WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #15

Problem 1. Consider the initial value problem $u_t + F(u)_x = 0$ subject to the initial condition $u(0,x) = g(x)$. Assuming that $g \in C^1(\mathbb{R})$ with $\sup_{x\in\mathbb{R}} |g'(x)| < \infty$ and that $F \in C^2(\mathbb{R})$ with $\sup_{x\in\mathbb{R}} |F''(x)| < \infty$, prove that there exists a unique classical solution $u \in C^1([0,T^*) \times \mathbb{R})$ where $T^* > 0$, possibly infinity. Give a formula for T^*.

Proof. Suppose that $u \in C^1([0,T],\mathbb{R})$ is a classical solution. Then u is constant a long the characteristic curves $(t, x(t))$, that is

$$\frac{d}{dt}u(t,x(t)) = u_t(t,x(t)) + u_x(t,x(t))x'(t) = 0$$

and $x'(t) = F'(u(t,x(t))$. This is an ordinary differential equation which can be solved in a unique fashion, provided an initial condition is given. Choose $x(0) = x_0 \in \mathbb{R}$. Since $u(t,x(t))$ is constant, we have

$$x(t) = F'(g(x_0))t + x_0 \ .$$

Note that the equation $x = F'(g(x_0)t + x_0$ can be solved for x_0 for small t in a unique fashion since the function $G(x,y) = x - y - F'(g(y))t$ satisfies

$$G_y(x,y) = -1 - F''(g(y))g'(y)t \neq 0 \ .$$

If $F''(g(y))g'(y) \geq 0$ for all $\in \mathbb{R}$, then a solution can be found for all $t > 0$. Otherwise, with

$$T^* = \frac{-1}{\inf_{y\in\mathbb{R}} F''(g(y))g'(y)} \ , \tag{1}$$

for given x there is a unique solution for every $t \in (0,T^*)$. This way one establishes the unique existence of a solution $u \in C^1([0,T^*) \times \mathbb{R})$ to the initial value problem

Finally, one can verify that it is not possible to extend u as a C^1 function to any time beyond T^*. Let $v(t) = F''(u(t,x(t))u_x(t,x(t))$ and differentiate this function. Using the fact that $u_{tx} = -F''(u)u_x^2 - F'(u)u_{xx}$ and that $F''(u)$ is constant along a characteristic curve, one obtains

$$\begin{aligned}\frac{d}{dt}[F''(u(t,x(t))u_x(t,x(t))] &= F''(u(t,x(t))u_{tx}(t,x(t)) \\ &+ u_{xx}(t,x(t))F'(u(t,x(t))F''(u(t,x(t)) = -[F''(u(t,x(t))]^2u_x^2(t,x(t)) \ ,\end{aligned}$$

which tells us that $v'(t) = -[v(t)]^2$. If T^* is finite there exists a value $x_0 \in \mathbb{R}$ such that $v(0) = F''(g(x_0))g'(x_0) < 0$. The solution to the initial value problem for the function v can be solved using separation of variables. One obtains

$$v(t) = \frac{1}{t + \dfrac{1}{v(0)}}$$

which shows that in view of (1) that there exist a point $x_0 \in \mathbb{R}$ such that

$$\lim_{t \nearrow T^*} v(t) = \infty \ .$$

Hence u cannot be of class C^1 for any $t \geq T^*$. However, it may be possible to extend you as an integral solution. The formation of a discontinuity at time T^* is one of the motivation to introduce the concept of weak solutions. □

Problem 2. a.) Prove that for $k \in \mathbb{R}$, the entropy $e(u) = |u - k|$ has the entropy-flux $f(u) = [F(u) - F(k)]\text{sgn}(u - k)$.

Proof. Note that $e, f \in W^1_\infty(a, b)$ for any interval $(a, b) \subset \mathbb{R}$. This is to say that the distributional derivatives are in $L_\infty(a, b)$. Indeed,

$$e'(u) = \text{sgn}(u-k) \text{ and } f'(u) = F'(u)\text{sgn}(u-k)+[F(u)-F(k)]\delta(u-k) = F'(u)\text{sgn}(u-k) \ ,$$

which establishes $f'(u) = F'(u)e'(u)$. □

b.) Suppose that u is a piece-wise differentiable entropy solution to the conservation law $u_t + F(u)_x = 0$ which is discontinuous along the C^1-curve $C = \{(t, x(t)) \ : \ t > 0\}$. Set

$$V_l = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} \ : \ x(t) < t\} \quad \text{and} \quad V_r = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} \ : \ x(t) > t\} \ .$$

Use the entropy/entropy-flux pair e, f of part a.) to prove the *Lax shock inequality*

$$F'(u_l) \geq x'(t) \geq F'(u_r) \qquad \text{for all } (t, x) = (t, x(t)) \ ,$$

where

$$u_l(t, x) = \lim_{V_l \ni (t_n, x_n) \to (t,x)} u(t_n, x_n) \ , \quad u_r(t, x) = \lim_{V_r \ni (t_n, x_n) \to (t,x)} u_n(t_n, x_n) \ , \qquad (t, x) \in C \ .$$

Hint: Use the definition of an entropy solution to derive the inequality

$$[F(u_l) + F(u_r) - 2F(k) - x'(t)(u_l + u_r - 2k)]\text{sgn}(u_l - u_r) \geq 0 \qquad (t, x) \in C$$

where k is between u_l and u_r. Then make use of the Rankine-Hugoniot condition and consider the limits for $k \to u_l$ and $k \to u_r$, respectively.

