^\infty v(x,x)\, dx - \int_0^\infty \int_0^t \frac{x}{t^2} v \, dx dt \\
&+ \frac{1}{2} \int_0^\infty v(t,t)\, dt + \int_0^\infty v(x,x)\, dx \\
&- \int_0^\infty v(0,x) dx - \int_0^\infty \frac{1}{2} v(t,t)\, dt = \int_0^\infty v(0,x)\, dx
\end{aligned}$$

which proves that u is an integral solution to the Riemann problem. For $\tilde{u}$ one has to compute the same integral

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}} [\tilde{u} v_t + \tilde{u}^2 v_x/2] dx dt &= \int_0^\infty \int_{t/2}^\infty v_t \, dx dt + \int_0^\infty \int_{t/2}^\infty \frac{1}{2} v_x \, dx dt \\
&= \int_0^\infty \frac{d}{dt} \int_{t/2}^\infty v \, dx dt + \int_0^\infty \frac{1}{2} v(t, t/2)\, dt - \frac{1}{2} \int_0^\infty v(t, t/2)\, dt \\
&= - \int_0^\infty v(0,x)\, dx
\end{aligned}$$

and the desired identity has been verified. □

b.) Find an integral solution to Burgers's equation with the initial condition

$$u(0,x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad 0 < x < 1 \\ 0 & \text{for} \quad x > 1 \end{cases} .$$

Does your solution satisfy the entropy condition $F'(u_l) > \sigma > F'(u_r)$?

Solution. Following the discussion on Burgers's equation from the lecture, it suggests that the entropy solution is given by

$$u(t,x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \frac{x}{t} & \text{for} \quad 0 < x < t \\ 1 & \text{for} \quad t < x < 1 + t/2 \\ 0 & \text{for} \quad x > 1 + t/2 \end{cases} ,$$

at least for $0 < t \leq 2$ since for $t > 2$ this function is not well-defined. The formula above suggests that for $t > 2$ the area where $u = 1$ disappears and that

$$u(t,x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \frac{x}{t} & \text{for} \quad 0 < x < s(t) \\ 0 & \text{for} \quad x > s(t) \end{cases} ,$$

where $s(t)$ is the curve along which the solution is discontinuous (shock curve). In order to have an integral solution the solution needs to satisfy the Rankine-Hugoniot condition $[F(u)] = x'(t)[u]$ where the shock curve is expressed as a function $x(t)$. In this case one as

$$\frac{F(u_l) - F(u_r)}{u_l - u_r} = \frac{x^2/(2t^2)}{x/t} = \frac{x}{2t}$$

Hence, $x(t) = C\sqrt{t}$ for $t \geq 2$ and the condition $x(2) = 2$ gives $x(t) = \sqrt{2t}$. The condition $x(2) = 1$ follows from the fact that $x(t) = 1 + t/2$ for $0 < t \leq 2$. In summary, the formula for the integral solution for $t \geq 2$ is given by

$$u(t,x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \frac{x}{t} & \text{for} \quad 0 < x < \sqrt{2t} \\ 0 & \text{for} \quad x > \sqrt{2t} \end{cases},$$

Finally one verifies that u satisfies the Lax shock condition $F'(u_l) > x'(t) > F'(u_r)$. Note that $F'(u) = u$ and that then

$$F'(u_l) = \frac{x}{t}, \qquad F'(u_r) = 0, \qquad x'(t) = \frac{x}{2t}\ .$$