WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #9 Key

Problem 1. Consider the homogeneous isotropic elastic wave equations with constant coefficients, that is

$$\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot [\mu(\nabla u + \nabla u^T)] - \nabla[\lambda \nabla \cdot u] = 0 \; .$$

Here ρ is the density, μ and λ are the Lamé parameters, and $u \; : \; [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ denotes the displacement. All three coefficients are assumed to be positive constant.

a.) Find solutions of the form $u(t,x) = ae^{i(\omega t - k\cdot x)}$ where ω is a positive constant and $a, k \in \mathbb{R}^3$. These solutions are called plane waves.

Solution. Suppose that $a, k \neq 0$. Compute

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u, \quad \nabla u + \nabla u^T = -i[ak^T + ka^T]e^{i(\omega t - k\cdot x)}, \quad \nabla \cdot u = -ia\cdot ke^{i(\omega t - k\cdot x)} \; ,$$

and hence

$$\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &- \nabla \cdot [\mu(\nabla u + \nabla u^T)] - \nabla[\lambda \nabla \cdot u] \\ &= -\rho\omega^2 ae^{i(\omega t - k\cdot x)} + \mu[ak^T + ka^T]ke^{i(\omega t - k\cdot x)} + \lambda[a^T k]ke^{i(\omega t - k\cdot x)} \\ &= e^{i(\omega t - k\cdot x)}[\omega^2 a - \mu a|k|^2 - (\mu+\lambda)k(a^T k)] \; . \end{aligned}$$

If $a = k$ then

$$\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot [\mu(\nabla u + \nabla u^T)] - \nabla[\lambda \nabla \cdot u] = ae^{i(\omega t - k\cdot x)}[-\rho\omega^2 + (2\mu+\lambda)|k|^2] = 0$$

if and only if $\omega = \sqrt{\frac{2\mu+\lambda}{\rho}}|k|$. If $a \perp k$, that is $a\cdot k = a^T k = 0$, then

$$\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot [\mu(\nabla u + \nabla u^T)] - \nabla[\lambda \nabla \cdot u] = ae^{i(\omega t - k\cdot x)}[-\rho\omega^2 + \mu|k|^2] = 0$$

if and only if $\omega = \sqrt{\frac{\mu}{\rho}}|k|$.

Comment. More generally, a plane wave is a solution to the differential equation of the form $u(t,x) = f(t, k\cdot x)$ for some $k \in \mathbb{R}^3$.

b.) Characterize your plane wave solutions as longitudinal ($k \parallel a$) or transversal ($k \perp a$).

Solution. For $a \parallel k$ with $|k| = 1$ the plane wave solution

$$u = ae^{i(t\sqrt{(2\mu+\lambda)/\rho} - k\cdot x)}$$

was obtained. This wave is longitudinal since its amplitude a points in the same direction as the wave vector k. This is to say, that the direction of propagation given by k coincides with the direction of the oscillation. In the context of elasticity, these waves a known as p-waves or pressure waves.

On the other hand, if $k \perp a$ with $|k| = 1$ one obtains the plane wave solution

$$u = ae^{i(t\sqrt{\mu/\rho} - k\cdot x)}$$

This time the oscillations move perpendicular to the direction of propagation. Hence, these waves are called transversal waves or in the setting of elasticity, shear waves. Since there are two linearly independent vectors a perpendicular to k, there are two linearly independent shear waves. The constant ω stands for the frequency of the waves. In our setting the pressure wave has a higher frequency than the shear wave.

Problem 2. Suppose that P is symmetric hyperbolic with coefficients in $W^1_\infty(Q)$ and that $A^0 = I_N$. Show that $u \in L_2(Q)$ and $Pu \in L_2(Q)$ implies that $u(t, \cdot) \in H^{-1/2}(\mathbb{R}^d)$ for all $t \in [0, T]$. Recall that $Q = (0, T) \times \mathbb{R}^d$.

Solution. Suppose that $w, v \in H^1(Q)$. Using integration by parts with respect to the time variable gives

$$\int_0^t \int_{\mathbb{R}^d} Pw \cdot \overline{v}\, dsdx = \int_0^t \int_{\mathbb{R}^d} w \cdot \overline{P^* v}\, dsdx + \int_\Omega w \cdot \overline{v}\, dx \Big|_{s=0}^{s=t},$$

for $0 < t \leq T$, where P^* is the adjoint operator introduced in Section 3.2. Choosing $v \in H^1(Q)$ such that $v(0, x) = 0$ for all $x \in \mathbb{R}^d$ gives

$$\big(w(t, \cdot), v(t \cdot)\big)_{L_2(\mathbb{R}^d)} = \int_0^t (Pw, v)_{L_2(\mathbb{R}^d)} dt - \int_0^t (w, P^* v)_{L_2(\mathbb{R}^d)} dt\ .$$

