WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #8 due 12/11/2015

Problem 1. Show that the stationary elastic equation

$$-D(\partial)^T \mathscr{A}(x) D(\partial) u = f \qquad \text{in } \Omega$$

with $a_{11} = a_{22} = a_{33}$, $a_{44} = a_{55} = a_{66} = \mu$, $a_{12} = a_{23} = a_{13} = \lambda = a_{11} - 2a_{44}$, and all other entries of the 6×6 matrix $\mathscr{A}$ being zero, results in the isotropic elastic equation

$$-\nabla \cdot [\mu(\nabla u + \nabla u^T)] - \nabla[\lambda \nabla \cdot u] = f\ .$$

In this context it may be useful to recall the operator

$$D(\partial) = \begin{bmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{bmatrix} .$$

Problem 2. a.) Show that the operator

$$P(D) = \frac{1}{4}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)^2$$

considered on the unit disk centered at the origin is not properly elliptic. This is to say that the symbol $p(\xi + in(x)\lambda)$ of the operator on the boundary S^1 with $\xi \perp n(x)$, $\xi \neq 0$ does not decompose into a product of polynomials p_+ and p_- with roots in λ having only positive real part (negative real part) with the degree of p_+ independent of ξ.

b.) Verify that the Dirichlet problem in the unit disk is not regular by constructing an infinite-dimensional space of solutions to the equation $Pu = 0$ with $u|_{S^1} = 0$.

Problem 3. Let $e, h : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ be vector-valued functions with three components each and suppose that ε, μ, σ are real symmetric 3×3 matrix functions ε, μ, σ : $[0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{3\times 3}$ and that ε and μ are uniformly positive definite.

a.) Show that the dynamic Maxwell equations

$$\partial_t(\varepsilon e) - \nabla \times h + \sigma e = f_1 \qquad \partial_t(\mu h) + \nabla \times e = f_2$$

form a symmetric hyperbolic system of order 1 in the sense of Definition 3.1.1.

b.) Can you find conditions on ε and μ which make this system constantly hyperbolic ? Hint: Make use of the solution of the first problem of homework #1.