WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #7 Key

Problem 1. Consider the biharmonic operator $P(x, D) = \Delta^2 = \Delta\Delta$ in a smooth domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$ together with the boundary conditions

$$B_1 u = u\ , \quad B_2 u = \frac{\partial u}{\partial n}\ , \quad B_3 = \Delta\ , \quad B_4 = \frac{\partial}{\partial n}\Delta\ .$$

a.) Show that exactly two scalar boundary conditions are needed in order to obtain a well-posed boundary value problem.

Solution. Note that $P(x, \xi + in(x)\lambda) = [(\xi + in(x)\lambda) \cdot (\xi + in(x)\lambda)^2$ for $x \in \partial\Omega$ and $\xi \perp n(x)$, $\xi \neq 0$. The polynomial equation $P(x, \xi + in(x)\lambda) = 0$ has exactly two zeros with negative real part. Hence there are two linearly independent bounded solutions to the ordinary differential equation

$$P\left(x, \xi + in(x)\frac{d}{dy}\right)\Phi(y) = 0 \tag{1}$$

on the interval $[0, \infty)$. Hence, one needs two initial condition in order to possibly solve the initial value problem uniquely. This translates into the need of two scalar boundary conditions.

b.) Which pairs of boundary conditions satisfy the Lopatinskii condition and which do not ?

Solution. The zeros of the polynomial equation

$$0 = [(\xi + in(x)\lambda) \cdot (\xi + in(x)\lambda)^2 = [|\xi|^2 - \lambda^2]^2$$

are $\lambda = \pm|\xi|$. Note that both roots are repeated roots, that is roots with multiplicity two. Hence, any bounded solution on the interval $[0, \infty)$ is of the form

$$\Phi(y) = C_1 e^{-y|\xi|} + C_2 y e^{-y|\xi|}$$

Compute

$$\begin{aligned}
B_1\left(\xi + in(x)\frac{d}{dy}\right)\Phi(0) &= \Phi(0) = C_1\\
B_2\left(\xi + in(x)\frac{d}{dy}\right)\Phi(0) &= n(x)\cdot\left(\xi + in(x)\frac{d}{dy}\right)\Phi(0) = i\Phi'(0) = i[-C_1|\xi| + C_2]\\
B_3\left(\xi + in(x)\frac{d}{dy}\right)\Phi(0) &= |\xi|^2\Phi(0) - \Phi''(0) = C_1|\xi|^2 - [C_1|\xi|^2 - 2C_2|\xi|] = 2C_2|\xi|\\
B_4\left(\xi + in(x)\frac{d}{dy}\right)\Phi(0) &= i|\xi|^2\Phi'(0) - i\Phi'''(0) = i[-C_1|\xi|^3 + C_2|\xi|^2 + C_1|\xi|^3 - 3C_2|\xi|^2]\\
&= -i2C_2|\xi|^2
\end{aligned}$$

According to part a.), the only way to satisfy the Lopatinskii condition is to use two scalar boundary condition. Hence, the unique solvability of the IVP for the ODE (1) reduces to the question of uniqueness in the case of two homogeneous boundary conditions at $y = 0$.
B_1, B_2: If $\Phi(0) = \Phi'(0) = 0$, then $C_1 = 0$ and $C_2 = 0$ and the only solution is $\Phi(y) \equiv 0$.
B_1, B_3: If $\Phi(0) = 0$ and $|\xi|^2\Phi(0) - \Phi''(0) = 0$, then $C_1 = 0$ and also $C_2 = 0$ since $\xi \neq 0$.
B_1, B_4: If $\Phi(0) = 0$ and $|\xi|^2\Phi'(0) - \Phi'''(0) = 0$, the $C_1 = 0$ and $C_2 = 0$.
B_2, B_3: If $\Phi'(0) = 0$, then $C_2 = C_1|\xi|$. If, in addition $|\xi|^2\Phi(0) - \Phi''(0) = 0$, then $C_2 = 0$ which gives also $C_1 = 0$.
B_2, B_4: If $\Phi'(0) = 0$ and, in addition, $|\xi|^2\Phi'(0) - \Phi'''(0) = 0$, then $C_2 = 0$ and because of $C_1|\xi| = C_2$, C_1 must vanish as well.
B_3, B_4: If $|\xi|^2\Phi(0) - \Phi''(0) = 0$, then $C_2 = 0$ and the condition $|\xi|^2\Phi'(0) - \Phi'''(0) = 0$ does not do anything to C_1. Hence, in this case the Lopatinskii condition does not hold.

