WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #4 due 11/13/2015

Problem 1. *An alternative proof of Lemma 2.5.2.* Prove the following statement. The number $i\eta$ is an eigenvalue of $K_1(\xi)$ if and only if $\det P_m(\xi,\eta) = 0$. Here $\xi \in \mathbb{R}^{d-1} \setminus \{0\}$. Hint: Use the fact that there must exist an eigenvector $w \in \mathbb{C}^{mN}$ and exploit the equation $K_1(\xi)w = i\eta w$. Recall that

$$K_1(\xi) = \begin{bmatrix} 0 & |\xi| I_N & 0 & \dots & \dots & 0 \\ 0 & 0 & |\xi| I_N & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & & & |\xi| I_N \\ \tilde{E}_1 & \tilde{E}_2 & \tilde{E}_3 & \dots & \dots & \tilde{E}_m \end{bmatrix} \quad \text{with} \quad \tilde{E}_j = -\tilde{A}_{j-1} |\xi|^{j-m} ,$$

where $\tilde{A}_j(\xi)$ denotes the principal part of $A_j(\xi)$, and that

$$P(D) = \frac{\partial^m}{\partial y^m} + \sum_{j=0}^{m-1} A_j(D_x) \frac{\partial^j}{\partial y^j} ,$$

where A_j is a tangential operator of order $m-j$ and $P_m(\xi,\eta) = (i\eta)^m + \sum_{j=0}^{m-1} \tilde{A}_j(\xi)(i\eta)^j$.

Problem 2. *The Dunford-Taylor integral.* The goal is to prove formula (2.5.2) for the spectral projection of the matrix $K_1(\xi)$. Let A be a square matrix and suppose that its spectrum (eigenvalues) $\sigma(A) \subset \omega$ where ω is open, bounded in $\mathbb{C}$ with a smooth boundary γ. Let f be a holomorphic function on a neighborhood of ω and introduce the complex line integral

$$f(A) = \frac{1}{2\pi i} \int_\gamma f(\zeta)(\zeta I - A)^{-1} d\zeta .$$

Here I denotes the identity matrix which is of the same type as A.

a.) Show that, if $f(z) = 1$, then $f(A) = I$ and that, if $f(z) = z$, then $f(A) = A$.

b.) Suppose now that $\omega = \omega_+ \cup \omega_-$ where

$$\omega_+ \subset \{z \in \mathbb{C} \,:\, \Re z > 0\}, \quad \text{and} \quad \omega_- \subset \{z \in \mathbb{C} \,:\, \Re z < 0\}$$

such that $\overline{\omega}_+ \cap \overline{\omega}_- = \emptyset$ and set

$$E_+ = \frac{1}{2\pi i} \int_{\gamma_+} (\zeta I - A)^{-1} d\zeta, \quad \text{and} \quad E_- = \frac{1}{2\pi i} \int_{\gamma_-} (\zeta I - A)^{-1} d\zeta ,$$

where $\gamma_\pm = \partial \omega_\pm$. Using the fact that $(fg)(A) = f(A)g(A)$ for holomorphic functions f and g defined on a neighborhood of ω, prove that

$$E_+ + E_- = I, \quad E_+^2 = E_+, \quad E_-^2 = E_-, \quad E_+E_- = E_-E_+ = 0 .$$

This shows that E_+ and E_- are complementary projections.
The formula $(fg)(A) = f(A)g(A)$ may be familiar from Linear Algebra or Functional Analysis. If not, do not worry, just use it.

Problem 3. From the proof of Theorem 2.3.3 recall the operator

$$R(x, D) = \chi_\lambda(x)[P(x, D) - P(\lambda, D)] = \chi_\lambda \sum_{|\alpha| \le m} [a_\alpha(x) - a_\alpha(\lambda)] D^\alpha$$

where χ_λ is a partition of unity subordinate to the lattice $\mathscr{O}_\varepsilon = \varepsilon\mathbb{Z}^d = \{\varepsilon j \,:\, j \in \mathbb{Z}^d\}$, that is $0 \le \chi_\lambda \le 1$, $\chi_\lambda \in C_0^\infty(\mathbb{R}^d)$, $\sum_{\lambda \in \mathscr{O}_\varepsilon} \chi_\lambda(x) \equiv 1$, and $\operatorname{supp} \chi_\lambda \subset \{x \in \mathbb{R}^d \,:\, |x - \lambda| \le \varepsilon\}$. Note that the operator R depends also on $\varepsilon > 0$.

a.) Prove the estimate (which is part of the Proof of Theorem 2.3.3)

$$\left\| \sum_{\lambda \in \mathscr{O}_\varepsilon} R_\lambda(x, D) u \right\|_{H^k(\mathbb{R}^d)} \le C(k)\varepsilon \|u\|_{H^{m+k}(\mathbb{R}^d)} + C(\varepsilon, k)\|u\|_{H^{m+k-1}(\mathbb{R}^d)}$$

for all $u \in H^{m+k}(\mathbb{R}^d)$ with $\operatorname{supp} u \subset V \subset\subset \Omega$. Here k is a non-negative integer and it is important that the first constant in the estimate does not depend on ε whereas the second constant will depend on ε.

b.) Recall the operators $E_\lambda(D)$ introduced in the proof of Theorem 2.3.3. (and discussed in the Homework #3) and prove the estimate

$$\left\| \sum_{\lambda \in \mathscr{O}_\varepsilon} E_\lambda(D) R_\lambda(x, D) u \right\|_{H^m(\mathbb{R}^d)} \le C\varepsilon \|u\|_{H^m(\mathbb{R}^d)} + C(\varepsilon)\|u\|_{H^{m-1}(\mathbb{R}^d)} \,.$$

for all $u \in H^{m+k}(\mathbb{R}^d)$ with $\operatorname{supp} u \subset V \subset\subset \Omega$.