WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #3 Key

Problem 1. Consider the 4×4 first-order differential operator in $\Omega \subset \mathbb{R}^3$

$$P(\partial)u = \begin{bmatrix} \nabla \times v + \nabla w \\ \nabla \cdot v \end{bmatrix} .$$

Here $u = \begin{bmatrix} v \\ w \end{bmatrix}$ is a vector-valued function with four components, v is a vector-valued function with three components, and the function w is scalar-valued.

a.) Write out the (principal) symbol $P(\xi)$.

Solution.

$$P(\xi) = \begin{bmatrix} 0 & -i\xi_3 & i\xi_2 & i\xi_1 \\ i\xi_3 & 0 & -i\xi_1 & i\xi_2 \\ -i\xi_2 & i\xi_1 & 0 & i\xi_3 \\ i\xi_1 & i\xi_2 & i\xi_3 & 0 \end{bmatrix}$$

b.) Prove that P is elliptic.

Solution. Compute

$$\det P(\xi) = -|\xi|^4$$

c.) Let $\alpha \in C^\infty(\overline{\Omega}, \mathbb{C}^{3\times 3})$ be a Hermitian matrix, i.e. $\alpha^H = \alpha$. Give sufficient and necessary conditions such that the operator

$$P_\alpha(x, \partial)u = \begin{bmatrix} \nabla \times v + \alpha(x)\nabla w \\ \nabla \cdot (\alpha(x)v) \end{bmatrix}$$

is an elliptic operator in the sense of Definition 2.3.1.

Solution. Similar to part b.), one computes

$$\det P_\alpha(x, \xi)u = -[\xi^T \alpha(x)\xi]^2 .$$

Hence, P is elliptic if and only if α has non-zero eigenvalues.

Problem 2. Suppose that $P(D)$ is a constant coeffcient elliptic operator. As discussed in the proof of Theorem 2.3.2, there exists a $K \in \mathbb{R}$ such that $P(\xi)^{-1}$ exists for all $|\xi| \geq K$. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ satisfying $\varphi(\xi) = 1$ for all $|\xi| \leq K$.

a.) Prove that the operator with symbol $E(\xi) = (1 - \varphi(\xi))P(\xi)^{-1}$ is a continuous operator from $H^\sigma(\mathbb{R}^d)$ into $H^{m+\sigma}(\mathbb{R}^d)$ for all $\sigma \in \mathbb{R}$. Here

$$E(D)u(x) = \frac{1}{(2\pi)^{d/2}} \int e^{ix\cdot\xi} E(\xi)\hat{u}(\xi)d\xi .$$

where $\hat{u}$ is the Fourier transform of u.

Solution. The ellipticity of P implies that there exists a constant such that $|P(\xi)| \geq C|\xi|^m$. Here $|\cdot|$ denotes a matrix norm (when applied to matrices), e.g. the spectral norm. (In the

L_2 setting the spectral norm is usually preferred since it is compatible with the Euclidean scalar product. However, all matrix norms are equivalent.)

To understand the inequality $|P(\xi)| \geq C|\xi|^m$ for large $|\xi|$ one looks at first at the principal symbol which is homogeneous of degree m in ξ and elliptic, hence $P_m(\xi) \geq c|\xi|^m$ for all $\xi \in \mathbb{R}^d$ where c is again a positive constant. The lower order terms can be estimated by some constant times $|\xi|^{m-1}$.

Using the definition of E gives then $|E(\xi)| \leq C|\xi|^{-m} \leq C\langle\xi\rangle^{-m}$. Then

$$\|E(D)u\|^2_{H^{m+\sigma}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |E(\xi)\hat{u}(\xi)|^2 \langle\xi\rangle^{2m+2\sigma} d\xi \leq C \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \langle\xi\rangle^{2s} d\xi = C\|u\|^2_{H^\sigma(\mathbb{R}^d)}$$

b.) Show that

$$E(D)P(D) = I + \rho(D)$$

where $\rho \in C_0^\infty(\mathbb{R}^d)$ and I is the identity mapping.
Solution. We have

$$\begin{aligned} E(D)P(D)u &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} E(\xi)\widehat{P(D)u}(\xi) d\xi = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} E(\xi)P(\xi)\hat{u}(\xi) d\xi \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} [1-\varphi(\xi)]\hat{u}(\xi) d\xi = u - \varphi(D)u \end{aligned}$$

which proves the statement with $\rho = -\varphi$.

