WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #3 due 11/06/2015

Problem 1. Consider the 4×4 first-order differential operator in $\Omega \subset \mathbb{R}^3$

$$P(\partial)u = \begin{bmatrix} \nabla \times v + \nabla w \\ \nabla \cdot v \end{bmatrix} .$$

Here $u = \begin{bmatrix} v \\ w \end{bmatrix}$ is a vector-valued function with four components, v is a vector-valued function with three components, and the function w is scalar-valued.

a.) Write out the (principal) symbol $P(\xi)$.

b.) Prove that P is elliptic.

c.) Let $\alpha \in C^\infty(\overline{\Omega}, \mathbb{C}^{3\times 3})$ be a Hermitian matrix, i.e. $\alpha^H = \alpha$. Give sufficient and necessary conditions such that the operator

$$P_\alpha(x, \partial)u = \begin{bmatrix} \nabla \times v + \alpha(x)\nabla w \\ \nabla \cdot (\alpha(x)v) \end{bmatrix}$$

an elliptic operator in the sense of Definition 2.3.1.

Problem 2. Suppose that $P(D)$ is a constant coeffcient elliptic operator. As discussed in the proof of Theorem 2.3.2, there exists a $K \in \mathbb{R}$ such that $P(\xi)^{-1}$ exists for all $|\xi| \geq K$. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ satisfying $\varphi(\xi) = 1$ for all $|\xi| \leq K$.

a.) Prove that the operator with symbol $E(\xi) = (1-\varphi(\xi))P(\xi)^{-1}$ is a continuous operator from $H^\sigma(\mathbb{R}^d)$ into $H^{m+\sigma}(\mathbb{R}^d)$ for all $\sigma \in \mathbb{R}$. Here

$$E(D)u(x) = \frac{1}{(2\pi)^{d/2}} \int e^{ix\cdot\xi} E(\xi)\hat{u}(\xi) d\xi .$$

where $\hat{u}$ is the Fourier transform of u.

b.) Show that

$$E(D)P(D) = I + \rho(D)$$

where $\rho \in C_0^\infty(\mathbb{R}^d)$ and I is the identity mapping.

c.) Let $\varphi \in C_0^\infty(\mathbb{R}^d)$. Prove that $\varphi(D)$ is a continuous operator from $H^s(\mathbb{R}^d)$ into $H^t(\mathbb{R}^d)$ for all real numbers $s, t \in \mathbb{R}$.

Problem 3. This problem has connection with Problem 3 of Homework #2. Let $\mathbb{T}^d$ denote the d-dimensional torus. If f is integrable on $\mathbb{T}^d$, then the Fourier coefficients of f are given by

$$\mathcal{F}[f](k) = \hat{f}(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(\theta) e^{-ik\cdot\theta} d\theta , \qquad k \in \mathbb{Z}^d.$$

For $s \in \mathbb{R}$, $s \geq 0$ we define

$$H^s(\mathbb{T}^d) = \left\{ u \in L_2(\mathbb{T}^d) \ : \ \sum_{k \in \mathbb{Z}^d} |\hat{u}(k)|^2 \langle k \rangle^{2s} < \infty \right\}$$

where $\langle k \rangle = \sqrt{1 + |k|^2}$.

a.) Show that for $k \in \mathbb{N}$

$$H^k(\mathbb{T}^d) = \left\{ u \in L_2(\mathbb{T}^d) \ : \ D^\alpha u \in L_2(\mathbb{T}^d) \text{ for } |\alpha| \leq k \right\} .$$

b.) Use Problem 3b from Homework #2 to prove Theorem 2.1.2, also known as Rellich's Theorem.