WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #7 due 12/04/2015

Problem 1. Consider the biharmonic operator $P(x, D) = \Delta^2 = \Delta\Delta$ in a smooth domain Ω together with the boundary conditions

$$B_1 u = u \ , \quad B_2 u = \frac{\partial u}{\partial n} \ , \quad B_3 = \Delta \ , \quad B_4 = \frac{\partial}{\partial n}\Delta \ .$$

a.) Show that exactly two scalar boundary conditions are needed in order to obtain a well-posed boundary value problem.

b.) Which pairs of boundary conditions satisfy the Lopatinskii condition and which do not ?

Problem 2. Let $P(x, \xi)$ be a *scalar* operator of order m on a smooth domain $\Omega \subset \mathbb{R}^d$ and suppose $d \geq 3$. Then for all $x \in \partial\Omega$ and $\xi \perp n(x)$, $\xi \neq 0$ we have the factorization $P_m(x, \xi + i\lambda n(x)) = P_+(x, \xi, \lambda)P_-(x, \xi, \lambda)$ where the polynomial P_+ has only roots with positive real part in λ and the polynomial P_- has only roots with negative real part in λ. (One can show that the the polynomials P_+ and P_- can be chosen to have smooth coefficients.) Let $B_1, ..., B_l$ be l scalar boundary conditions where $l = \deg P_-$.

a.) Using the fact that the zeros of P are continuous functions of (x, ξ), show that l is independent of (x, ξ). Here the condition $d \geq 3$ is important.

b.) Show that the Lopatinskii condition of Definition 2.5.8 is equivalent to the following condition. The polynomials

$$B_j(x, \xi + i\lambda n(x)) \ , \quad j = 1, 2, .., l$$

are a basis of $\mathbb{C}[\lambda]/P_-(x, \xi, \lambda)$ for all $\xi \perp n(x)$, $\xi \neq 0$. Here $\mathbb{C}[\lambda]$ denotes the ring of polynomials with complex coefficients in λ and $\mathbb{C}[\lambda]/P_-(x, \xi, \lambda)$ is the quotient ring of $\mathbb{C}[\lambda]$ by the ideal generated by $P_-(x, \xi, \lambda)$.

Problem 3. Consider the boundary value problem

$$P(\partial)u = \begin{bmatrix} \nabla \times v + \nabla w \\ -\nabla \cdot v \end{bmatrix} = f \quad \text{in } \Omega \subset \mathbb{R}^3 \ ,$$

$$n \times v = g \quad \text{in } \partial\Omega \ .$$

Here $u = \begin{bmatrix} v \\ w \end{bmatrix}$ is a vector-valued function with four components, v is a vector-valued function with three components, and the function w is scalar-valued.

a.) Use the Divergence Theorem (Gauss's Theorem) to find an *integration by parts formula* for the curl operator. To be more precise, the task is to express the integral

$$\int_\Omega (\nabla \times v) \cdot q \, dx$$

where v and q are smooth vector-valued function with three components each, by integrals which do not contain any derivative of v.

b.) Define an unbounded operator $\mathscr{P} : L_2(\Omega)^4 \to L_2(\Omega)^4$ with

$$\mathscr{D}(\mathscr{P}) = \{u \in H^1(\Omega)^4 \;:\; n \times v = 0 \text{ in } \partial\Omega\}$$

and $\mathscr{P}u = P(\partial)u$. Find the (Hilbert space) adjoint $\mathscr{P}^*$ in the sense of unbounded operators. Recall that

$$\mathscr{D}(\mathscr{P}^*) = \{y \in L_2(\Omega)^4 \;:\; u \mapsto (\mathscr{P}u, y) \text{ is a bounded linear functional on } \mathscr{D}(\mathscr{P})\} \;.$$

c.) Suppose that Ω is simply connected. Find $\ker T$ and coker T where T is the operator is the continuous linear operator $T : H^{k+1}(\Omega)^4 \to H^k(\Omega)^4 \times H^{1/2+k}(\partial\Omega)^3$ defined by

$$Tu = \left(P(\partial)u, n \times v\Big|_{\partial\Omega}\right) \;.$$

d.) Use your answer of c.) to describe the solvability of the boundary value problem above. In particular, answer the following question. Is the boundary value problem solvable for all $f \in L_2(\Omega)^4$ and $g \in H^{1/2}(\partial\Omega)^3$? Is the solution unique ? What regularity does the solution posses ? If the data are more regular, say $f \in H^k(\Omega)^4$ and $g \in H^{k+1/2}(\partial\Omega)^3$ where k is a positive integer, what can you say about the solution ?