SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #1 due 4/26

Problem 1. Suppose that $\Omega \subset \mathbb{R}^d$ is an open, bounded and connected set with a C^1-boundary and exterior unit normal field $\nu : \partial\Omega \mapsto \mathbb{R}^d$. Suppose that $u \in C^2(\overline{\Omega}, \mathbb{R})$ satisfies

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \nu \cdot \nabla u &= \psi \quad \text{in } \partial\Omega. \end{aligned}$$

Prove that

$$\int_{\partial\Omega} \psi(x)\, dS(x) = -\int_{\Omega} f(x)\, dx.$$

Problem 2. Suppose that $u \in C(\Omega, \mathbb{R})$ and suppose there exists $R > 0$ such that $B_r(x) \subset \Omega$ for all $r \leq R$. Here $B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r\}$ is the open ball with center at x and radius r. Prove that

$$\lim_{r \to 0^+} \frac{1}{|B_r|} \int_{B_r(x)} u(y)\, dy = u(x)\ .$$

Here $|B_r|$ denotes the volume of the d-dimensional ball with radius r.

Problem 3. Suppose that $u \in C^1(\mathbb{R}^d, \mathbb{R})$. Prove that

$$\int_{B_R(0)} u(x)\, dx = \int_0^R \left[\int_{\partial B_r(0)} u(x) dS(x) \right] dr = \int_0^R r^{d-1} \left[\int_{\partial B_1(0)} u(ry) dS(y) \right] dr.\ .$$

Hint: The equality of the second and the third integral is established by means of an identity established in the lecture. The first integral can be transformed into an integral over the unit ball (that is the ball with radius 1) and then differentiated with respect to R.