WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #5 due 11/20/2015

Problem 1. Consider the Neumann problem for the Laplacian in the 3-dimensional unit ball, that is the boundary value problem

$$\Delta u = f \text{ in } B(0,1)\ , \qquad \frac{\partial u}{\partial n} = g \text{ in } S^{d-1}$$

Here $B(0,1) = \{x \in \mathbb{R}^3 \ : \ |x| < 1\}$ and $S^2 = \partial B(0,1) = \{x \in \mathbb{R}^3 \ : \ |x| = 1\}$, and n is the exterior unit norm vector of $B(0,1)$ on S^2.

a.) Show that $B(0,1)$ has a C^∞ boundary.

b.) For a given neighborhood of $\mathscr{U}(x)$ with $x \in S^2$ and a coordinate mapping $\varphi \in C^\infty(\mathscr{U})$ found in part a.), give an explicit transformation of this boundary value problem to the half space.

Problem 2. Consider the *stationary isotropic system of elasticity* with constant coefficients in the half space, that is 3×3 system of second order

$$\mu\Delta u + (\lambda + \mu)\nabla\nabla \cdot u = f \qquad \text{in } \mathbb{R}^3_+\ ,$$

where u and f are vector valued functions with 3 components each and λ and μ are real constants, called Lamé parameters.

a.) Under which conditions on μ and λ is this system elliptic ?

b.) Reduce this system to a first order system of the form $\partial v/\partial y - K(D_x)v = F$.

c.) Reduce the boundary condition $[\nabla u]^s n = g$ on $\partial\mathbb{R}^3_+ = \mathbb{R}^2$ to a boundary condition of order zero for the function v introduced in problem a.). Here $n = -e_3$ is the exterior unit normal vector (which coincides with the opposite of the last standard basis vector in $\mathbb{R}^3$) and $[\nabla u]^s$ is the 'symmetric gradient' of u, that is $[\nabla u]^s = [\nabla u + \nabla u^T]/2$. (It may be better to call this expression the symmetric Jacobian since ∇u is the Jacobian matrix of u.)

Problem 3. *The Sobolev space $H_{(k,s)}(\mathbb{R}^d)$.* For a distribution $u \in \mathcal{S}'(\mathbb{R}^d)$, k and $s \in \mathbb{R}$, one defines the norm

$$\|u\|^2_{(k,s)} = \int_{-\infty}^{\infty}\int_{\mathbb{R}^{d-1}} |\hat{u}(\xi,\eta)|^2 \sqrt{1+|\xi|^2+|\eta|^2}^{2k} \langle\xi\rangle^{2s} d\xi d\eta$$

where $\xi = (\xi_1, ..., \xi_{d-1})$, $\langle \xi' \rangle = \sqrt{1+|\xi'|^2}$, $\eta \in \mathbb{R}$, and $\hat{u}$ is the Fourier transform of u with respect to all d variables. Then one introduces the Sobolev space $H_{(k,s)}(\mathbb{R}^d)$ as the set

$$\{u \in \mathcal{S}'(\mathbb{R}^d) \ : \ \|u\|_{(k,s)} < \infty\}\ .$$

Show that for $k > 1/2$ there exists a linear continuous operator T from $H_{(k,s)}(\mathbb{R}^d)$ into $H^{k+s-1/2}(\mathbb{R}^{d-1})$ such that $(Tu)(x) = u(x,0)$ for all $u \in C_0^\infty(\mathbb{R}^d)$.