Proof. Since u is an entropy solution we know that

$$\int_0^\infty \int_{\mathbb{R}} \{e(u)\varphi_t + f(u)\varphi_x\} \ dxdt \geq 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, $\varphi \geq 0$ and $k \in \mathbb{R}$. Here e and f are a differentiable entropy/entropy flux pair. Splitting the integral along the curve C and using integration by parts gives

$$
\begin{aligned}
0 \le& \int_0^\infty \int_{-\infty}^{x(t)} \{e(u)\varphi_t + f(u)\varphi_x\}\ dxdt + \int_0^\infty \int_{x(t)}^{\infty} \{e(u)\varphi_t + f(u)\varphi_x\}\ dxdt \\
=& \int_0^\infty \frac{d}{dt}\int_{-\infty}^{x(t)} e(u)\varphi\, dxdt - \int_0^\infty x'(t)e(u_l(t,x(t)))\varphi(t,x(t))\, dxdt \\
&- \int_0^\infty \int_{-\infty}^{x(t)} \{e(u)_t + f(u)_x\}\,\varphi\, dxdt + \int_0^\infty f(u_l(t,x(t)))\varphi(t,x(t))\, dt \\
&+ \int_0^\infty \frac{d}{dt}\int_{x(t)}^{\infty} e(u)\varphi\, dxdt - \int_0^\infty x'(t)e(u_r(t,x(t)))\varphi(t,x(t))\, dxdt \\
&- \int_0^\infty \int_{x(t)}^{\infty} \{e(u)_t + f(u)_x\}\,\varphi\, dxdt - \int_0^\infty f(u_r(t,x(t)))\varphi(t,x(t))\, dt\ .
\end{aligned}
$$

The first integral on the right hand side vanishes since φ has compact support in $\mathbb{R}_+ \times \mathbb{R}$. Hence

$$
\begin{aligned}
&\int_0^\infty \{f(u_l(t,x(t))) - f(u_r(t,x(t))) - x'(t)[e(u_l(t,x(t))) - e(u_r(t,x(t)))]\}\,\varphi(t,x(t))\, dxdt \\
&\qquad - \int_0^\infty \int_{-\infty}^{x(t)} \{e(u)_t + f(u)_x\}\,\varphi\, dxdt - \int_0^\infty \int_{x(t)}^{\infty} \{e(u)_t + f(u)_x\}\,\varphi\, dxdt \ge 0\ .
\end{aligned}
$$

Since u is an entropy solution we know that $e(u)_t + f(u)_x = e'(u)[u_t + F'(u)u_x] = 0$ for all $(t,x) \notin C$ and consequently, since $\varphi \ge 0$

$$
[f(u)] - [e(u)]x'(t) \ge 0 \qquad \text{for all } (t,x) \in C\ .
$$

Here $[f(u)] = f(u_l) - f(u_r)$ and $[e(u)] = e(u_l) - e(u_r)$. At this point one uses now the entropy/entropy flux pair from part a.). This pair is not differentiable but one can approximate it by differentiable functions and justify this way the following computation. With $k \in \mathbb{R}$ between u_l and u_r one has

$$
[F(u_l) - F(k)]\mathrm{sgn}(u_l - k) - [F(u_r) - F(k)]\mathrm{sgn}(u_r - k) - x'(t)[|u_l - k| - |u_r - k|] \ge 0
$$

Since $u_l - k$ and $u_r - k$ have opposite sign, this simplifies to

(2) $$[F(u_l) + F(u_r) - 2F(k) - x'(t)(u_l + u_r - 2k)]\mathrm{sgn}(u_l - u_r) \ge 0 \qquad (t,x) \in C$$

The Rankine-Hugoniot condition is now used to eliminate $F(u_r)$ out of this inequality. We have

$$
F(u_r) = F(u_l) - (u_l - u_r)x'(t)
$$

and thus

(3) $$[F(u_l) - F(k) - x'(t)(u_l - k)]\mathrm{sgn}(u_l - u_r) \ge 0$$

which gives since k is between u_r and u_l

(4) $$\frac{F(u_l) - F(k)}{u_l - k} \ge x'(t)\ .$$

To make this argument more transparent one can distinguish cases.
Case 1. $u_r < u_l$, $k \in (u_r, u_l)$. Then (3) simplifies to $F(u_l) - F(k) \ge x'(t)(u_l - k)$.
Case 2. $u_l < u_r$, $k \in (u_l, u_r)$. Then formula (3) becomes $F(u_l) - F(k) \le x'(t)(u_l - k)$.
In both cases one obtains after division by $u_l - k$ formula (4). Taking the one-sided limit

$k \to u_l$ results in $F'(u_l) \geq x'(t)$. If one uses the Rankine-Hugoniot condition to replace $F(u_l)$ in formula (2) one gets

$$F(u_r) - F(k) - x'(t)(u_r - k)\text{sgn}(u_l - u_r) \geq 0 ,$$

which after a similar argument turns into $x'(t) \geq F'(u_r)$. □

c.) Suppose now that F is uniformly convex, that is $F''(z) \geq \theta > 0$ for some positive constant θ, for all $z \in \mathbb{R}$. What condition on the initial data u_l and u_r are needed to guarantee that the Riemann problem has a discontinuous entropy solution ?

Solution. In order to obtain an entropy solution with a discontinuity one needs that $F'(u_l) \geq F'(u_r)$. The convexity of F implies that F' is strictly increasing. Thus $u_l > u_r$.