WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #14 due 02/05/2016

Problem 1. Verify that the shallow water equations

$$\phi_t + (v\phi)_x = 0$$
$$v_t + \left(\frac{v}{2} + \phi\right)_x = 0$$

form a strictly hyperbolic system as long as $\phi > 0$.

Problem 2. Consider the matrix function

$$B(z) = \begin{cases} e^{-1/z^2} \begin{bmatrix} \cos(2/z) & \sin(2/z) \\ \sin(2/z) & -\cos(2/z) \end{bmatrix} & \text{for } z \neq 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{for } z = 0 \end{cases} .$$

a.) Show that $B \in C^\infty(\mathbb{R}; \mathbb{R}^{2\times 2})$.

b.) Prove that there do not exist eigenvectors $r_1(z)$, $r_2(z)$ depending continuously on z near 0. What happens to the eigenspaces as $z \to 0$?

Problem 3. a.) Consider the initial value problem (Riemann Problem) for Burgers's equation

$$u_t + uu_x = 0 \qquad \text{for } t > 0, x \in \mathbb{R} ,$$
$$u(0,x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases} .$$

Prove that

$$u(t,x) = \begin{cases} 0 & \text{for} & x < 0 \\ \frac{x}{t} & \text{for} & 0 < x < t \\ 1 & \text{for} & x > t \end{cases} \qquad \text{and} \qquad \tilde{u}(t,x) = \begin{cases} 0 & \text{for} & x < t/2 \\ 1 & \text{for} & x > t/2 \end{cases}$$

are both integral solutions to Burgers's equation.

b.) Find an integral solution to Burgers's equation with the initial condition

$$u(0,x) = \begin{cases} 0 & \text{for} & x < 0 \\ 1 & \text{for} & 0 < x < 1 \\ 0 & \text{for} & x > 1 \end{cases} .$$

Does your solution satisfy the entropy condition $F'(u_l) > \sigma > F'(u_r)$?