Choose now $g \in H^{1/2}(\mathbb{R}^d)$. By Proposition 2.2.5 there exists a $v \in H^1(Q)$ and a positive constant C independent of g such that $v(0, \cdot) = g$ and $\|v\|_{H^1(Q)} \leq C\|g\|_{H^{1/2}(\mathbb{R}^d)}$. In addition one can multiply this function v by a suitable smooth cutoff function $\varphi(t)$ such that $\varphi(0)v(0, \cdot) = 0$. For convenience we will denoted the resulting function by v. It satisfies still the bound $\|v\|_{H^1(Q)} \leq C\|g\|_{H^{1/2}(\mathbb{R}^d)}$, with for a larger constant C. Then, using the displayed formula above and the continuity of P^* as an operator from $H^1(Q)$ into $L_2(Q)$, one gets,

$$\begin{aligned}
\|w(t, \cdot)\|_{H^{-1/2}(\mathbb{R}^d)} &= \sup_{\|g\|_{H^{1/2}(\mathbb{R}^d)}} |\big(w(t, \cdot), g\big)_{L_2(\mathbb{R}^d)}| \\
&\leq \|Pw\|_{L_2(Q)}\|v\|_{L_2(Q)} + \|P^* v\|_{L_2(Q)}\|_{L_2(q)}\|w\|_{L_2(Q)} \\
&\leq \|Pw\|_{L_2(Q)}\|v\|_{L_2(Q)} + C\|v\|_{H^1(Q)}\|_{L_2(q)}\|w\|_{L_2(Q)} \\
&\leq C\|v\|_{H^1(Q)} \left[\|w\|_{L_2(Q)} + \|Pw\|_{L_2(Q)}\right] \\
&\leq C\|g\|_{H^{1/2}(\mathbb{R}^d)} \left[\|w\|_{L_2(Q)} + \|Pw\|_{L_2(Q)}\right]
\end{aligned}$$

for all $w \in H^1(Q)$. To complete the proof we use the following density statement which we will prove below: For $u \in L_2(Q)$ with $Pu \in L_2(Q)$ there exists a sequence of $w_n \in C_0^\infty(\mathbb{R}^{d+1})$, $n = 1, 2, ...$ such that

$$\|w_n - u\|_{L_2(Q)} \to 0 \quad \text{ and } \|Pw_n - Pu\|_{L_2(Q)} \to 0\ . \tag{1}$$

Using the inequality above shows that $\|w_n(t, \cdot)\|$ converges in $H^{-1/2}(\mathbb{R}^d)$ and the limit is defined to be the trace $u(t, \cdot)$. This proves the statement for $0 < t \leq T$. For $t = 0$ one chooses $t = T$ and $v \in H^1(Q)$ with $v(T, \cdot) = 0$.

Finally we will show that the restriction of functions in $C_0^\infty(\mathbb{R}^{d+1})$ is dense in the linear space

$$\mathscr{H} = \{u \in L_2(Q) \ : \ Pu \in L_2(Q)\} \ .$$

Suppose that $u \in \mathscr{H}$ is orthogonal to all functions in $v \in C_0^\infty(\mathbb{R}^{d+1})$, that is

$$0 = (u, v)_{L_2(Q)} + (Pu, Pv)_{L_2(Q)} \quad \text{for all } v \in C_0^\infty(\mathbb{R}^{d+1}) \ .$$

Then $u_0 = -Pu$ satisfies $P^* u_0 = -u$ in the sense of distributions. Since $u_0 \in L_2(Q)$ we know that $u_0 \in \mathscr{H}$ as well since P is symmetric hyperbolic. Indeed, one can show that for $w \in L_2(Q)$ we have $Pw \in L_2(Q)$ if and only if $P^* w \in L_2(Q)$. Hence we know that

$$(P^* u_0, v)_{L_2(Q)} = (u_0, Pv)_{L_2(Q)} \qquad \text{for all } v \in C_0^\infty(\mathbb{R}^{d+1}) \ .$$

Hence $u_0 \in \mathscr{H}_0$ which is the closure of the space $C_0^\infty(Q)$ with respect to the topology in $\mathscr{H}$. There must exist a sequence $\psi_k \in C_0^\infty(Q)$ $(k = 1, 2, ...)$ such that

$$(u, v)_{L_2(Q)} + (Pu, Pv)_{L_2(Q)} = \lim_{k\to\infty} \left\{(-P^* \psi_k, v)_{L_2(Q)} + (\psi_k, Pv)_{L_2(Q)}\right\} = 0 \ ,$$

for all $v \in \mathscr{H}$. Thus $u \equiv 0$ and consequently restriction of $C_0^\infty(\mathbb{R}^{d+1})$ functions to Q is dense in $\mathscr{H}$.