Problem 2. Let $P(x, \xi)$ be a *scalar* elliptic operator of order m on a smooth domain $\Omega \subset \mathbb{R}^d$ and suppose $d \geq 3$. Then for all $x \in \partial\Omega$ and $\xi \perp n(x)$, $\xi \neq 0$ we have the factorization $P_m(x, \xi + i\lambda n(x)) = P_+(x, \xi, \lambda)P_-(x, \xi, \lambda)$ where the polynomial P_+ has only roots with positive real part in λ and the polynomial P_- has only roots with negative real part in λ. (One can show that the the polynomials P_+ and P_- can be chosen to have smooth coefficients.) Let $B_1, ..., B_l$ be l scalar boundary conditions where $l = \deg P_-$.

a.) Using the fact that the zeros of P are continuous functions of (x, ξ), show that l is independent of (x, ξ). Here the condition $d \geq 3$ is important.

Solution. We will make the additional assumption that $\partial\Omega$ is connected. This assumption may not be necessary, however, it will simplify the proof and makes us avoid certain topological consideration. Without loss of generality we will assume that

$$P_m(x, \xi + i\lambda n(x)) = \lambda^m + \sum_{j=0}^{m} a_j(x, \xi)\lambda^j \ .$$

Otherwise, we divide by the leading coefficient of the polynomial which is cannot vanish since P is elliptic.

Fix $(\underline{x}, \underline{\xi})$ with $\underline{x} \in \partial\Omega$ and $\underline{\xi} \neq 0$, $\underline{\xi} \perp n(\underline{x})$, the polynomial $P_m(\underline{x}, \underline{\xi} + i\lambda n(\underline{x}))$ has l zeros λ_j, $j = 1, ..., l$ with negative real part and $m - l$ zeros λ_j, $j = l + 1, ..., m$ with positive real part. This follows from the fundamental theorem of algebra and from the ellipticity condition. Hence we can write

$$P_m(\underline{x}, \underline{\xi} + i\lambda n(\underline{x})) = \prod_{j=1}^{m}(\lambda - \lambda_j) = P_-(\lambda)P_+(\lambda)$$

where $P_-(\lambda) = \prod_{j=1}^{l}(\lambda - \lambda_j)$ and $P_+(\lambda) = \prod_{j=l+1}^{m}(\lambda - \lambda_j)$. This procedure can be performed at any point $(x, \xi) \in \partial\Omega \times \mathbb{R}^d \setminus \{0\}$ with $\xi \perp n(x)$. Hence

$$P_m(x, \xi + i\lambda n(x)) = \prod_{j=1}^{m}(\lambda - \lambda_j(x, \xi))$$

Since the roots λ_j are continuous functions of (x, ξ) which cannot be purely imaginary and the set

$$\mathscr{W} = \{(x, \xi) \in \partial\Omega \times \mathbb{R}^d \setminus \{0\} \ : \ \xi \perp n(x)\}$$

is connected because of $d \geq 3$, no root λ_j with $\Re\lambda_j > 0$ for some $(x, \xi) \in \mathscr{W}$ can have negative real part at another point in $\mathscr{W}$. Hence, the number l is independent of the choice of $(x, \xi) \in \mathscr{W}$.

To understand why $\mathscr{W}$ is connected it is important to understand that for a given $\underline{x} \in \partial\Omega$ the set $\{\xi \ : \ \xi \neq 0 \text{ and } \xi \perp n(\underline{x}\}$ is of dimension $d-1$. If $d = 2$, this is a punctured straight line (that is a line with one point removed) and hence not connected. On the other hand, a punctured plane is connected.

b.) Show that the Lopatinskii condition of Definition 2.5.8 is equivalent to the following condition. The polynomials

$$B_j(x, \xi + i\lambda n(x)) \ , \quad j = 1, 2, .., l$$

are a basis of $\mathbb{C}[\lambda]/P_-(x, \xi, \lambda)$ for all $\xi \perp n(x)$, $\xi \neq 0$. Here $\mathbb{C}[\lambda]$ denotes the ring of polynomials with complex coefficients in λ and $\mathbb{C}[\lambda]/P_-(x, \xi, \lambda)$ is the quotient ring of $\mathbb{C}[\lambda]$ by the ideal generated by $P_-(x, \xi, \lambda)$.

