SOMMERSEMESTER 2016 - HÖHERE ANALYSIS II
LINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #2 due 5/03

Problem 1. Suppose that $\Sigma \subset \mathbb{R}^d$ is a regular hypersurface, that is there exists an open set $U \subset \mathbb{R}^{d-1}$ and an injective function $x \in C^1(U, \mathbb{R}^d)$ such that $x(U) = \Sigma$ and that the rank of the Jacobian matrix Dx is equal to $d-1$ for all $u \in U$. The pair (U, x) is a *parametrization* of Σ.

Suppose that (V, y) is another parametrization of Σ. Prove that for all $f \in C(\Sigma)$ we have

$$\int_U f(x(u))\sqrt{\det[Dx(u)^T Dx(u)]}\, du = \int_V f(y(v))\sqrt{\det[Dy(v)^T Dy(v)]}\, dv \ .$$

Note that this identity shows that the definition of the surface integral $\int_\Sigma f\, dS$ given in the lecture is independent of the parametrization used.

Problem 2. Prove the converse of Theorem 2.2: Suppose that $\Omega \subset \mathbb{R}^d$ open, $u \in C^2(\Omega, \mathbb{R})$, and that

$$u(x) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\, dS(y)$$

for all $B_r(x) \subset \Omega$. Then $\Delta u = 0$ in Ω.

Problem 3. Suppose that $a \in \mathbb{R}$, $u_0 \in C^1(\mathbb{R})$, and that $f \in C^1(\mathbb{R}^2)$. Use the method of characteristics to find a solution to the initial value problem

$$\begin{cases} \partial_t u + a\partial_x u = f(t,x) & (t,x) \in \mathbb{R}^2 \ , \\ u(0,x) = u_0(x) & x \in \mathbb{R} \ . \end{cases}$$