Problem 3. Prove the following simplified version of Lemma 3.2.1: A function $u \in L_\infty(Q)$ satisfies $u \in W^1_\infty(\mathbb{R}^d)$ if and only if u is Lipschitz, i.e., there exists a constant $L > 0$ such that $|u(x) - u(y)| \le L|x - y|$ for all $x, y \in \mathbb{R}^d$. (Hint: Use the regularization of functions as described in the lecture note before Lemma 3.3.4.)

Proof. Suppose that $u \in W^1_\infty(\mathbb{R}^d)$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $\phi \equiv 1$ for $|x| \le R$ and consider the function $\phi u \in W^1_\infty(\mathbb{R}^d)$ with compact support. For brevity, this function is denoted by u and $u^{(\varepsilon)}$ is its regularization. Then $u^{(\varepsilon)} \to u$ in $L_1(\mathbb{R}^d)$ as $\varepsilon \to 0$ which implies the existence of a sequence $\varepsilon_l \to 0$ as $l \to \infty$ with $u^{(\varepsilon_l)} \to u$ almost everywhere (with respect to the Lebesgue measure in $\mathbb{R}^d$).

Furtheremore, for $x, y \in \mathbb{R}^d$ we have

$$u^{(\varepsilon)}(x) - u^{(\varepsilon)}(y) = \int_0^1 \nabla u^{(\varepsilon)}(tx + (1-t)y)dt \cdot (x - y) \ ,$$

which implies

$$|u^{(\varepsilon)}(x) - u^{(\varepsilon)}(y)| \le sup_{z\in\mathbb{R}^d} |\nabla u^{(\varepsilon)}(z)||x - y| \ .$$

Note that

$$\frac{\partial u^{(\varepsilon)}}{\partial x_j}(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x - y) \frac{\partial u}{\partial y_j}(y) \, dy$$

gives

$$\sup_{x\in\mathbb{R}^d} \left|\frac{\partial u^{(\varepsilon)}}{\partial x_j}(x)\right| \le \sup_{x\in\mathbb{R}^d} \left|\frac{\partial u}{\partial x_j}(x)\right|$$

and hence,

$$|u^{(\varepsilon)}(x) - u^{(\varepsilon)}(y)| \le L|x - y| \quad \text{where} \quad L = \sup_{x\in\mathbb{R}^d} |\nabla u(x)| \ ,$$

which is to say that the regularization on u is uniformly Lipschitz continuous. Since $u^{(\varepsilon_l)}$ converges almost everywhere to u we know that

$$|u(x) - u(y)| \le L|x - y| \quad \text{for all } x, y \in \mathbb{R}^d \setminus \mathcal{N} \ , \tag{2}$$

where $\mathcal{N}$ is a set of Lebesgue measure zero. Recall that functions in L_∞ are equivalence classes of functions which may differ from each other only on a set of Lebesgue measure zero. For $x \in \mathcal{N}$ one considers the representative which satisfies

$$u(x) = \lim_{x_n \to x} u(x_n) \quad x_n \in \mathbb{R}^d \setminus \mathcal{N} \, .$$

In view of (2), the value $u(x)$ is independent of the chosen sequence and the chosen representative satisfies the Lipschitz condition for all $x, y \in \mathbb{R}^d$. Finally, since R was an arbitrary positive number, one obtains that each equivalence class $u \in W^1_\infty(\mathbb{R}^d)$ has a Lipschitz continuous representative.

To prove the converse, assume that $u \in L_\infty$ and that there exists a constant $L > 0$ such that $|u(x) - u(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}^d$. For fixed j, the difference quotients $D_{j,h}u(x) = (u(x + he_j) - u(x))/h$ where e_j is the jth standard basis vector and $h \neq 0$ are uniformly bounded. There must exist a sequence h_l, $l = 1, 2, 3, ...$ and $v \in L_\infty(\mathbb{R}^d)$ such that $D_{j,h_l}u \to v$ weakly star in $L_\infty(\mathbb{R}^d)$. This is to say that for all $w \in L_1(\mathbb{R}^d)$ we have

$$\lim_{l \to \infty} \int_{\mathbb{R}^d} w D_{j,h_l} u \, dx = \int_{\mathbb{R}^d} wv \, dx$$

Finally, we will show that v is the distributional derivative of u in the jth coordinate direction. Let $w \in C_0^\infty(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} u \frac{\partial w}{\partial x_j} \, dx = \lim_{l \to \infty} \int_{\mathbb{R}^d} u D_{j,-h_l} w \, dx = - \lim_{l \to \infty} \int_{\mathbb{R}^d} D_{j,h_l} u \, w \, dx = - \int_{\mathbb{R}^d} vw \, dx \ ,$$

which proves that v is the first derivative of u with respect to x_j.

$\square$