WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #4 Key

Problem 1. *An alternative proof of Lemma 2.5.2.* Prove the following statement. The number $i\eta$ is an eigenvalue of $K_1(\xi)$ if and only if $\det P_m(\xi,\eta) = 0$. Here $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$. Hint: Use the fact that there must exist an eigenvector $w \in \mathbb{C}^{mN}$ and exploit the equation $K_1(\xi)w = i\eta w$. Recall that

$$K_1(\xi) = \begin{bmatrix} 0 & |\xi| I_N & 0 & \dots & \dots & 0 \\ 0 & 0 & |\xi| I_N & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & & & |\xi| I_N \\ \tilde{E}_1 & \tilde{E}_2 & \tilde{E}_3 & \dots & \dots & \tilde{E}_m \end{bmatrix} \qquad \text{with} \qquad \tilde{E}_j = -\tilde{A}_{j-1}|\xi|^{j-m} \ ,$$

where $\tilde{A}_j(\xi)$ denotes the principal part of $A_j(\xi)$, and that

$$P(D) = \frac{\partial^m}{\partial y^m} + \sum_{j=0}^{m-1} A_j(D_x) \frac{\partial^j}{\partial y^j} \ ,$$

where A_j is a tangential operator of order $m-j$ and $P_m(\xi,\eta) = (i\eta)^m + \sum_{j=0}^{m-1} \tilde{A}_j(\xi)(i\eta)^j$.

Proof. Suppose that $\xi \in \mathbb{R}^{d-1}$ is a non-zero vector. Note that in the case $m = 1$ there is nothing to prove. In this case the principal symbol is

$$P_1(\xi,\eta) = i\eta - \tilde{A}_0(\xi) = i\eta - \tilde{E}_1 = i\eta - K_1(\xi) \ ,$$

which shows that $\det P(\xi,\eta) = 0$ is equivalent to $i\eta$ being an eigenvalue of $K_1(\xi)$. So in what follows we assume $m > 1$ which is the interesting case.

Suppose $i\eta$ is a eigenvalue of the matrix $K_1(\xi)$. Then there exists a non-zero vector $w \in \mathbb{C}^{Nm}$ such that $i\eta w = K_1(\xi) w$. Matching the block structure of $K_1(\xi)$ we write

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} \ , \qquad w_j \in \mathbb{C}^N \text{ for } j = 1, 2, ..., m$$

and from the formula for $K_1(\xi)$ we conclude that

(1)

$$i\eta w_j = |\xi| w_{j+1} \text{ for } j = 1,2,...,m-1, \qquad \text{and} \qquad i\eta w_m = \sum_{k=1}^m \tilde{E}_k w_k = -\sum_{k=1}^m \tilde{A}_{k-1} |\xi|^{k-m} w_k \ .$$

Note that $\eta = 0$ implies $w_j = 0$ for $j = 1, 2, ..., m-1$ is equivalent to $\tilde{A}_0 w_m = 0$ which in turn is equivalent to $\det P_m(\xi, 0) = 0$. This proves the statement in the case $i\eta = 0$. Now assume that $i\eta \neq 0$ which implies that $w_j \neq 0$ for all $j = 1, 2, ..., m$. Define $u = w_1/|\xi|^{m-1}$ which is a non-zero vector in $\mathbb{C}^N$. This gives

$$w_j = (i\eta)^{j-1}|\xi|^{m-j}u\ , \qquad \text{for } j = 1, ..., m \tag{2}$$

Then, the second equation in (1) can be rewritten as

$$(i\eta)^m u + \sum_{j=0}^{m-1} \tilde{A}_j(\xi)(i\eta)^j u = 0 \tag{3}$$

which show that u is a solution to the equation $P_m(\xi, \eta)u = 0$ and thus $\det P_m(\xi, \eta) = 0$.

Conversely, if $\det P_m(\xi, \eta) = 0$ there exists a non-zero vector u such that equation (3) is true. Then one defines a vector $w \in \mathbb{C}^N$ with the "blocks" defined by formula (2). This vector satisfies formula (1) and hence, the vector $w \neq 0$ must be an eigenvector of $K_1(\xi)$ with eigenvalue $i\eta$. □

Problem 2. *The Dunford-Taylor integral.* The goal is to prove formula (2.5.2) for the spectral projection of the matrix $K_1(\xi)$. Let A be a square matrix and suppose that its spectrum (eigenvalues) $\sigma(A) \subset \omega$ where ω is open, bounded in $\mathbb{C}$ with a smooth boundary γ. Let f be a holomorphic function on a neighborhood of ω and introduce the complex line integral

$$f(A) = \frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta I - A)^{-1} d\zeta\ .$$

Here I denotes the identity matrix which is of the same type as A.

a.) Show that, if $f(z) = 1$, then $f(A) = I$ and that, if $f(z) = z$, then $f(A) = A$.

