WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #15

Problem 1. Consider the initial value problem $u_t + F(u)_x = 0$ subject to the initial condition $u(0,x) = g(x)$. Assuming that $g \in C^1(\mathbb{R})$ with $\sup_{x\in\mathbb{R}} |g'(x)| < \infty$ and that $F \in C^2(\mathbb{R})$ with $\sup_{x\in\mathbb{R}} |F''(x)| < \infty$, prove that there exists a unique classical solution $u \in C^1([0,T^*) \times \mathbb{R})$ where $T^* > 0$, possibly infinity. Give a formula for T^*.

Problem 2. a.) Prove that for $k \in \mathbb{R}$, the entropy $e(u) = |u - k|$ has the entropy-flux $f(u) = [F(u) - F(k)]\mathrm{sgn}(u - k)$.

b.) Suppose that u is a piece-wise differentiable entropy solution to the conservation law $u_t + F(u)_x = 0$ which is discontinuous along the C^1-curve $C = \{(t, x(t)) \,:\, t > 0\}$. Set

$$V_l = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \,:\, x(t) < t\} \quad \text{and} \quad V_r = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \,:\, x(t) > t\} \,.$$

Use the entropy/entropy-flux pair e, f of part a.) to prove the *Lax shock inequality*

$$F'(u_l) \geq x'(t) \geq F'(u_r) \qquad \text{for all } (t,x) = (t, x(t)) \,,$$

where

$$u_l(t,x) = \lim_{V_l \ni (t_n,x_n) \to (t,x)} u(t_n, x_n) \,, \quad u_r(t,x) = \lim_{V_r \ni (t_n,x_n) \to (t,x)} u_n(t_n, x_n) \,, \qquad (t,x) \in C \,.$$

Hint: Use the definition of an entropy solution to derive the inequality

$$[F(u_l) + F(u_r) - 2F(k) - x'(t)(u_l + u_r - 2k)]\mathrm{sgn}(u_l - u_r) \geq 0 \qquad (t,x) \in C$$

where k is between u_l and u_r. Then make use of the Rankine-Hugoniot condition and consider the limits for $k \to u_l$ and $k \to u_r$, respectively.

c.) Suppose now that F is uniformly convex, that is $F''(z) \geq \theta > 0$ for some positive constant θ, for all $z \in \mathbb{R}$. What condition on the initial data u_l and u_r are needed to guarantee that the Riemann problem has a discontinuous entropy solution ?