Solution. Note that the quotien ring $\mathbb{C}[\lambda]/P_-(x, \xi, \lambda)$ is isomorphic to the ring of polynomials with complex coefficients of degree less or equal to $l-1$. A basis of this ring is given by the monomials $1, \lambda, ..., \lambda^{d-1}$. The bounded solutions of

$$P_m\left(x, \xi + i\lambda n(x)\right) \Phi(y) = 0$$

is the solution set to the differential equation

$$P_-\left(x, \xi, \frac{d}{dy}\right) \Phi(y) = 0 \ .$$

Since $\deg P_- = l$, the solution space is l dimensional. In order to determine a solution uniquely, one will need initial conditions for the first $l-1$ derivatives. The l boundary conditions

$$B_j\left(x, \xi + in(x)\frac{d}{dy}\right) \Phi(0) \ , \qquad j = 1, 2, ..., l \ ,$$

determine the first $l-1$ derivatives of Φ at zero if the polynomials $B_j(x, \xi + i\lambda n(x))$, $j = 1, .., l$ are linearly independent and are of degree less than l. However, if a certain $B_j(x, \xi + i\lambda n(x))$ has degree greater than $l-1$, one has

$$B_j(x, \xi + i\lambda n(x)) = Q_j(x, \xi, \lambda) P_-(x, \xi + in(x)\lambda) + R_j(x, \xi, \lambda) \ ,$$

where R_j is a polynomial in λ of degree less than l. Observe that since Φ solves the differential equation above, one has

$$B_j\left(x, \xi + in(x)\frac{d}{dy}\right) \Phi(0) = R_j\left(x, \xi, \frac{d}{dy}\right) \Phi(0) \ .$$

This step shows that the Lopatinskii condition is satisfied if and only if the polynomail $R_j(x, \xi, \lambda)$ form a basis of the ring of polynomials of degree less than l. This finishes the proof since R_j is the representant of $B_j \in \mathbb{C}[\lambda]/P_-(x, \xi, \lambda)$ in the ring of polynomials of degree less than l.

Problem 3. Consider the boundary value problem

$$P(\partial)u = \begin{bmatrix} \nabla \times v + \nabla w \\ -\nabla \cdot v \end{bmatrix} = f \quad \text{in } \Omega \subset \mathbb{R}^3 \ ,$$
$$n \times v = g \quad \text{in } \partial\Omega \ .$$

Here $u = \begin{bmatrix} v \\ w \end{bmatrix}$ is a vector-valued function with four components, v is a vector-valued function with three components, and the function w is scalar-valued.

a.) Use the Divergence Theorem (Gauss's Theorem) to find an *integration by parts formula* for the curl operator. To be more precise, the task is to express the integral

$$\int_\Omega (\nabla \times v) \cdot q \, dx$$

where v and q are smooth vector-valued function with three components each, by integrals which do not contain any derivative of v.

Solution. Note that $\nabla \cdot (v \times q) = (\nabla \times v) \cdot q - v \cdot (\nabla \times q)$. Hence, using Gauss's Theorem gives

$$\int_\Omega (\nabla \times v) \cdot q \, dx = \int_\Omega v \cdot (\nabla \times q) \, dx + \int_{\partial\Omega} n \cdot (v \times q) \, ds \, .$$

The boundary integral may be written as

$$\int_{\partial\Omega} (n \times v) \cdot q \, ds = - \int_{\partial\Omega} v \cdot (n \times q) \, ds \, .$$

b.) Define an unbounded operator $\mathscr{P} \, : \, L_2(\Omega)^4 \to L_2(\Omega)^4$ with

$$\mathscr{D}(\mathscr{P}) = \{ u \in H^1(\Omega)^4 \, : \, n \times v = 0 \text{ in } \partial\Omega \}$$

and $\mathscr{P} u = P(\partial) u$. Find the (Hilbert space) adjoint $\mathscr{P}^*$ in the sense of unbounded operators. Recall that

$$\mathscr{D}(\mathscr{P}^*) = \{ y \in L_2(\Omega)^4 \, : \, u \mapsto (\mathscr{P} u, y) \text{ is a bounded linear functional on } \mathscr{D}(\mathscr{P}) \} \, .$$

