WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #2 due 10/30/2015

Problem 1.

Let $p(x_1, ..., x_N)$ be the $N \times N$ Vandermonde determinant

$$p(x_1, ..., x_N) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \cdots & x_N^{N-1} \end{bmatrix} .$$

a.) Show that $p(x_1, ..., x_N)$ is a polynomial of degree $N-1$ in x_N with roots equal $x_1, x_2, ..., x_{N-1}$.

b.) Show that the $N \times N$ linear system

$$\sum_{j=1}^{N} a_j(-j)^k = 1, \qquad k = 0, 1, ..., N-1$$

has a unique solution.

Problem 2. Let $s \in (0,1)$. Show that an equivalent norm in $H^s(\mathbb{R}^d)$ is given by

$$\|u\|_s^2 = \int_{\mathbb{R}^d} |u(x)|^2 dx + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{2s+d}} dxdy .$$

Recall that the norm in the Sobolev space $H^s(\mathbb{R}^d)$ is defined using the Fourier transform

$$\|u\|_{H^s(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$.

Problem 3. *Sobolev space on the torus* $\mathbb{T}^d$. The torus $\mathbb{T}^d$ is the Cartesian product of d copies of the unit circle S^1. A function defined on the torus is a 2π periodic function with respect to each independent variable. If f is integrable on $\mathbb{T}^d$, then the Fourier coefficients of f are given by

$$\mathcal{F}[f](k) = \hat{f}(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(\theta) e^{-ik\cdot\theta} d\theta , \qquad k \in \mathbb{Z}^d.$$

The set of Fourier coefficients plays for the same role as the Fourier transform in $\mathbb{R}^d$. The expansion of a periodic function into a Fourier series is the analogue of the Fourier inversion formula, that is

$$f(\theta) = \frac{1}{(2\pi)^{d/2}} \sum_{k \in \mathbb{Z}^d} \hat{f}(k) e^{ik\cdot\theta}, \qquad \theta \in \mathbb{T}^d .$$

One can show that the map $\mathcal{F}$ is a isomorphic between the function spaces $L_2(\mathbb{T}^d)$ and $l_2(\mathbb{Z}^d)$. Then, for $s \in \mathbb{R}$, $s \geq 0$ we define

$$H^s(\mathbb{T}^d) = \left\{ u \in L_2(\mathbb{T}^d) \ : \ \sum_{k \in \mathbb{Z}^d} |\hat{u}(k)|^2 \langle k \rangle^{2s} < \infty \right\}$$

where $\langle k \rangle = \sqrt{1 + |k|^2}$.

a.) Show that $H^s(\mathbb{T}^d)' \approx H^{-s}(\mathbb{T}^d)$ where

$$H^{-s}(\mathbb{T}^d) = \left\{ u \in \mathcal{D}'(\mathbb{T}^d) \ : \ \sum_{k \in \mathbb{Z}^d} |\hat{u}(k)|^2 \langle k \rangle^{-2s} < \infty \right\} .$$

Here $H^s(\mathbb{T}^d)'$ denotes the dual space of $H^s(\mathbb{T}^d)$ with respect to the L_2 inner product.

b.) Define the operator Λ^σ on $\mathcal{D}'(\mathbb{T}^d)$ by

$$[\Lambda^\sigma u](\theta) = \sum_{k \in \mathbb{Z}^d} \hat{u}(k) \langle k \rangle^\sigma e^{ik \cdot \theta} , \qquad \sigma \in \mathbb{R} .$$

Then $H^s(\mathbb{T}^d) = \Lambda^{-s} L_2(\mathbb{T}^d)$ for all $s \in \mathbb{R}$. Show that, for any $s \in \mathbb{R}$, the natural injection operator

$$j : H^{s+\sigma}(\mathbb{T}^d) \to H^s(\mathbb{T}^d)$$

is compact for all $\sigma > 0$. Hint: Note that the mapping $\Lambda^\sigma \ : \ H^{s+\sigma}(\mathbb{T}^d) \to H^s(\mathbb{T}^d)$ is continuous and that $j = \Lambda^{-\sigma} \circ \Lambda^\sigma$. Hence, to prove this statement it will suffice to show that $\Lambda^{-\sigma} \ : \ H^s(\mathbb{T}^d) \to H^s(\mathbb{T}^d)$ is a compact operator whenever $\sigma > 0$.