c.) Let $\varphi \in C_0^\infty(\mathbb{R}^d)$. Prove that $\varphi(D)$ is a continuous operator from $H^s(\mathbb{R}^d)$ into $H^t(\mathbb{R}^d)$ for all real numbers $s, t \in \mathbb{R}$.
Solution. Let $K = \operatorname{supp}\varphi$ which is a compact set in $\mathbb{R}^d$. Then

$$\|\varphi(D)u\|^2_{H^t(\mathbb{R}^d)} = \int_K \langle\xi\rangle^{2t} |\varphi(\xi)\hat{u}(\xi)|^2 d\xi \leq C(s,t) \int_K \langle\xi\rangle^{2s} |\hat{u}(\xi)|^2 d\xi = \|u\|^2_{H^s(\mathbb{R}^d)}$$

since $|\varphi(\xi)|\langle\xi\rangle^{t-s}$ is a continuous function on K and hence bounded.

Problem 3. This problem has connection with Problem 3 of Homework #2. Let $\mathbb{T}^d$ denote the d-dimensional torus. If f is integrable on $\mathbb{T}^d$, then the Fourier coefficients of f are given by

$$\mathcal{F}[f](k) = \hat{f}(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(\theta) e^{-ik\cdot\theta} d\theta\,, \qquad k \in \mathbb{Z}^d.$$

For $s \in \mathbb{R}$, $s \geq 0$ we define

$$H^s(\mathbb{T}^d) = \left\{ u \in L_2(\mathbb{T}^d) \ : \ \sum_{k\in\mathbb{Z}^d} |\hat{u}(k)|^2 \langle k\rangle^{2s} < \infty \right\}$$

where $\langle k\rangle = \sqrt{1+|k|^2}$.

a.) Show that for $m \in \mathbb{N}$

$$H^m(\mathbb{T}^d) = \left\{ u \in L_2(\mathbb{T}^d) \ : \ D^\alpha u \in L_2(\mathbb{T}^d) \text{ for } |\alpha| \leq m \right\}\ .$$

Solution. Note that $[\mathcal{F}(D^\alpha u](k) = k^\alpha \hat{u}(k)$. Hence $D^\alpha u \in L_2(\mathbb{T}^d)$ for all $|\alpha| \leq m$ if and only if $k^\alpha \hat{u}(k) \in l_2$ for $|\alpha| \leq m$. One can find positive constants c_1 and c_2 such that

$$c_1\langle k\rangle^{2m} \leq \sum_{|\alpha|\leq m} k^{2\alpha} \leq c_2\langle k\rangle^{2m}$$

which proves that $u \in H^m(\mathbb{T}^d)$ is equivalent to $D^\alpha u \in L_2(\mathbb{T}^d)$ for all $|\alpha| \leq m$.

b.) Use Problem 3b from Homework #2 to prove Theorem 2.1.2, also known as Rellich's Theorem.
Solution. Note that since the region Ω is bounded it can be put inside of a (scaled) torus $\mathbb{T}^d$. By scaled torus we mean a d dimensional cube of side length large enough so that Ω can be placed inside. The natural injection j from $H^{s+\sigma}(\Omega)$ into $H^s(\Omega)$ can be written as follows.

$$j = E \circ i \circ R$$

where E is an extension operator for $H^{s+\sigma}(\Omega)$ into $H^{s+\sigma}(\mathbb{T}^d)$ and R is the restriction operator from $H^s(\mathbb{T}^d)$ to $H^s(\Omega)$, and i is the natural injection from $H^{s+\sigma}(\mathbb{T}^d)$ into $H^s(\mathbb{T}^d)$ which was proved to be compact in Problem 3 in the previous homework. Note that the operators E and R are continuous. Hence, the operator j written as a composition of continuous and compact operators is compact.

To be honest, the continuity of E is not entirely trivial. With the equivalent definition of $H^s(\Omega)$ as restrictions of functions in $H^s(\mathbb{R}^d)$ it remains to be shown that the multiplication of u with a cutoff function χ is a continuous operation. This operation is needed to obtain a function which can be extended as a periodic function on $\mathbb{R}^d$.