Solution. For $u, y \in H^1(\Omega)^4$ compute using part a.) and the Divergence Theorem

$$\begin{aligned}
(P(\partial)u, y) =& \int_\Omega (\nabla \times v) \cdot \overline{q} \, dx + \int_\Omega \nabla w \cdot \overline{q} \, dx - \int_\Omega (\nabla \cdot u) \overline{z} \, dx \\
=& \int_\Omega v \cdot (\nabla \times \overline{q}) \, dx + \int_\Omega u \cdot \nabla \overline{z} \, dx - \int_\Omega w \nabla \overline{q} \, dx \\
&+ \int_{\partial\Omega} (n \times u) \cdot \overline{q} \, ds + \int_{\partial\Omega} w \, n \cdot \overline{q} \, ds - \int_{\partial\Omega} n \cdot u \, \overline{z} \, ds \\
=& (u, P(\partial)y) + + \int_{\partial\Omega} (n \times u) \cdot \overline{q} \, ds + \int_{\partial\Omega} w \, n \cdot \overline{q} \, ds - \int_{\partial\Omega} n \cdot u \, \overline{z} \, ds \, ,
\end{aligned}$$

where we write $y = (q, z)^T$ similar to $u = (v, w)^T$. Hence, the only way to obtain the estimate in the definition for the dual is that $y \in L_2(\Omega)^4$, $P(\partial)y \in L_2(\Omega)^4$, and that $z = n \cdot y = 0$ in $\partial\Omega$. According to Corollary 2.8.3. this means that $y \in H^1(\Omega)^4$ since the boundary conditions $B_1 y = n \cdot y$, $B_2 y = z$ satisfy the Lopatinskii condition. This has been verified in last weeks homework. Hence,

$$\mathscr{D}(\mathscr{P}^*) = \{ y \in H^1(\Omega)^4 \, : \, z = n \cdot y = 0 \text{ in } \partial\Omega \}$$

and $\mathscr{P}^* y = P(\partial) y$. The unbounded operator $\mathscr{P}$ is not self adjoint.

c.) Suppose that Ω is simply connected. Find $\ker T$ and $\operatorname{coker} \mathrm{T}$ where T is the operator is the continuous linear operator $T \, : \, H^{k+1}(\Omega)^4 \to H^k(\Omega)^4 \times H^{1/2+k}(\partial\Omega)^3$ defined by

$$Tu = \Big(P(\partial)u, n \times v\Big|_{\partial\Omega} \Big) \, .$$

Solution. Note that $n\times v|_{\partial\Omega}$ is a tangential vector field. Indeed $n\cdot(n\times v) = v\cdot(n\times n) = 0$. Hence, the operator T should be defined as an operator from $H^{k+1}(\Omega)^4$ into $H^k(\Omega)^4 \times H_t^{1/2+k}(\partial\Omega)^3$ where $H_t^{1/2+k}(\partial\Omega)^3$ denotes the tangential vector fields on $\partial\Omega$, that is

$$H_t^{1/2+k}(\partial\Omega)^3 = \{v \in H^{1/2+k}(\partial\Omega)^3 \ : \ n\cdot v = 0 \text{ in } \partial\Omega\}$$

From Corollary 2.6.9 we know that $u \in \ker T$ implies $u \in C^\infty(\overline{\Omega})$. In addition, the linear operator T is certainly injective in its second component. Hence, it will suffice to study $\ker \mathscr{P}$. Taking the curl of the first (vector) equation gives

$$\nabla\times(\nabla\times v) + \nabla\times\nabla w = 0$$

which because of $\nabla\times\nabla w = 0$ implies $\nabla\times(\nabla\times v) = 0$. Then, using part a.) and the boundary condition $n\times v = 0$ gives

$$0 = \int_\Omega \nabla\times(\nabla\times v)\cdot\overline{v}\,dx = \int_\Omega |\nabla\times v|^2\,dx$$

and thus $\nabla\times v = 0$. Since Ω is simply connected we have $v = \nabla p$ for some smooth scalar-valued function p. Furthermore, the boundary condition $n\times v = 0$ gives $n\times\nabla p =$ which means that the *tangential gradient* of p vanishes on $\partial\Omega$. This can happen only inf p is constant on each connected component $\partial\Omega_j$ $(j = 1,..,r)$ of the boundary.