Proof. Note that in both cases the curve can be deformed to a large circle with radius R such that

$$(\zeta I - A)^{-1} = \frac{1}{\zeta}(I - A/\zeta)^{-1} = \frac{1}{\zeta}\sum_{k=0}^{\infty}\frac{A^k}{\zeta^k} \qquad \text{for } |\zeta| = R\ ,$$

where the series is absolutely (i.e. in norm) convergent. If $f(z) = 1$, then

$$\frac{1}{2\pi i}\int_{|\zeta|=R}(\zeta I - A)^{-1}\, d\zeta = \frac{1}{2\pi i}\int_{|\zeta|=R}\frac{1}{\zeta}\sum_{k=0}^{\infty}\frac{A^k}{\zeta^k} d\zeta\ .$$

Since the series is absolutely convergent, the summation and integration can be interchanged. All integrals of the form

$$\int_{|\zeta|=R}\zeta^k d\zeta$$

vanish, except for $k = -1$. In that case the integral is equal to one.

The case $f(z) = z$ is similar. One obtains the integral

$$\frac{1}{2\pi i}\int_{|\zeta|=R}\zeta(\zeta I - A)^{-1}\, d\zeta = \frac{1}{2\pi i}\int_{|\zeta|=R}\sum_{k=0}^{\infty}\frac{A^k}{\zeta^k} d\zeta = A\ ,$$

where the last equal sign follows with the same reasoning as in the case $f(z) = 1$. □

b.) Suppose now that $\omega = \omega_+ \cup \omega_-$ where

$$\omega_+ \subset \{z \in \mathbb{C} \ : \ \Re z > 0\}, \qquad \text{and} \qquad \omega_- \subset \{z \in \mathbb{C} \ : \ \Re z < 0\}$$

such that $\overline{\omega}_+ \cap \overline{\omega}_- = \emptyset$ and set

$$E_+ = \frac{1}{2\pi i} \int_{\gamma_+} (\zeta I - A)^{-1} d\zeta, \qquad \text{and} \qquad E_- = \frac{1}{2\pi i} \int_{\gamma_-} (\zeta I - A)^{-1} d\zeta \ ,$$

where $\gamma_\pm = \partial \omega_\pm$. Using the fact that $(fg)(A) = f(A)g(A)$ for holomorphic functions f and g defined on a neighborhood of ω, prove that

$$E_+ + E_- = I, \quad E_+^2 = E_+, \quad E_-^2 = E_-, \quad E_+E_- = E_-E_+ = 0 \ .$$

This shows that E_+ and E_- are complementary projections.
The formula $(fg)(A) = f(A)g(A)$ may be familiar from Linear Algebra or Functional Analysis. If not, do not worry, just use it

Proof. Choose f to be holomorphic on ω such that $f(z) = 1$ on ω_+ and $f(z) = 0$ on ω_- and set $g(z) = 1 - f(z)$ which is also holomorphic on ω. Then

$$E_+ = \frac{1}{2\pi i} \int_{\partial\omega} f(z)(\zeta I - A)^{-1} d\zeta \quad \text{and} \quad E_- = \frac{1}{2\pi i} \int_{\partial\omega} g(z)(\zeta I - A)^{-1} d\zeta$$

Since $f(z)g(z) = 0$ in ω we have $E_+E_- = E_-E_+ = 0$ and since $f(z)^2 = f(z)$ in ω we have $E_+^2 = E_+$. Likewise one obtains $E_-^2 = E_-$. Finally, $f(z) + g(z) = 1$ in ω which implies that $E_+ + E_- = I$. ☐

Problem 3. From the proof of Theorem 2.3.3 recall the operator

$$R(x, D) = \chi_\lambda(x)[P(x, D) - P(\lambda, D)] = \chi_\lambda \sum_{|\alpha| \le m} [a_\alpha(x) - a_\alpha(\lambda)] D^\alpha$$

where χ_λ is a partition of unity subordinate to the lattice $\mathscr{O}_\varepsilon = \varepsilon \mathbb{Z}^d = \{\varepsilon j \ : \ j \in \mathbb{Z}^d\}$, that is $0 \le \chi_\lambda \le 1$, $\chi_\lambda \in C_0^\infty(\mathbb{R}^d)$, $\sum_{\lambda \in \mathscr{O}_\varepsilon} \chi_\lambda(x) \equiv 1$, and $\operatorname{supp} \chi_\lambda \subset \{x \in \mathbb{R}^d \ : \ |x - \lambda| \le \varepsilon\}$. Note that the operator R depends also on $\varepsilon > 0$.

a.) Prove the estimate (which is part of the Proof of Theorem 2.3.3)

$$\left\| \sum_{\lambda \in \mathscr{O}_\varepsilon} R_\lambda(x, D) u \right\|_{H^k(\mathbb{R}^d)} \le C(k) \varepsilon \|u\|_{H^{m+k}(\mathbb{R}^d)} + C(\varepsilon, k) \|u\|_{H^{m+k-1}(\mathbb{R}^d)}$$

for all $u \in H^{m+k}(\mathbb{R}^d)$ with $\operatorname{supp} u \subset V \subset\subset \Omega$. Here k is a non-negative integer and it is important that the first constant in the estimate does not depend on ε whereas the second constant will depend on ε.

