SOMMERSEMESTER 2015 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #1 KEY

Problem 1. Suppose that $\Omega \subset \mathbb{R}^d$ is an open, bounded and connected set with a C^1-boundary and exterior unit normal field $\nu : \partial\Omega \mapsto \mathbb{R}^d$. Suppose that $u \in C^2(\overline{\Omega}, \mathbb{R})$ satisfies

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \nu \cdot \nabla u &= \psi \quad \text{in } \partial\Omega. \end{aligned}$$

Prove that

$$\int_{\partial\Omega} \psi(x)\, dS(x) = -\int_{\Omega} f(x)\, dx.$$

Proof. Use the divergence theorem

$$\int_{\Omega} \nabla \cdot w\, dx = \int_{\partial\Omega} w \cdot \nu\, dS$$

which is valid for all vector fields $w \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Choose $w = \nabla u$. Then, using $\nabla \cdot (\nabla u) = \Delta u$ one obtains

$$\int_{\Omega} \Delta u\, dx = \int_{\partial\Omega} \nabla u \cdot \nu\, dS\ .$$

□

Problem 2. Suppose that $u \in C(\Omega, \mathbb{R})$ and suppose there exists $R > 0$ such that $B_r(x) \subset \Omega$ for all $r \leq R$. Here $B_r(x) = \{y \in \mathbb{R}^d \ :\ |y - x| < r\}$ is the open ball with center at x and radius r. Prove that

$$\lim_{r \to 0^+} \frac{1}{|B_r|} \int_{B_r(x)} u(y)\, dy = u(x)\ .$$

Here $|B_r|$ denotes the volume of the d-dimensional ball with radius r.

Proof. Fix $\varepsilon > 0$. Since u is continuous, there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|u(y) - u(x)| < \varepsilon$. Hence, for $r < \delta$ we have

$$\begin{aligned} \frac{1}{|B_r|} \int_{B_r(x)} u(y)\, dy \leq& \frac{1}{|B_r|} \int_{B_r(x)} u(x)\, dy + \frac{1}{|B_r|} \int_{B_r(x)} [u(y) - u(x)]\, dy \\ =& u(x) \frac{1}{|B_r(x)|} \int_{B_r(x)} dx + \frac{1}{|B_r|} \int_{B_r(x)} [u(y) - u(x)]\, dy \\ =& u(x) + \frac{1}{|B_r|} \int_{B_r(x)} [u(y) - u(x)]\, dy\ . \end{aligned}$$

Thus, for $r < \delta$ we have, using the triangle inequality,

$$\left| \frac{1}{|B_r|} \int_{B_r(x)} u(y)\, dy - u(x) \right| \leq \frac{1}{|B_r|} \int_{B_r(x)} |u(y) - u(x)|\, dy < \varepsilon \; .$$

Since the choice of ε was arbitrary, the proof is complete. $\square$

Problem 3. Suppose that $u \in C^1(\mathbb{R}^d, \mathbb{R})$. Prove that

$$\int_{B_R(0)} u(x)\, dx = \int_0^R \left[\int_{\partial B_r(0)} u(x) dS(x) \right] dr = \int_0^R r^{d-1} \left[\int_{\partial B_1(0)} u(ry) dS(y) \right] dr. \; .$$

Hint: The equality of the second and the third integral is established by means of an identity established in the lecture. The first integral can be transformed into an integral over the unit ball (that is the ball with radius 1) and then differentiated with respect to R.

Proof. Introduce a function $F\,:\,[0,\infty) \mapsto \mathbb{R}$ by

$$F(R) = \int_{B_R(0)} u(x)\, dx = R^d \int_{B_1(0)} u(Ry)\, dy \; .$$

Here a change of variables in the volume integral was performed $x = Ry$, $dx = R^d\, dy$. Then, using the fact that u is differentiable, one computes using the chain rule

$$F'(R) = dR^{d-1} \int_{B_1(0)} u(Ry)\, dy + R^d \int_{B_1(0)} \sum_{j=1}^{d} \frac{\partial u}{\partial x_j}(Ry) \cdot y_j\, dy$$

Define now a vector field $w(y) = u(Ry)y$. Then $\nabla \cdot w(y) = u(Ry)d + \nabla u(Ry)R \cdot y$ and we see that using the divergence theorem

$$F'(r) = R^{d-1} \int_{\partial B_1(0)} w(y) {\cdot} \nu\, dS(y) = R^{d-1} \int_{\partial B_1(0)} w(y) {\cdot} y\, dS(y) = R^{d-1} \int_{\partial B_1(0)} u(Ry)\, dS(y) \; .$$

By the fundamental theorem of calculus,

$$\frac{d}{dR} \int_0^R r^{d-1} \left[\int_{\partial B_1(0)} u(ry) dS(y) \right] dr = R^{d-1} \int_{\partial B_1(0)} u(Ry) dS(y)$$

which proves that the derivative of the first and the third integral coincide. Since both expressions are zero for $R = 0$, they have to be the same. It remains to show that the second integral is equal to the third integral. This follows from the properties of the surface integral discussed in the lecture on April 21, in particular

$$\int_{r\Sigma} f(x)\, dS(x) = r^{d-1} \int_{\Sigma} f(ry)\, dS(y) \; ,$$

where Σ is a regular surface in $\mathbb{R}^d$. $\square$