Example. If $\Omega = B(0,2)\setminus\overline{B(0,1)}$ where $B(x,R)$ is the open ball centered at x with radius R, then Ω is simply connected and $\partial\Omega$ has two connected components, $\partial\Omega_1 = \partial B(0,2)$ and $\partial\Omega_2 = \partial B(0,1)$.

Furthermore, since $\nabla\cdot(\nabla\times v) = 0$ in we obtain also $\nabla\cdot\nabla p = \Delta p = 0$. Hence, p is the solution to the following boundary value problem for the Laplace equation

$$\Delta p = 0 \text{ in } \Omega\ , \qquad p = c_j \text{ on } \Omega_j\ , j = 1,...,r\ . \tag{2}$$

claim. The solution space of this problem has dimension r.

Proof. Each solution to the boundary value problem for p can be written as a linear combination of r linearly independent function p_j which are given as unique solutions to the Dirichlet problem

$$\Delta p_j = 0 \text{ in } \Omega\ , \qquad p_j = 1 \text{ on } \partial\Omega_j \text{ and } p_j = 0 \text{ on } \partial\Omega_k \text{ for } k\neq j\ .$$

One verifies that $p = \sum_{j=1}^r c_j p_j$ solves the problem (2). ☐

Since $\nabla\times v = 0$ we also have that $\nabla w = 0$ which shows that w must be constant. A basis of $\ker T$ is given by the vectors $u_j = (\nabla p_j, 1)^T \in C^\infty(\overline{\Omega})^4$ for $j = 1,2,...,r$. This proves $\dim\ker T = r$ where r is the number of connected components on $\partial\Omega$.

To determine the cokernel of T, note that T is surjective in its second component. Hence, it will suffice to study the cokernel of $\mathscr{P}$ which is isomorphic to the orthogonal complement of the range of $\mathscr{P}$ which in turn is the same as the kernel of $\mathscr{P}^*$. This is done as follows. Suppose that $y = (q,z)^T \in \mathscr{P}^*$, that is

$$\nabla\times y + \nabla z = 0, \quad \nabla\cdot y = 0 \text{ in } \Omega\ , \qquad n\cdot y = z = 0 \text{ in } \partial\Omega\ .$$

Applying the divergence to the first vector equation gives $\Delta z = 0$ which together with the boundary condition results in $z\equiv 0$ in Ω. Then $\nabla\times y = 0$ which gives again $y = \nabla p$ for some smooth scalar-valued p and the equation $\nabla\cdot y = 0$ gives $\Delta p = 0$ and the first boundary condition provides $n\cdot\nabla p = 0$. Since Ω is simply connected, we conclude that p

has to be constant which results in $v = 0$. Hence, $\ker \mathscr{P}^* = \{0\}$ and hence, the operator T is surjective.

Observe that the operator T has the following interesting property: The index of T is equal to the number of connected components of the boundary of Ω. The index of the operator describes topological properties of Ω. The assumption that Ω is simply connected can be removed but it will change the index of the operator. The kernel and the cokernel of the operator T are known to be isomorphic to the *cohomology spaces* for differential forms.

d.) Use your answer of c.) to describe the solvability of the boundary value problem above. In particular, answer the following question. Is the boundary value problem solvable for all $f \in L_2(\Omega)^4$ and $g \in H^{1/2}(\partial\Omega)^3$? Is the solution unique ? What regularity does the solution posses ? If the data are more regular, say $f \in H^k(\Omega)^4$ and $g \in H^{k+1/2}(\partial\Omega)^3$ where k is a positive integer, what can you say about the solution ?

The work in part c.) shows that the boundary value problem has solutions for all $f \in L_2(\Omega)^4$ and $g \in H^{1/2}(\partial\Omega)^3$. However, the solutions are not unique. To every solution we can add an element from $\ker T$. In order to produce a unique solution one has to restrict the solution space to the quotient space

$$H^1(\Omega)/\ker T \approx (\ker T)^\perp = \{u \in H^1(\Omega) \ : \ (u, U)_\Omega = 0 \text{ for all } U \in \ker T\} \ .$$

For more regular data, say $f \in H^k(\Omega)^4$ and $g \in H^{k+1/2}(\partial\Omega)^3$, one obtains a unique solution in the linear space

$$\{u \in H^k(\Omega) \ : \ (u, U)_\Omega = 0 \text{ for all } U \in \ker T\} \ .$$