Proof. Since the coefficients are in $C^\infty(\overline{\Omega})$, we know that the a_α are Lipschitz continuous, that is there exists a constant L such that

$$|a_\alpha(x) - a_\alpha(\lambda)| \le L\varepsilon$$

for all $x \in \operatorname{supp}\chi_\lambda \cap \overline{\Omega}$, $|\alpha| \le m$, and $\lambda \in \mathscr{O} \cap \Omega$ where $\varepsilon > 0$. Compute now by mean of the product rule and the triangle inequality

$$(4) \quad \left\| \sum_{\lambda\in\mathscr{O}_\varepsilon} R_\lambda(x,D)u \right\|^2_{H^k(\mathbb{R}^d)} = \sum_{|\beta|\le k} \left\| D^\beta \left[\sum_{\lambda\in\mathscr{O}_\varepsilon} \chi_\lambda \sum_{|\alpha|\le m} [a_\alpha(x) - a_\alpha(\lambda)] D^\alpha u \right] \right\|^2_{L_2(\mathbb{R}^d)}$$

$$\le C \sum_{|\beta|\le k} \sum_{\lambda\in\mathscr{O}_\varepsilon} \sum_{|\alpha|\le m} \int_{\mathbb{R}^d} \chi_\lambda^2 |a_\alpha(x) - a_\alpha(\lambda)|^2 |D^{\alpha+\beta}u|^2 dx + C(\varepsilon,k)\|u\|^2_{H^{k+m-1}(\mathbb{R}^d)}$$

Now one makes use of the Lipschitz continuity of the coefficients and the fact that

$$\sum_{\lambda\in\mathscr{O}_\varepsilon} \chi_\lambda^2 < 1 \sum_{\lambda\in\mathscr{O}_\varepsilon} \chi_\lambda = 1$$

which allows to estimate the first term on the right-hand side in (4) by $C\varepsilon^2\|u\|^2_{H^{k+m}(\mathbb{R}^d)}$. Note that the proof implies that the estimate

$$\sum_{\lambda\in\mathscr{O}_\varepsilon} \|R_\lambda(x,D)u\|_{H^k(\mathbb{R}^d)} \le C(k)\varepsilon\|u\|_{H^{m+k}(\mathbb{R}^d)} + C(\varepsilon,k)\|u\|_{H^{m+k-1}(\mathbb{R}^d)}$$

is also true. This will be of significance in the proof of part b. □

b.) Recall the operators $E_\lambda(D)$ introduced in the proof of Theorem 2.3.3. (and discussed in the Homework #3) and prove the estimate

$$(5) \qquad \left\| \sum_{\lambda\in\mathscr{O}_\varepsilon} E_\lambda(D) R_\lambda(x,D)u \right\|_{H^m(\mathbb{R}^d)} \le C\varepsilon\|u\|_{H^m(\mathbb{R}^d)} + C(\varepsilon)\|u\|_{H^{m-1}(\mathbb{R}^d)} .$$

for all $u \in H^{m+k}(\mathbb{R}^d)$ with $\operatorname{supp} u \subset V \subset\subset \Omega$. Observe that the constant in front of the second term on the right-hand side in (4) vanishes for $k = 0$.

Proof. From Homework #3 we know that

$$\|E_\lambda(D)v\|_{H^m(\mathbb{R}^d} \le C\|v\|_{L_2(\mathbb{R}^d)} .$$

with a constant independent of ε, for all $\lambda \in \mathscr{O}_\varepsilon \cap \Omega$. Hence, by the triangle inequality

$$\left\| \sum_{\lambda\in\mathscr{O}_\varepsilon} E_\lambda(D) R_\lambda(x,D)u \right\|_{H^m(\mathbb{R}^d)} \le \sum_{\lambda\in\mathscr{O}_\varepsilon} \|E_\lambda(D) R_\lambda(x,D)u\|_{H^m(\mathbb{R}^d)} \le C \sum_{\lambda\in\mathscr{O}_\varepsilon} \|R_\lambda(x,D)u\|_{L_2(\mathbb{R}^d)}$$

By the proof of part a.) we know that the last term can be estimate by $C\varepsilon\|u\|_{H^m(\mathbb{R}^d)}$ which shows that the desired inequality is true even without the second term on the right-hand side. □